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On Numerical Semigroups of Genus 9

神奈川工科大学・基礎・教養教育センター　米田 二良 (Jiryo Komeda)
Center for Basic Education and Integrated Learning
Kanagawa Institute of Technology

§1. Introduction.

Let $\mathbb{N}_0$ be the additive semigroup of non-negative integers. A subsemigroup H of $\mathbb{N}_0$ is called a *numerical semigroup* if the complement $\mathbb{N}_0 \backslash H$ of H in $\mathbb{N}_0$ is a finite set. The cardinality $g(H)$ of the set $\mathbb{N}_0 \backslash H$ is called the *genus* of H. In this paper we are interested in numerical semigroups of genus 9. For a non-singular complete irreducible curve C over an algebraically closed field k of characteristic 0 (which is called a *curve* in this paper) and its point P we set

$$H(P) = \{n \in \mathbb{N}_0 | \exists \text{ a rational function } f \text{ on } C \text{ with } (f)_\infty = nP\}.$$

A numerical semigroup is *Weierstrass* if there exists a curve C with its point P such that $H(P) = H$. We have the following results:

Fact 1. *Every numerical semigroup of genus $g \leqq 8$ is Weierstrass.* (See Lax [10], Komeda [4] and Komeda-Ohbuchi [8] for the case $g = 4$, $5 \leqq g \leqq 7$ and $g = 8$ respectively.)

We note that for any $g \geqq 16$ there exists a non-Weierstrass numerical semigroup of genus g (see Buchweitz [1].) A numerical semigroup H is *primitive* if the largest positive integer not in H is less than twice the least positive integer in H. Then we know the following fact:

Fact 2. *Every primitive numerical semigroup of genus 9 is Weierstrass.* (See Komeda [6].)

We want to study non-primitive numerical semigroups of genus 9.

§2. Non-primitive numerical semigroups of genus 9.

An *n-semigroup*, i.e., a numerical semigroup in which the least positive integer is n. When n is lower, we have the following result:

Fact 3. *For $1 \leqq n \leqq 5$ every n-semigroup is Weierstrass.* (See Maclachlan [11], Komeda [2] and [3] for the case $n = 3$, $n = 4$ and $n = 5$ respectively.)

Moreover, we have the following facts for two kinds of numerical semigroups:

Fact 4. *Every g-semigroup of genus g is Weierstrass.* (See Pinkham [12].)

Fact 5. *There is a unique non-primitive* $(g-1)$*-semigroup of genus* g*, which is Weierstrass.* (See Komeda [5].)

Therefore, we are interested in non-primitive n-semigroups of genus 9 for $n = 6, 7$.

§3. Non-primitive 6-semigroups of genus 9.

Definition 1. A numerical semigroup H with $\sharp M(H) = m$ is said to be *of toric type* if there are a positive integer l, monomials g_j's ($j = 1, \ldots, l+m-1$) in $k[X_1, \ldots, X_m]$ and a saturated subsemigroup S of $\mathbb{Z}^l$ generated by $b_1, \ldots, b_{l+m-1}$ which generates $\mathbb{Z}^l$ as a group such that

$$\begin{array}{ccc} \mathrm{Spec}\ k[H] & \hookrightarrow & \mathrm{Spec}\ k[X_1, \ldots, X_m] \\ \downarrow & \square & \downarrow \\ \mathrm{Spec}\ k[S] & \hookrightarrow & \mathrm{Spec}\ k[Y_1, \ldots, Y_{l+m-1}] \end{array}$$

where the horizontal maps are the embeddings through the generators and the right vertical map is induced by the k-algebra morphism from $k[Y_1, \ldots, Y_{l+m-1}]$ to $k[X_1, \ldots, X_m]$ sending Y_j to g_j.

Definition 2. A $2m$-semigroup H is *of double covering type* if there is a double covering $\pi : C \longrightarrow C_0$ of curves with ramification point P such that $H(P) = H$.

We can show the following:

Theorem 1. *Every non-primitive* 6*-semigroup of genus* 9 *is either of toric type or double covering type, hence Weierstrass.* (See Komeda [9].)

§4. Non-primitive 7-semigroups of genus 9.

We know that every non-primitive 7-semigroup of genus 9 is generated by 5 or 6 elements. We list up all non-primitive 7-semigroups of genus 9.

Remark 2. A non-primitive 7-semigroup of genus 9 generated by 5 elements is one of the following:

$\langle 7,9,10,11,13\rangle$, $\langle 7,9,10,11,12\rangle$, $\langle 7,9,10,12,13\rangle$, $\langle 7,8,11,12,13\rangle$.

Theorem 3. *Every non-primitive* 7*-semigroup of genus* 9 *generated by* 5 *elements is of toric type, hence Weierstrass.* (See Komeda [7])

Remark 4. A non-primitive 7-semigroup of genus 9 generated by 6 elements is one of the following:

$\langle 7,9,11,12,13,17\rangle$, $\langle 7,9,11,12,13,15\rangle$, $\langle 7,10,11,12,13,16\rangle$, $\langle 7,10,11,12,13,15\rangle$.

First, we shall show that $\langle 7,9,11,12,13,17\rangle$ is of toric type. We set $a_1 = 7$, $a_2 = 9$, $a_3 = 11$, $a_4 = 12$, $a_5 = 13$, $a_6 = 17$. Then we have a generating system of relations among a_1, a_2, a_3, a_4, a_5 and a_6 as follows:

$$3a_1 = a_2 + a_4,\ 2a_2 = a_1 + a_3,\ 2a_3 = a_2 + a_5,\ 2a_4 = a_1 + a_6,\ 2a_5 = a_2 + a_6,$$

$$2a_6 = a_2 + a_4 + a_5,\ a_1 + a_5 = a_2 + a_3,\ a_1 + a_6 = a_3 + a_5,\ 2a_1 + a_3 = a_4 + a_5,$$

$$a_3 + a_6 = a_1 + a_2 + a_4,\ a_5 + a_6 = a_1 + a_3 + a_4,\ 2a_1 + a_2 = a_3 + a_4,$$

$$2a_1 + a_4 = a_2 + a_6,\ a_4 + a_6 = a_1 + a_2 + a_5.$$

We set

$$\mathbf{b}_i = \mathbf{e}_i \in \mathbb{Z}^6,\ i = 1, \dots, 6,\ \mathbf{b}_7 = (1,1,-1,0,0,0),\ \mathbf{b}_8 = (1,0,-1,1,0,0),$$

$$\mathbf{b}_9 = (1,0,1,-1,1,0),\ \mathbf{b}_{10} = (-1,0,-1,1,0,1),\ \mathbf{b}_{11} = (0,0,2,-1,0,0).$$

Let S be the subsemigroup of $\mathbb{Z}^6$ generated by $\mathbf{b}_1, \dots, \mathbf{b}_{11}$. Then Spec $k[S]$ is a 6-dimensional affine toric variety. We have a fiber product

$$\begin{array}{ccc} \mathrm{Spec}\ k[H] & \hookrightarrow & \mathrm{Spec}\ k[X_1, \dots, X_6] = \mathbb{A}^6 \\ \downarrow & \square & \downarrow^{a\eta} \\ \mathrm{Spec}\ k[S] & \hookrightarrow & \mathrm{Spec}\ k[Y_1, \dots, Y_{11}] = \mathbb{A}^{11} \end{array}$$

where $\eta : k[Y_1, \dots, Y_{11}] \longrightarrow k[X_1, \dots, X_6]$ is the k-algebra homomorphism sending Y_i to ξ_i for $1 \leqq i \leqq 11$ where

$$\xi_1 = X_1,\ \xi_2 = X_6,\ \xi_3 = X_3,\ \xi_4 = X_5,\ \xi_5 = X_1,\ \xi_6 = X_6,$$

$$\xi_7 = X_5,\ \xi_8 = X_2,\ \xi_9 = X_4,\ \xi_{10} = X_4,\ \xi_{11} = X_2.$$

Hence, the numerical semigroup $\langle 7, 9, 11, 12, 13, 17 \rangle$ is Weierstrass.

Second, we shall show that $\langle 7, 9, 11, 12, 13, 15 \rangle$ is of toric type. We set $a_1 = 7$, $a_2 = 9$, $a_3 = 11$, $a_4 = 12$, $a_5 = 13$, $a_6 = 15$. Then we have a generating system of relations among a_1, a_2, a_3, a_4, a_5 and a_6 as follows:

$$3a_1 = a_2 + a_4,\ 2a_2 = a_1 + a_3,\ 2a_3 = a_1 + a_6,\ 2a_4 = a_3 + a_5,\ 2a_5 = a_3 + a_6,$$

$$2a_6 = a_1 + a_3 + a_4,\ a_1 + a_5 = a_2 + a_3,\ a_1 + a_6 = a_2 + a_5,$$

$$2a_1 + a_3 = a_4 + a_5,\ a_3 + a_6 = 2a_1 + a_4,\ a_5 + a_6 = a_1 + a_2 + a_4,$$

$$2a_1 + a_2 = a_3 + a_4,\ 2a_1 + a_5 = a_4 + a_6,\ a_2 + a_6 = a_3 + a_5.$$

We set

$$\mathbf{b}_i = \mathbf{e}_i \in \mathbb{Z}^4,\ i = 1, \dots, 4,\ \mathbf{b}_5 = (1,1,-1,0),\ \mathbf{b}_6 = (-1,1,1,0),$$

$$\mathbf{b}_7 = (-1,0,2,0),\ \mathbf{b}_8 = (2,0,-1,1),\ \mathbf{b}_9 = (-1,2,0,-1).$$

Let S be the subsemigroup of $\mathbb{Z}^4$ generated by $\mathbf{b}_1, \ldots, \mathbf{b}_9$. Then Spec $k[S]$ is a 4-dimensional affine toric variety. We have a fiber product

$$\begin{array}{ccc} \operatorname{Spec} k[H] & \hookrightarrow & \operatorname{Spec} k[X_1, \ldots, X_6] = \mathbb{A}^6 \\ \downarrow & \square & \downarrow^{a}\eta \\ \operatorname{Spec} k[S] & \hookrightarrow & \operatorname{Spec} k[Y_1, \ldots, Y_9] = \mathbb{A}^9 \end{array}$$

where $\eta : k[Y_1, \ldots, Y_9] \longrightarrow k[X_1, \ldots, X_6]$ is the k-algebra homomorphism sending Y_i to ξ_i for $1 \leqq i \leqq 9$ where

$$\xi_1 = X_1,\ \xi_2 = X_5,\ \xi_3 = X_2,\ \xi_4 = X_1,\ \xi_5 = X_3,\ \xi_6 = X_6,\ \xi_7 = X_3,\ \xi_8 = X_4,\ \xi_9 = X_4.$$

Hence, the numerical semigroup $\langle 7, 9, 11, 12, 13, 15 \rangle$ is Weierstrass.

Third, we consider the semigroup $\langle 7, 10, 11, 12, 13, 16 \rangle$. We set

$$a_1 = 7,\ a_2 = 10,\ a_3 = 11,\ a_4 = 12,\ a_5 = 13,\ a_6 = 16.$$

Then we have a generating system of relations among a_1, a_2, a_3, a_4, a_5 and a_6 as follows:

$$3a_1 = a_2 + a_3,\ 2a_2 = a_1 + a_5,\ 2a_3 = a_2 + a_4,\ 2a_4 = a_3 + a_5,$$

$$2a_5 = a_2 + a_6,\ 2a_6 = a_1 + a_4 + a_5,\ a_1 + a_6 = a_2 + a_5,\ a_1 + a_6 = a_3 + a_4,$$

$$2a_1 + a_2 = a_3 + a_5,\ 2a_1 + a_3 = a_4 + a_5,\ 2a_1 + a_4 = a_2 + a_6,$$

$$2a_1 + a_5 = a_3 + a_6,\ a_4 + a_6 = a_1 + a_2 + a_3,\ a_5 + a_6 = a_1 + a_2 + a_4.$$

Let S be the subsemigroup of $\mathbb{Z}^4$ generated by

$$\mathbf{b}_i = \mathbf{e}_i \in \mathbb{Z}^4,\ i = 1, \ldots, 4,\ \mathbf{b}_5 = (2, -1, 0, 0),\ \mathbf{b}_6 = (3, -2, 0, 0),$$

$$\mathbf{b}_7 = (-1, 2, 1, 0),\ \mathbf{b}_8 = (-2, 2, 1, 1),\ \mathbf{b}_9 = (4, -3, -1, 0).$$

Then Spec $k[S]$ is a 4-dimensional non-normal variety such that we have a fiber product

$$\begin{array}{ccc} \operatorname{Spec} k[H] & \hookrightarrow & \operatorname{Spec} k[X_1, \ldots, X_6] = \mathbb{A}^6 \\ \downarrow & \square & \downarrow^{a}\eta \\ \operatorname{Spec} k[S] & \hookrightarrow & \operatorname{Spec} k[Y_1, \ldots, Y_9] = \mathbb{A}^9 \end{array}.$$

Here $\eta : k[Y_1, \ldots, Y_9] \longrightarrow k[X_1, \ldots, X_6]$ is the k-algebra homomorphism sending Y_i to ξ_i for $1 \leqq i \leqq 9$ where

$$\xi_1 = X_2,\ \xi_2 = X_1,\ \xi_3 = X_1,\ \xi_4 = X_3,\ \xi_5 = X_5,\ \xi_6 = X_6,\ \xi_7 = X_3,\ \xi_8 = X_4,\ \xi_9 = X_4.$$

Lastly we investigate the semigroup $\langle 7, 10, 11, 12, 13, 15 \rangle$. We set

$$a_1 = 7,\ a_2 = 10,\ a_3 = 11,\ a_4 = 12,\ a_5 = 13,\ a_6 = 15.$$

Then we have a generating system of relations among a_1, a_2, a_3, a_4, a_5 and a_6 as follows:

$$3a_1 = a_2 + a_3,\ 2a_2 = a_1 + a_5,\ 2a_3 = a_1 + a_6,\ 2a_4 = 2a_1 + a_2,$$

$$2a_5 = 2a_1 + a_4,\ 2a_6 = a_1 + a_3 + a_4,\ a_1 + a_6 = a_2 + a_4,\ a_2 + a_6 = a_4 + a_5,$$

$$2a_1 + a_2 = a_3 + a_5,\ 2a_1 + a_3 = a_4 + a_5,\ 2a_1 + a_4 = a_3 + a_6,$$

$$2a_1 + a_5 = a_4 + a_6,\ a_2 + a_5 = a_3 + a_4,\ a_5 + a_6 = a_1 + a_2 + a_3.$$

Let S be the subsemigroup of $\mathbb{Z}^3$ generated by

$$\mathbf{b}_i = \mathbf{e}_i \in \mathbb{Z}^3,\ i = 1, \ldots, 3,\ \mathbf{b}_4 = (1, 1, -1),\ \mathbf{b}_5 = (1, -1, 1),$$

$$\mathbf{b}_6 = (2, -2, 1),\ \mathbf{b}_7 = (-2, 1, 1),\ \mathbf{b}_8 = (-1, 3, -1).$$

Then Spec $k[S]$ is a 3-dimensional non-normal variety where we have a fiber product

$$\begin{array}{ccc} \text{Spec } k[H] & \hookrightarrow & \text{Spec } k[X_1, \ldots, X_6] = \mathbb{A}^6 \\ \downarrow & \square & \downarrow^{a}\eta \\ \text{Spec } k[S] & \hookrightarrow & \text{Spec } k[Y_1, \ldots, Y_8] = \mathbb{A}^8 \end{array}$$

Here $\eta : k[Y_1, \ldots, Y_8] \longrightarrow k[X_1, \ldots, X_6]$ is the k-algebra homomorphism sending Y_i to ξ_i for $1 \leqq i \leqq 8$ where

$$\xi_1 = X_2,\ \xi_2 = X_4,\ \xi_3 = X_6,\ \xi_4 = X_1,\ \xi_5 = X_5,\ \xi_6 = X_3,\ \xi_7 = X_1,\ \xi_8 = X_3.$$

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