Title	Cross-diffusion induced instability and Turing's diffusion induced instability(Theory of Bio-Mathematics and Its Applications)
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Citation	数理解析研究所講究録 (2006), 1499: 50-72
Issue Date	2006-07
URL	http://hdl.handle.net/2433/58399
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
1499 巻 2006 年 50-72

Cross-diffusion induced instability and Turing's diffusion induced instability

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Abstract

The cross-diffusion competition systems were introduced by Shigesada et al. to describe the population pressure by other species. In this paper, introducing the densities of the active individuals and the less active ones, we show that the cross-diffusion competition system can be approximated by the reaction-diffusion system which only includes the linear diffusion. The linearized stability around the constant equilibrium solution is also studied, which implies that the cross-diffusion induced instability can be regarded as Turing's instability of the corresponding reaction-diffusion system.

1 Introduction

Understanding of spatial and/or temporal behaviors of interacting species in ecological systems is a central problem in population ecology. As one of several ecological interactions of multi-species, competitive dynamics of interacting species have been investigated from theoretical as well as field works. Various types of mathematical models have been proposed to study problems of coexistence or exclusion of competing species.

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Among many mathematical models, reaction-diffusion systems have been used to understand the evolutional behavior of spatially segregating regions of competing species. Let $u_i(t,x)$ be the population density of the i-th species U_i $(i = 1, 2, ..., M)$ at time $t > 0$ and the position $x \in \Omega$, where Ω is a bounded domain in $\mathbf{R}^2$. The competition-diffusion system of Lotka-Volterra-Gause type for u_i $(i = 1, 2, ..., M)$ is given by

$$u_{it} = d_i \Delta u_i + \left(r_i - \sum_{j=1}^{M} a_{ij} u_j \right) u_i, \qquad t > 0, x \in \Omega, \tag{1}$$

where d_i is the diffusion rate of u_i, r_i is the intrinsic growth rate, a_{ii} and a_{ij} $(i \neq j)$ are respectively the intra-specific and the inter-specific competition rates $(i, j = 1, 2, ..., M)$. All of the rates are positive constants. We impose the zero-flux boundary conditions

$$\frac{\partial u_i}{\partial \nu} = 0, \qquad t > 0, x \in \partial\Omega, \tag{2}$$

on (1) where ν is the outward normal unit vector on $\partial\Omega$. The initial conditions are

$$u_i(0, x) = u_{0i}(x) \geq 0, \qquad x \in \Omega. \tag{3}$$

The simplest system of (1) is the case when $M = 2$, that is,

$$\begin{cases} u_{1t} &= d_1 \Delta u_1 + (r_1 - a_1 u_1 - b_1 u_2) u_1, \qquad t > 0, x \in \Omega, \\ u_{2t} &= d_2 \Delta u_2 + (r_2 - b_2 u_1 - a_2 u_2) u_2, \qquad t > 0, x \in \Omega, \end{cases} \tag{4}$$

where r_i, a_i, b_i $(i = 1, 2)$ are positive constants. With the similar boundary and initial conditions to (2) and (3), qualitative properties of non-negative solutions (u_1, u_2) of the system (4) have been intensively studied in mathematical communities. The first remark is that the stable attractors of (4) should be equilibrium solutions ([3]). This information indicates that existence and stability of non-negative equilibrium solutions is important for the study of asymptotic behavior of solutions. If the domain Ω is convex, it is proved in [7] that any spatially non-constant equilibrium solutions are unstable, even if they exist. This implies that solutions of (4) become spatially homogeneous asymptotically, in other words, stable equilibrium solutions of the diffusion-less system of (4)

$$\begin{cases} u_{1t} &= (r_1 - a_1 u_1 - b_1 u_2) u_1, \qquad t > 0, \\ u_{2t} &= (r_2 - b_2 u_1 - a_2 u_2) u_2, \qquad t > 0 \end{cases} \tag{5}$$

are very important to know the asymptotic behavior of the solutions of (4) even if the diffusion term is present. In fact, if we assume the situation where the inter-specific competition is strong, that is,

$$\frac{a_1}{b_2} < \frac{r_1}{r_2} < \frac{b_1}{a_2}, \tag{6}$$

stable spatially constant equilibrium solutions of (4) are $(u_1, u_2) = (r_1/a_1, 0)$ and $(0, r_2/a_2)$. This means that one of the competing species survives and the other is extinct. We can say ecologically that competitive exclusion occurs for two species, if they are strongly competing.

On the other hand, it is observed in fields that individuals do not necessarily move around randomly [6]. Among them, one example is that the movement depends on the population pressures caused by interacting other species. This situation can not be described by (4) any more. Along this line, one model for the competitive interaction between two species is proposed in [15], which is given by

$$\begin{cases} u_{1t} &= \Delta[(d_1 + \alpha u_2)u_1] + (r_1 - a_1 u_1 - b_1 u_2)u_1, \qquad t > 0, x \in \Omega, \\ u_{2t} &= \Delta[(d_2 + \beta u_1)u_2] + (r_2 - b_2 u_1 - a_2 u_2)u_2, \qquad t > 0, x \in \Omega, \end{cases} \tag{7}$$

where α, β stand for the cross-diffusion pressures and are non-negative constants. We simply call α and β cross-diffusion coefficients. From the nonlinearity viewpoint, (7) falls into quasi-linear parabolic systems so that even the existence problem of solutions is not trivial and has been investigated by several authors (for instance [1, 10, 2] and the references therein). On the other hand, the stationary problem for (7) has been studied from spatial-segregation viewpoints, for which three basically different approaches were employed:

(i) Bifurcation approach ([12]);

(ii) Singular perturbation approach ([13, 5, 9]);

(iii) Elliptic approach ([8]).

Integrating the results above, it turns out that the structure of equilibrium solutions of (7) with the boundary conditions similar to (2) sensitively depends on parameters in the systems and is extremely complicated, even in one dimension. As a simplified system of (7), let us consider the following system:

$$\begin{cases} u_{1t} &= \Delta[(d_1 + \alpha u_2)u_1] + (r_1 - a_1 u_1 - b_1 u_2)u_1, \qquad t > 0, x \in \Omega, \\ u_{2t} &= d_2 \Delta u_2 + (r_2 - b_2 u_1 - a_2 u_2)u_2, \qquad t > 0, x \in \Omega, \end{cases} \tag{8}$$

where we simply put $\beta = 0$. Suppose that

$$\frac{b_1}{a_2} < \frac{r_1}{r_2} < \frac{a_1}{b_2}, \tag{9}$$

for which, there is only one stable spatially constant equilibrium solutions $(u_1^*, u_2^*) = ((a_2 r_1 - b_1 r_2)/(a_1 a_2 - b_1 b_2), (-b_2 r_1 + a_1 r_2)/(a_1 a_2 - b_1 b_2))$ while others $(u_1, u_2) = (0, 0)$, $(r_1/a_1, 0)$ and $(0, r_2/a_2)$ are unstable when α is small enough. Mimura and Kawasaki [12] reported that (u_1^*, u_2^*) loses its stability as α increases (see also [11]). For example, consider Ω to be one dimensional interval and fix $\alpha(= 3)$ and the values of the competition rates to satisfy (9). Taking $d = d_1 = d_2$ as the bifurcation parameter, we can

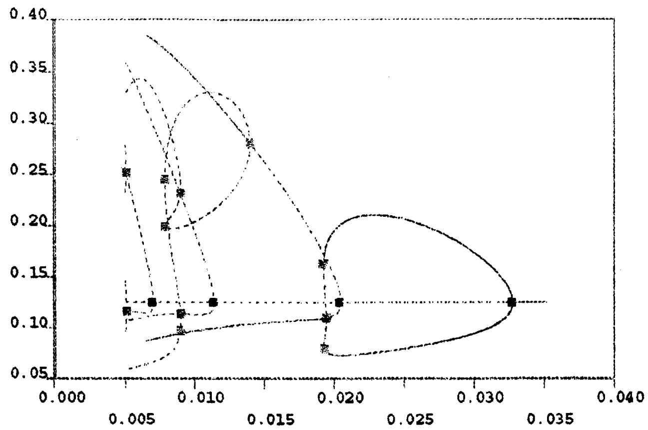

Figure 1: Bifurcation diagram for (8) when $d_1 = d_2 = d$ varies with $r_1 = 5.0$, $a_1 = 3.0$, $b_1 = 1.0$, $r_2 = 2.0$, $a_2 = 3.0$, $b_2 = 1.0$ and $\alpha = 3.0$. The vertical axis indicates the values of u_2 at $x = 0$; the horizontal axis does the ones of d. The solid curves indicate the branches of the stable equilibrium solutions and the dotted curves do the unstable ones.

see the bifurcation diagram of equilibrium solutions in Fig. 1 where (u_1^*, u_2^*) is used as the trivial solution. The local bifurcation theory with complementary numerics says that there are stable non-constant equilibrium solutions which exhibit spatial segregating patterns between two competing species, depending on the values of parameters d and α. This means that the spatially segregating coexistence of two competing species occurs by the cross-diffusion effect, which is called *cross-diffusion induced instability.* This result shows a remarkable contrast with the fact of (4) that (u_1^*, u_2^*) is always stable under (9).

It is obvious that the cross-diffusion term in (8) is separated into two terms:

$$u_{1,t} = \nabla\Big[(d_1 + \alpha u_2)\nabla u_1\Big] + \alpha\nabla(u_1\nabla u_2) + \cdots .$$

One may interpret "each term" as follows: the first term of the right hand side looks like "Fickian diffusion" where the rate depends on u_2, while the second does "direct movement" due to the gradient of u_2. Ecologically speaking, U_1 moves to avoid the congestion of U_2. One knows that the second term is essential to generate cross-diffusion induced instability. The question which we would like to address here is "Is the cross-diffusion mechanism only one way to avoid the congestion of the other species ?", in other words, "Is a direct movement necessary to avoid the congestion of the others ?" What we would like to discuss in this paper is to answer these questions.

In order to understand the meaning of cross-diffusion effect in (7) (or (8)), let us consider the movement of a single species (say N) on the inhomogeneous medium which is specified by the given function $V(x)$. Suppose that the place x where $V(x)$ is larger is more unfavorable for N. By replacing u_2 in (7) with $V(x)$ and neglecting the growth term, we obtain the following scalar linear diffusion equation for the population density n of N:

$$n_t = \Delta\left[\left(d+\alpha V(x)\right)n\right], \qquad t>0, x\in\Omega, \tag{10}$$

$$n(0,x) = n_0(x), \qquad x\in\Omega, \tag{11}$$

$$\frac{\partial n}{\partial \nu} = 0, \qquad t>0, x\in\partial\Omega, \tag{12}$$

(cf. [14] or [15]). We can ecologically say that the species avoids the unfavorable habitat due to $V(x)$.

We now consider the following situation on the movement of species: the species is split into two types of the states; one is less active (resp. active) and the other is active (resp. more active). The state of each individual can convert into the other depending on the environmental inhomogeneity $V(x)$.

We denote the population density of each type by n_1 and n_2 where $n = n_1+n_2$, and suppose that they move randomly with the rates d or $d+\alpha$. Thus the dynamics of n_1 and n_2 can be described by

$$\begin{cases} n_{1t} &= d\Delta n_1 + \dfrac{1}{\varepsilon}[h(x)n_2 - k(x)n_1], \\ n_{2t} &= (d+\alpha)\Delta n_2 + \dfrac{1}{\varepsilon}[k(x)n_1 - h(x)n_2], \end{cases} \tag{13}$$

where $1/\varepsilon$ is the conversion rate between n_1 and n_2, which is a positive constant. The non-negative functions $h(x)$ and $k(x)$ are specified later. Adding two equations, we obtain

$$\begin{aligned} n_t &= d\Delta n + \alpha\Delta n_2, \\ n_{2t} &= (d+\alpha)\Delta n_2 + \frac{1}{\varepsilon}[k(x)n_1 - h(x)n_2], \end{aligned}$$

where $n = n_1 + n_2$. Letting $\varepsilon \to +0$ in the second equation, one can formally expect that

$$h(x)n_2 = k(x)n_1.$$

This implies that

$$n_1 = \frac{h(x)}{h(x)+k(x)}n, \qquad n_2 = \frac{k(x)}{h(x)+k(x)}n.$$

Therefore, if we set $h(x) = 1-V(x)$ (we assume $0\le V(x)\le 1$) and $k(x) = V(x)$ for instance, it turns out that

$$n_2 = V(x)n$$

and

$$n_t = d\Delta n + \alpha\Delta(V(x)n) = \Delta\left[\Big(d + \alpha V(x)\Big)n\right].$$

This formal discussion indicates that the direct movement can be represented by random walking if individuals of the species possess the mechanism of switching from one type of random walking to the other, depending on environmental inhomogeneity.

We now apply this limiting procedure as $\varepsilon \to +0$ to (8) where one species U_1 is split into two types and each individual of U_1 converts its state into the other state, depending on the spatial distribution of the competitor U_2 at x. We denote by $v_1(t,x)$ and $v_2(t,x)$ the population densities of the two types of U_1, and rewrite the population density of the species U_2 as $v_3(t,x)$. We will construct the reaction-diffusion system for (v_1, v_2, v_3), which approximates bounded solutions (u_1, u_2) in $\Omega \times [0,T]$, and thus we assume

$$0 \le u_1(t,x) \le M_1, \quad 0 \le u_2(t,x) \le M_2 \qquad \text{on } [0,T]\times\Omega. \tag{14}$$

Following the discussion above, the population dynamics of U_1 and U_2 can be formally described by the following three component reaction-diffusion system

$$\left\{\begin{array}{rcll} v_{1t} &=& d_1\Delta v_1 + (r_1 - a_1(v_1+v_2) - b_1 v_3)v_1 & \\ && +\dfrac{1}{\varepsilon}[h(v_3)v_2 - k(v_3)v_1], & t>0,\ x\in\Omega, \\ v_{2t} &=& (d_1+\alpha M_2)\Delta v_2 + (r_1 - a_1(v_1+v_2) - b_1 v_3)v_2 & \\ && +\dfrac{1}{\varepsilon}[k(v_3)v_1 - h(v_3)v_2], & t>0,\ x\in\Omega, \\ v_{3t} &=& d_2\Delta v_3 + (r_2 - b_2(v_1+v_2) - a_2 v_3)v_3, & t>0,\ x\in\Omega, \end{array}\right. \tag{15}$$

where we regard h and k as certain functions of $v_3(t,x)$. We impose the boundary conditions

$$\frac{\partial v_i}{\partial \nu} = 0 \quad (i=1,2,3), \qquad t>0,\ x\in\partial\Omega \tag{16}$$

and the initial conditions

$$v_i(0,x) = v_{0i}(x) \ge 0 \quad (i=1,2,3), \qquad x\in\Omega \tag{17}$$

which will be determined later. Setting $\rho = v_1 + v_2$, we rewrite (15) as

$$\left\{\begin{array}{rcl} \rho_t &=& d_1\Delta\rho + \alpha M_2\Delta v_2 + (r_1 - a_1\rho - b_1 v_3)\rho, \\ v_{2t} &=& (d_1+\alpha M_2)\Delta v_2 + (r_1 - a_1\rho - b_1 v_3)v_2 \\ && +\dfrac{1}{\varepsilon}\left[k(v_3)\rho - \Big(h(v_3)+k(v_3)\Big)v_2\right], \\ v_{3t} &=& d_2\Delta v_3 + (r_2 - b_2\rho - a_2 v_3)v_3. \end{array}\right. \tag{18}$$

We can expect that v_2 converges to $k(v_3)\rho/(h(v_3)+k(v_3))$ as $\varepsilon \to +0$ from the second equation of (18). Replacing v_2 in the first equation of (18) by this limit, we formally get a cross-diffusion system (8), provided that

$$k(s) \equiv \Big(h(s)+k(s)\Big)\frac{s}{M_2} \tag{19}$$

for $s \geq 0$. As the simplest example of h and k, we can take

$$h(s) = 1 - \frac{s}{M_2}, \quad k(s) = \frac{s}{M_2}.$$

We will discuss other examples in §5. Our result is as follows:

Theorem 1 *Let $(u_1, u_2) = (u_1(t,x), u_2(t,x))$ be the solution of (8),(2),(3). Suppose that (u_1, u_2) is sufficiently smooth on $[0,T] \times \overline{\Omega}$ and satisfies (14) for some positive numbers T, M_1 and M_2 with $M_2 \geq r_2/a_2$. Choose smooth functions h and k satisfying (19) and*

$$h(s) \geq 0 \quad \text{on } [0, M_2], \tag{20}$$
$$k(s) \geq 0 \quad \text{on } [0, \infty), \tag{21}$$
$$h(s) + k(s) > 0 \quad \text{on } [0, \infty). \tag{22}$$

Determine the initial datum (v_{01}, v_{02}, v_{03}) by

$$\begin{cases} v_{01}(x) \equiv \left\{1 - \dfrac{u_{02}(x)}{M_2}\right\} u_{01}(x), \\ v_{02}(x) \equiv \dfrac{u_{02}(x)}{M_2} u_{01}(x), \\ v_{03}(x) \equiv u_{02}(x) \end{cases} \tag{23}$$

over Ω. Let $(v_1, v_2, v_3) = (v_1(t,x;\varepsilon), v_2(t,x;\varepsilon), v_3(t,x;\varepsilon))$ be the solution of (15),(16),(17) depending on a positive number ε. Suppose that there exists positive numbers ε_0 and M_0 satisfying

$$|v_1(t,x;\varepsilon)| + |v_2(t,x;\varepsilon)| + |v_3(t,x;\varepsilon)| \leq M_0 \tag{24}$$

for $(t,x) \in [0,T] \times \overline{\Omega}$ and $\varepsilon \in (0, \varepsilon_0]$. Then the difference between $(v_1 + v_2, v_3)$ and (u_1, u_2) is estimated to be

$$\begin{cases} \sup\limits_{t\in[0,T]} \|v_1(t,\cdot;\varepsilon) + v_2(t,\cdot;\varepsilon) - u_1(t,\cdot)\|_{L^2(\Omega)} \leq C\varepsilon, \\ \sup\limits_{t\in[0,T]} \|v_3(t,\cdot;\varepsilon) - u_2(t,\cdot)\|_{L^2(\Omega)} \leq C\varepsilon \end{cases} \tag{25}$$

for $\varepsilon \in (0, \varepsilon_0]$. Here $C = C(u_1, u_2, \varepsilon_0, M_0, T)$ is a positive constant independent of ε.

This theorem shows that the solutions of (8) can be approximated by those of (15) in a finite time interval if the solutions are bounded. We will prove this theorem in §4.

As the next problem, we address the question on the asymptotic behavior of solutions. In the successive sections we consider the stationary problems of (8) and (15) and numerically discuss the resemblance between the mathematical structure of the equilibrium solutions of (8) and that of (15) as $\varepsilon \to +0$.

2 Stationary problem

In the previous section, we showed that for fixed $T > 0$, solutions of the cross-diffusion system (8) can be approximated by those of the reaction-diffusion system (15) for $0 < t < T$ if ε is sufficiently small. However, this result does not give any information on the equilibrium solutions of (8) and (15). In this section we consider the one-dimensional stationary problem of (8) and (15) with the zero-flux boundary conditions, assuming the weak competition condition (9).

Take d as a bifurcation parameter ($0.005 \leq d \leq 0.0351$) with $\alpha = 3$. The global structure of equilibrium solutions to (8) is already demonstrated in Fig. 1. Here we numerically show the global structures of equilibrium solutions to (15). The interval of d which is computed is the same as in Fig. 1. For $\varepsilon = 0.01$, the structure of equilibrium solution to (15) is demonstrated in Fig. 2. It is obvious to see that two structures in Figs. 1 and 2 are quantitatively different. However, it is surprising that these seem to be qualitatively similar. In fact, noting Fig. 3 ($\varepsilon = 0.001$) and Fig. 4 ($\varepsilon = 0.0001$), the global structure of equilibrium solutions seems to converge to the one in Fig. 1 as ε becomes smaller. These results strongly support that the stationary problem to (8) is also approximated by the one to (15) if ε is sufficiently small.

3 Turing's instability and cross-diffusion induced instability

One of the most important mechanisms of pattern formation is *Turing's instability* or *diffusion-induced instability* in short range activator-long range inhibitor type reaction-diffusion systems ([16]). This phenomenon occurs under the situation where the inhibitor diffuses faster than the activator. If the activator increases locally, then it generates the inhibitor at the same time. Because of the large diffusivity, the inhibitor also increases outside of its neighborhood of the high concentration of the activator. This keeps the activator below outside and the inhomogeneity of the distribution of the activator forms. Mathematically speaking, the stable equilibrium points of some ordinary differential equations become unstable by adding the diffusion. Consider the competition-diffusion system (4) under the weak competition condition (9). As was seen in §1, the diffusion-less system (5) has the stable equilibrium point (u_1^*, u_2^*). For

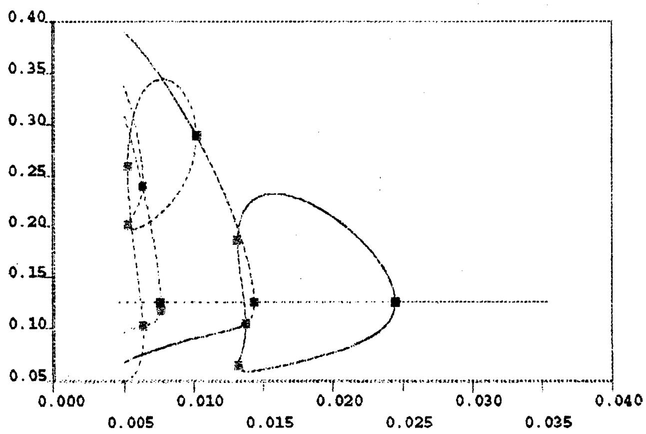

Figure 2: Bifurcation diagram for (15) when $d_1 = d_2 = d$ varies with $\varepsilon = 0.01$ and the other parameters as in Fig. 1. The vertical axis indicates the value of v_3 at $x = 0$; the horizontal axis does that of d.

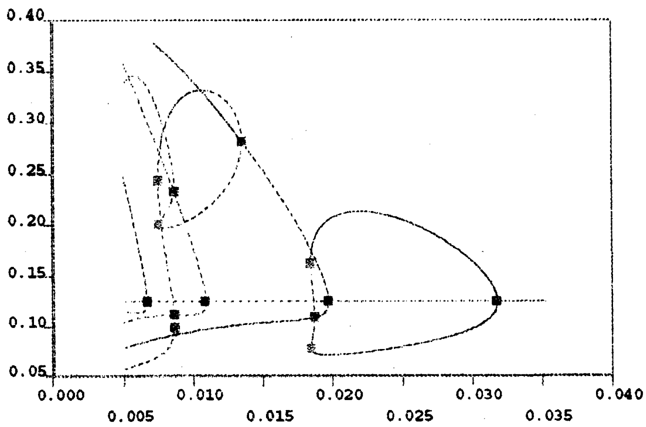

Figure 3: Bifurcation diagram for (15) when $d_1 = d_2 = d$ varies with $\varepsilon = 0.001$ and the other parameters as in Fig. 1.

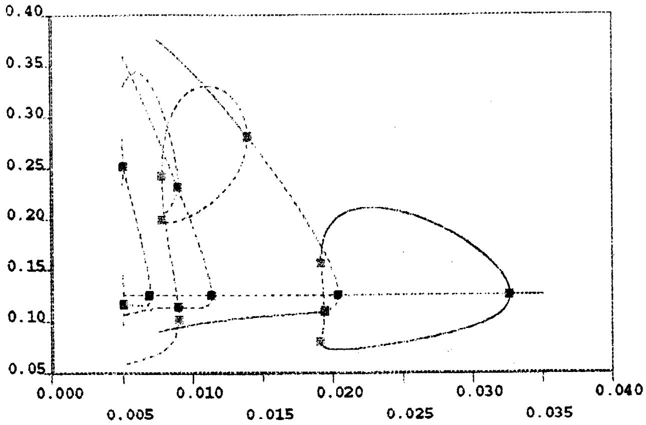

Figure 4: Bifurcation diagram for (15) when $d_1 = d_2 = d$ varies with $\varepsilon = 0.0001$ and the other parameters as in Fig. 1.

the corresponding competition-diffusion system (4), it is well known that Turing's instability never occurs, that is, the spatially constant equilibrium solution (u_1^*, u_2^*) is always stable. Actually the comparison principle directly implies that all the solutions converge to the constant solution (u_1^*, u_2^*) when both components of initial data are positive. On the other hand, as was seen in §2, stable non-constant stationary solutions of the cross-diffusion system (8) bifurcate from the stable constant solution (u_1^*, u_2^*) under the weak competition condition. In this section we will make clear the relationship between Turing's instability and the cross-diffusion induced instability for (8).

First we consider the linearized stability of the constant stationary solution (u_1^*, u_2^*) for (8) with $\alpha > 0$. For simplicity of notation, we set

$$\begin{cases} f(u_1, u_2) &= (r_1 - a_1 u_1 - b_1 u_2) u_1, \\ g(u_1, u_2) &= (r_2 - b_2 u_1 - a_2 u_2) u_2. \end{cases} \tag{26}$$

Then the linearized operator for the right hand side of (8) in a neighborhood of (u_1^*, u_2^*) is

$$\begin{pmatrix} d_1 \Delta + \alpha u_2^* \Delta + f_{u_1}(u_1^*, u_2^*) & \alpha u_1^* \Delta + f_{u_2}(u_1^*, u_2^*) \\ g_{u_1}(u_1^*, u_2^*) & d_2 \Delta + g_{u_2}(u_1^*, u_2^*) \end{pmatrix}.$$

Thus the eigenvalues μ of the linearized operator are characterized by $P(\mu) = 0$ where

$$P(\mu) := \begin{vmatrix} -d_1 \sigma - \alpha u_2^* \sigma + f_{u_1}(u_1^*, u_2^*) - \mu & -\alpha u_1^* \sigma + f_{u_2}(u_1^*, u_2^*) \\ g_{u_1}(u_1^*, u_2^*) & -d_2 \sigma + g_{u_2}(u_1^*, u_2^*) - \mu \end{vmatrix},$$

and σ is one of the eigenvalues of $-\Delta$ with the Neumann boundary condition (2).

Next we consider the reaction-diffusion system (15). Here we also define

$$\begin{cases} f_1(v_1, v_2, v_3) &= [r_1 - a_1(v_1 + v_2) - b_1 v_3] v_1, \\ f_2(v_1, v_2, v_3) &= [r_1 - a_1(v_1 + v_2) - b_1 v_3] v_2, \\ f_3(v_1, v_2, v_3) &= [r_2 - b_2(v_1 + v_2) - a_2 v_3] v_3. \end{cases}$$

It is easily seen that

$$\left\{\begin{array}{rcll} f(v_1+v_2,v_3) & = & f_1(v_1,v_2,v_3)+f_2(v_1,v_2,v_3), & \\ g(v_1+v_2,v_3) & = & f_3(v_1,v_2,v_3), & \\ f_{u_1}(v_1+v_2,v_3) & = & f_{1,v_i}(v_1,v_2,v_3)+f_{2,v_i}(v_1,v_2,v_3) & (i=1,2), \\ f_{u_2}(v_1+v_2,v_3) & = & f_{1,v_3}(v_1,v_2,v_3)+f_{2,v_3}(v_1,v_2,v_3), & \\ g_{u_1}(v_1+v_2,v_3) & = & f_{3,v_i}(v_1,v_2,v_3) & (i=1,2), \\ g_{u_2}(v_1+v_2,v_3) & = & f_{3,v_3}(v_1,v_2,v_3). & \end{array}\right. \tag{27}$$

Recall that (15) can be rewritten into (18) with $\rho=v_1+v_2$. We see from this and (19) that (18) also possesses the constant equilibrium (ρ^*,v_2^*,v_3^*) where $\rho^*=u_1^*$, $v_3^*=u_2^*$ and

$$v_2^*=\frac{k(v_3^*)\rho^*}{h(v_3^*)+k(v_3^*)}=\frac{v_3^*}{M_2}\rho^*=\frac{u_2^*}{M_2}u_1^*, \tag{28}$$

and that (15) possesses the equilibrium solution $(v_1^*,v_2^*,v_3^*)=(\rho^*-v_2^*,v_2^*,v_3^*)$. The linearized eigenvalue problem for (15) around (v_1^*,v_2^*,v_3^*) is reduced to $Q^\varepsilon(\mu)=0$ where

$$\begin{array}{rcl} Q^\varepsilon & := & \left|\begin{array}{ccc} \xi_{11} & \xi_{12} & \xi_{13} \\ \xi_{21} & \xi_{22} & \xi_{23} \\ f_{3,v_1}(v_1^*,v_2^*,v_3^*) & f_{3,v_2}(v_1^*,v_2^*,v_3^*) & -d_2\sigma+f_{3,v_3}(v_1^*,v_2^*,v_3^*)-\mu \end{array}\right|, \\ \xi_{11} & := & -d_1\sigma+f_{1,v_1}(v_1^*,v_2^*,v_3^*)-\frac{1}{\varepsilon}k(v_3^*)-\mu, \\ \xi_{12} & := & f_{1,v_2}(v_1^*,v_2^*,v_3^*)+\frac{1}{\varepsilon}h(v_3^*), \\ \xi_{13} & := & f_{1,v_3}(v_1^*,v_2^*,v_3^*)+\frac{1}{\varepsilon}(h'(v_3^*)v_2^*-k'(v_3^*)v_1^*), \\ \xi_{21} & := & f_{2,v_1}(v_1^*,v_2^*,v_3^*)+\frac{1}{\varepsilon}k(v_3^*), \\ \xi_{22} & := & -(d_1+\alpha M_2)\sigma+f_{2,v_2}(v_1^*,v_2^*,v_3^*)-\frac{1}{\varepsilon}h(v_3^*)-\mu, \\ \xi_{23} & := & f_{2,v_3}(v_1^*,v_2^*,v_3^*)+\frac{1}{\varepsilon}(k'(v_3^*)v_1^*-h'(v_3^*)v_2^*). \end{array}$$

Using (27), we see

$$\begin{array}{rcl} \xi_{11}+\xi_{21} & = & -d_1\sigma+f_{u_1}(u_1^*,u_2^*)-\mu, \\ \xi_{12}+\xi_{22} & = & -(d_1+\alpha M_2)\sigma+f_{u_1}(u_1^*,u_2^*)-\mu, \\ \xi_{13}+\xi_{23} & = & f_{u_2}(u_1^*,u_2^*). \end{array}$$

Then, adding the second row of Q^ε to the first one and subtracting the first column from the second one, we have

$$
\begin{aligned}
Q^\varepsilon &= \begin{vmatrix} \xi_{11}+\xi_{21} & \xi_{12}+\xi_{22} & \xi_{13}+\xi_{23} \\ \xi_{21} & \xi_{22} & \xi_{23} \\ g_{u_1}(u_1^*,u_2^*) & g_{u_1}(u_1^*,u_2^*) & -d_2\sigma + g_{u_2}(u_1^*,u_2^*) - \mu \end{vmatrix} \\
&= \begin{vmatrix} \xi_{11}+\xi_{21} & \xi_{12}+\xi_{22}-\xi_{11}-\xi_{21} & \xi_{13}+\xi_{23} \\ \xi_{21} & \xi_{22}-\xi_{21} & \xi_{23} \\ g_{u_1}(u_1^*,u_2^*) & 0 & -d_2\sigma + g_{u_2}(u_1^*,u_2^*) - \mu \end{vmatrix} \\
&= \begin{vmatrix} -d_1\sigma + f_{u_1}(u_1^*,u_2^*) - \mu & -\alpha M_2\sigma & f_{u_2}(u_1^*,u_2^*) \\ \xi_{21} & \xi_{22}-\xi_{21} & \xi_{23} \\ g_{u_1}(u_1^*,u_2^*) & 0 & -d_2\sigma + g_{u_2}(u_1^*,u_2^*) - \mu \end{vmatrix}.
\end{aligned}
$$

We note that the last determinant corresponds to the eigenvalue problem of (18). We can rewrite it as follows:

$$
\begin{aligned}
Q^\varepsilon &= \frac{1}{\varepsilon}\begin{vmatrix} -d_1\sigma + f_{u_1}(u_1^*,u_2^*) - \mu & -\alpha M_2\sigma & f_{u_2}(u_1^*,u_2^*) \\ k(v_3^*) & -h(v_3^*) - k(v_3^*) & k'(v_3^*)v_1^* - h'(v_3^*)v_2^* \\ g_{u_1}(u_1^*,u_2^*) & 0 & -d_2\sigma + g_{u_2}(u_1^*,u_2^*) - \mu \end{vmatrix} \\
&\quad +O(1)
\end{aligned}
$$

as $\varepsilon \to +0$. By (19), we have

$$
k'(s) = (h'(s)+k'(s))\frac{s}{M_2} + \frac{h(s)+k(s)}{M_2}, \qquad \frac{k(s)+h(s)}{M_2} = \frac{k(s)}{s}
$$

and then

$$
\begin{aligned}
k'(v_3^*)v_1^* - h'(v_3^*)v_2^* &= k'(u_2^*)\left(u_1^* - \frac{u_2^*}{M_2}u_1^*\right) - h'(u_2^*)\frac{u_2^*}{M_2}u_1^* \\
&= \frac{h(u_2^*)+k(u_2^*)}{M_2}u_1^* = \frac{k(u_2^*)u_1^*}{u_2^*}.
\end{aligned}
$$

The principal part of Q^ε can be reduced to

$$
\frac{1}{\varepsilon}\begin{vmatrix} -d_1\sigma + f_{u_1}(u_1^*,u_2^*) - \mu & -\alpha M_2\sigma & f_{u_2}(u_1^*,u_2^*) \\ k(u_2^*) & -\dfrac{M_2 k(u_2^*)}{u_2^*} & \dfrac{k(u_2^*)u_1^*}{u_2^*} \\ g_{u_1}(u_1^*,u_2^*) & 0 & -d_2\sigma + g_{u_2}(u_1^*,u_2^*) - \mu \end{vmatrix}.
$$

Multiplying the second row by $-\alpha\sigma u_2^*/k(u_2^*)$ and adding the product to the first one, we can calculate the principal part as follows:

$$
\frac{1}{\varepsilon}\begin{vmatrix} -(d_1+\alpha u_2^*)\sigma + f_{u_1}(u_1^*,u_2^*) - \mu & 0 & -\alpha\sigma u_1^* + f_{u_2}(u_1^*,u_2^*) \\ k(u_2^*) & -\dfrac{M_2 k(u_2^*)}{u_2^*} & \dfrac{k(u_2^*)u_1^*}{u_2^*} \\ g_{u_1}(u_1^*,u_2^*) & 0 & -d_2\sigma + g_{u_2}(u_1^*,u_2^*) - \mu \end{vmatrix}.
$$

Thus we obtain

$$Q^{\varepsilon}(\mu) = -\frac{M_2 k(u_2^*)}{\varepsilon u_2^*} P(\mu) + R(\mu) = -\frac{h(u_2^*) + k(u_2^*)}{\varepsilon} P(\mu) + R(\mu), \qquad (29)$$

where $R(\mu)$ is independent of ε. It is easy to see that

$$P(\mu) = \mu^2 + O(\mu), \qquad R(\mu) = -\mu^3 + O(\mu^2)$$

as $|\mu| \to \infty$, which together with (22), implies that the eigenvalues of the linearized operator of (15) converges to those of (8) in a half plane $\{\mu \in \mathbf{C} \mid \mathrm{Re}\, \mu > -\gamma\}$ for an arbitrary positive number γ. Thus, one finds that cross-diffusion induced instability of (8) can be regarded as Turing's instability of (15) if ε tends to zero.

More precisely speaking to the cross-diffusion system (8), we can say that in the approximating reaction-diffusion system (15), v_3 plays a role of activator, while v_2 does a role of inhibitor. Since $(v_1^* + v_2^*, v_3^*)$ is a stable equilibrium point of (8) and (29) implies that two roots of $Q^{\varepsilon}(\mu) = 0$ are close to those of $P(\mu) = 0$ and that the other is negative, the spatially constant equilibrium point (v_1^*, v_2^*, v_3^*) is stable if α vanishes. However, if α is large, then $P(\mu) = 0$ possesses a positive root because of the cross-diffusion induced instability, which implies that the equilibrium point of (15) destabilizes for suitably small ε when the diffusivity of one inhibitor v_2 is large enough. Thus it turns out that the reaction-diffusion system (15) includes the framework of short range activator-long range inhibitor reaction-diffusion system for (v_3, v_2) and that the cross-diffusion induced instability of (8) can be regarded as Turing's instability of (15).

4 RD-approximation to cross-diffusion systems

To prove Theorem 1 in §1, we consider an auxiliary problem in this section:

$$\begin{cases} u_t = d_1 \Delta u + \tilde{\alpha} \Delta w + f(u,v) + \eta_1(u,v,w), & t > 0,\ x \in \Omega, \\ v_t = d_2 \Delta v + g(u,v) + \eta_2(u,v,w), & t > 0,\ x \in \Omega, \\ w_t = (d_1 + \tilde{\alpha}) \Delta w + \eta_3(u,v,w) + \dfrac{1}{\varepsilon}\kappa(u,v,w), & t > 0,\ x \in \Omega \end{cases} \qquad (30)$$

to approximate a cross-diffusion system

$$\begin{cases} \tilde{u}_t = \Delta[(d_1 + \tilde{\alpha}\phi(\tilde{v}))\tilde{u}] + f(\tilde{u}, \tilde{v}), & t > 0,\ x \in \Omega, \\ \tilde{v}_t = d_2 \Delta \tilde{v} + g(\tilde{u}, \tilde{v}), & t > 0,\ x \in \Omega. \end{cases} \qquad (31)$$

For simplicity we assume that the functions ϕ, f, g, η_1, η_2, η_3 and κ allow all nonnegative numbers as their independent variables. We impose on (31) the boundary condition

$$\frac{\partial \tilde{u}}{\partial \nu} = \frac{\partial \tilde{v}}{\partial \nu} = 0, \qquad t > 0,\ x \in \partial\Omega \qquad (32)$$

and the initial condition

$$\tilde{u}(0,x) = u_0(x), \quad \tilde{v}(0,x) = v_0(x), \qquad x \in \Omega, \tag{33}$$

where we assume

$$u_0(x) \geq 0, \quad v_0(x) \geq 0 \qquad \text{on } \Omega.$$

On (30) we impose the boundary condition

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \qquad t > 0, \ \ x \in \partial\Omega \tag{34}$$

and the initial condition

$$u(0,x) = u_0(x), \ \ v(0,x) = v_0(x), \ \ w(0,x) = \phi(v_0(x))u_0(x), \quad x \in \Omega. \tag{35}$$

We will deduce Theorem 1 at the end of this section from some a priori estimates for the difference between the solutions of (30), (34), (35) and those of (31), (32), (33).

Theorem 2 *Let d_1, d_2, $\tilde{\alpha}$ be positive numbers and let ϕ, f, g, η_1, η_2, η_3 and κ be smooth functions satisfying*

$$\phi(s_2) \geq 0, \tag{36}$$

$$\eta_1(s_1, s_2, \phi(s_2)s_1) \equiv \eta_2(s_1, s_2, \phi(s_2)s_1) \equiv \kappa(s_1, s_2, \phi(s_2)s_1) \equiv 0, \tag{37}$$

$$\kappa_u(s_1, s_2, s_3) \geq 0, \tag{38}$$

$$\kappa_w(s_1, s_2, s_3) < 0, \tag{39}$$

$$\kappa_{uw}(s_1, s_2, s_3) \equiv 0 \tag{40}$$

for $(s_1, s_2, s_3) \in [0,\infty)^3$. Denote by $(\tilde{u},\tilde{v}) = (\tilde{u}(t,x), \tilde{v}(t,x))$ the solution of (31), (32), (33), *and by $(u,v,w) = (u(t,x;\varepsilon), v(t,x;\varepsilon), w(t,x;\varepsilon))$ the solution of* (30), (34), (35) *parametrized by a positive number ε. Suppose that $(\tilde{u},\tilde{v})$ is nonnegative and sufficiently smooth at least on $[0,T] \times \overline{\Omega}$, where T is a positive number. Also suppose the existence of positive numbers ε_0 and $\tilde{M}_0$ satisfying the following: (u,v,w) is sufficiently smooth and*

$$(u(t,x;\varepsilon), \ v(t,x;\varepsilon), \ w(t,x;\varepsilon)) \in [0,\tilde{M}_0]^3 \tag{41}$$

on $[0,T] \times \overline{\Omega}$ for $\varepsilon \in (0,\varepsilon_0]$. Then the difference between (u,v,w) and $(\tilde{u},\tilde{v})$ is estimated to be

$$\begin{cases} \sup\limits_{t\in[0,T]} \|u(t,\cdot;\varepsilon) - \tilde{u}(t,\cdot)\|_{L^2(\Omega)} \leq C\varepsilon, \\ \sup\limits_{t\in[0,T]} \|v(t,\cdot;\varepsilon) - \tilde{v}(t,\cdot)\|_{L^2(\Omega)} \leq C\varepsilon, \\ \sup\limits_{t\in[0,T]} \|w(t,\cdot;\varepsilon) - \phi(\tilde{v}(t,\cdot))\tilde{u}(t,\cdot)\|_{L^2(\Omega)} \leq C\varepsilon \end{cases} \tag{42}$$

for $\varepsilon \in (0,\varepsilon_0]$. Here $C = C(\tilde{u},\tilde{v},\varepsilon_0,\tilde{M}_0,T)$ is a positive constant independent of ε.

A similar theorem is shown in [4], where the convergence in $H^1(\Omega)$ as $\varepsilon \to +0$ is proved instead of (42), while κ_w is assumed to be a negative constant. However, since this assumption is quite strong, we extend this result to Theorem 2 for more general systems (30) which include η_1, η_2 and κ where κ_w is not assumed to be a constant.

Proof. Fix a positive number T satisfying the assumption in the theorem. Due to (41) we may assume that f, g, η_1, η_2, η_3, κ, ϕ and their derivatives appearing in this proof are uniformly bounded independently of (t, x) and in particular ε. (For a preciser argument we need suitable truncation of f, g, etc. around $[0, \tilde{M}_0]^3$. See, e.g., §2 of [4].) We denote the positive constants independent of ε and (t, x) by c_i $(i = 1, 2, \cdots)$. Moreover it follows from (39) that

$$\kappa_0 := \inf_{(s_1,s_2,s_3)\in[0,\tilde{M}_0]^3} \{-\kappa_w(s_1, s_2, s_3)\} > 0. \tag{43}$$

Set

$$\tilde{w} := \phi(\tilde{v})\tilde{u},\ U := u - \tilde{u},\ V := v - \tilde{v},\ W := w - \phi(\tilde{v})\tilde{u} = w - \tilde{w}.$$

Using $\eta_1(\tilde{u}, \tilde{v}, \tilde{w}) = \eta_2(\tilde{u}, \tilde{v}, \tilde{w}) = \kappa(\tilde{u}, \tilde{v}, \tilde{w}) = 0$ which follows from (37), we have

$$\begin{cases} U_t &= d_1\Delta U + \tilde{\alpha}\Delta W + f(\tilde{u}+U, \tilde{v}+V) - f(\tilde{u}, \tilde{v}) \\ & \quad +\eta_1(\tilde{u}+U, \tilde{v}+V, \tilde{w}+W) - \eta_1(\tilde{u}, \tilde{v}, \tilde{w}), \\ V_t &= d_2\Delta V + g(\tilde{u}+U, \tilde{v}+V) - g(\tilde{u}, \tilde{v}) \\ & \quad +\eta_2(\tilde{u}+U, \tilde{v}+V, \tilde{w}+W) - \eta_2(\tilde{u}, \tilde{v}, \tilde{w}), \\ W_t &= (d_1+\tilde{\alpha})\Delta W \\ & \quad +\frac{1}{\varepsilon}\{\kappa(\tilde{u}+U, \tilde{v}+V, \tilde{w}+W) - \kappa(\tilde{u}+U, \tilde{v}+V, \tilde{w})\} \\ & \quad +\frac{1}{\varepsilon}\{\kappa(\tilde{u}+U, \tilde{v}+V, \tilde{w}) - \kappa(\tilde{u}, \tilde{v}, \tilde{w})\} \\ & \quad +\eta_3(\tilde{u}+U, \tilde{v}+V, \tilde{w}+W) - \tilde{w}_t + (d_1+\tilde{\alpha})\Delta\tilde{w} \end{cases} \tag{44}$$

for $t > 0$ and $x \in \Omega$. The difference (U, V, W) also satisfies the boundary condition

$$\frac{\partial U}{\partial \nu} = \frac{\partial V}{\partial \nu} = \frac{\partial W}{\partial \nu} = 0, \qquad t > 0,\ x \in \partial\Omega$$

and the initial condition

$$U(0, x) = V(0, x) = W(0, x) = 0, \qquad x \in \Omega.$$

We denote a primitive function of $\kappa(u, v, \tilde{w})$ with respect to u by

$$K(u, v; \tilde{w}) := \int_0^u \kappa(s, v, \tilde{w})\, ds,$$

where we regard K as a function of $(u, v) \in [0, \infty)^2$ parametrized by $\tilde{w} \in [0, \infty)$. Using K, we define a quadratic form $E(t; \varepsilon)$ of $(U(t, \cdot; \varepsilon),\ V(t, \cdot; \varepsilon))$ in $L^2(\Omega)$ by

$$E(t) = E(t;\varepsilon) := \int_\Omega \left\{ K(\tilde{u}+U, \tilde{v}+V; \tilde{w}) - \tilde{K} - \tilde{K}_u U - \tilde{K}_v V \right\} dx,$$

where we abbreviate $K(\tilde{u}, \tilde{v}; \tilde{w})$, $K_u(\tilde{u}, \tilde{v}; \tilde{w})$, etc. to $\tilde{K}$, $\tilde{K}_u$, etc.; hereafter we often use similar abbreviation for simplicity of notation. We denote the standard norm and inner product in $L^2(\Omega)$ by $\|\cdot\|$ and $(\cdot, \cdot)$. Differentiating $E(t)$ in t, we have

$$\begin{aligned}
\frac{dE}{dt} &= \Big(K_u(\tilde{u}+U, \tilde{v}+V; \tilde{w}) - \tilde{K}_u - \tilde{K}_{uu}U - \tilde{K}_{vu}V,\ \tilde{u}_t\Big) \\
&\quad + \Big(K_v(\tilde{u}+U, \tilde{v}+V; \tilde{w}) - \tilde{K}_v - \tilde{K}_{uv}U - \tilde{K}_{vv}V,\ \tilde{v}_t\Big) \\
&\quad + \Big(K_{\tilde{w}}(\tilde{u}+U, \tilde{v}+V; \tilde{w}) - \tilde{K}_{\tilde{w}} - \tilde{K}_{u\tilde{w}}U - \tilde{K}_{v\tilde{w}}V,\ \tilde{w}_t\Big) \\
&\quad + \Big(K_u(\tilde{u}+U, \tilde{v}+V; \tilde{w}) - \tilde{K}_u,\ U_t\Big) \\
&\quad + \Big(K_v(\tilde{u}+U, \tilde{v}+V; \tilde{w}) - \tilde{K}_v,\ V_t\Big) \\
&\le c_1(\|U\|^2 + \|V\|^2 + \|W\|^2) \\
&\quad + \Big(\kappa(\tilde{u}+U, \tilde{v}+V, \tilde{w}) - \tilde{\kappa},\ d_1\Delta U + \tilde{\alpha}\Delta W\Big) \\
&\quad + \Big(K_v(\tilde{u}+U, \tilde{v}+V; \tilde{w}) - \tilde{K}_v,\ d_2\Delta V\Big).
\end{aligned}$$

Observe that

$$\begin{aligned}
&\Big(K_v(\tilde{u}+U, \tilde{v}+V; \tilde{w}) - \tilde{K}_v,\ d_2\Delta V\Big) \\
&= -d_2\Big[\Big(\{K_{vu}(\tilde{u}+U, \tilde{v}+V; \tilde{w}) - \tilde{K}_{vu}\}\nabla\tilde{u},\ \nabla V\Big) \\
&\quad + \Big(\{K_{vv}(\tilde{u}+U, \tilde{v}+V; \tilde{w}) - \tilde{K}_{vv}\}\nabla\tilde{v},\ \nabla V\Big) \\
&\quad + \Big(\{K_{v\tilde{w}}(\tilde{u}+U, \tilde{v}+V; \tilde{w}) - \tilde{K}_{v\tilde{w}}\}\nabla\tilde{w},\ \nabla V\Big) \\
&\quad + \Big(K_{vu}(\tilde{u}+U, \tilde{v}+V; \tilde{w})\nabla U,\ \nabla V\Big) \\
&\quad + \Big(K_{vv}(\tilde{u}+U, \tilde{v}+V; \tilde{w})\nabla V,\ \nabla V\Big)\Big] \\
&\le c_2(\|U\| + \|V\| + \|\nabla U\| + \|\nabla V\|)\|\nabla V\|
\end{aligned}$$

and that

$$
\begin{aligned}
&\Big(\kappa(\tilde u+U,\tilde v+V,\tilde w)-\tilde\kappa,\ d_1\Delta U\Big)\\
&= -d_1\Big[\Big(\{\kappa_u(\tilde u+U,\tilde v+V,\tilde w)-\tilde\kappa_u\}\nabla\tilde u,\ \nabla U\Big)\\
&\quad+\Big(\{\kappa_v(\tilde u+U,\tilde v+V,\tilde w)-\tilde\kappa_v\}\nabla\tilde v,\ \nabla U\Big)\\
&\quad+\Big(\{\kappa_w(\tilde u+U,\tilde v+V,\tilde w)-\tilde\kappa_w\}\nabla\tilde w,\ \nabla U\Big)\\
&\quad+\Big(\kappa_u(\tilde u+U,\tilde v+V,\tilde w)\nabla U,\ \nabla U\Big)+\Big(\kappa_v(\tilde u+U,\tilde v+V,\tilde w)\nabla V,\ \nabla U\Big)\Big]\\
&\le -d_1\Big(\kappa_u(\tilde u+U,\tilde v+V,\tilde w)\nabla U,\ \nabla U\Big)+c_3(\|U\|+\|V\|+\|\nabla V\|)\|\nabla U\|\\
&\le c_3(\|U\|+\|V\|+\|\nabla V\|)\|\nabla U\|,
\end{aligned}
$$

where we used (38). Thus we obtain

$$
\begin{aligned}
\frac{dE}{dt} &\le \tilde\alpha\Big(\kappa(\tilde u+U,\tilde v+V,\tilde w)-\tilde\kappa,\ \Delta W\Big)+c_1(\|U\|^2+\|V\|^2+\|W\|^2)\\
&\quad+c_2(\|U\|+\|V\|+\|\nabla U\|+\|\nabla V\|)\|\nabla V\|\\
&\quad+c_3(\|U\|+\|V\|+\|\nabla V\|)\|\nabla U\|. \qquad (45)
\end{aligned}
$$

Recalling (40), we have

$$\kappa_u(\tilde u+U,\tilde v+V,\tilde w+W)-\kappa_u(\tilde u+U,\tilde v+V,\tilde w)=0.$$

Multiply the third equation of (44) by $-\Delta W$ and integrate it by parts over Ω. Then we can derive

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla W\|^2=(W_t,\ -\Delta W)\\
&= -(d_1+\tilde\alpha)\|\Delta W\|^2\\
&\quad+\frac{1}{\varepsilon}\Big(\kappa_w(\tilde u+U,\tilde v+V,\tilde w+W)\nabla W,\ \nabla W\Big)\\
&\quad+\frac{1}{\varepsilon}\Big(\{\kappa_v(\tilde u+U,\tilde v+V,\tilde w+W)-\kappa_v(\tilde u+U,\tilde v+V,\tilde w)\}(\nabla\tilde v+\nabla V),\ \nabla W\Big)\\
&\quad+\frac{1}{\varepsilon}\Big(\{\kappa_w(\tilde u+U,\tilde v+V,\tilde w+W)-\kappa_w(\tilde u+U,\tilde v+V,\tilde w)\}\nabla\tilde w,\ \nabla W\Big)\\
&\quad-\frac{1}{\varepsilon}\Big(\kappa(\tilde u+U,\tilde v+V,\tilde w)-\tilde\kappa,\ \Delta W\Big)\\
&\quad-\Big(\eta_3(\tilde u+U,\tilde v+V,\tilde w+W)-\tilde\eta_3,\ \Delta W\Big)\\
&\quad+\Big(\nabla\{(d_1+\tilde\alpha)\Delta\tilde w-\tilde w_t+\tilde\eta_3\},\ \nabla W\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\le -(d_1+\tilde{\alpha})\|\Delta W\|^2 - \frac{1}{\varepsilon}\Big(\kappa(\tilde{u}+U,\tilde{v}+V,\tilde{w})-\tilde{\kappa},\,\Delta W\Big)\\
&\quad -\frac{\kappa_0}{\varepsilon}\|\nabla W\|^2 + \frac{c_4}{\varepsilon}(\|W\|+\|\nabla V\|)\|\nabla W\|\\
&\quad +c_5(\|U\|+\|V\|+\|W\|)\|\Delta W\| + c_6\|\nabla W\|\\
&\le -\frac{\kappa_0}{2\varepsilon}\|\nabla W\|^2 - \frac{1}{\varepsilon}\Big(\kappa(\tilde{u}+U,\tilde{v}+V,\tilde{w})-\tilde{\kappa},\,\Delta W\Big)\\
&\quad +c_7\varepsilon + c_8(\|U\|^2+\|V\|^2+\|W\|^2) + \frac{c_9}{\varepsilon}(\|W\|^2+\|\nabla V\|^2), \qquad (46)
\end{aligned}
$$

where we used (43). On the other hand, it follows from the first and second equations of (44) that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|U\|^2 &= -d_1\|\nabla U\|^2 - \tilde{\alpha}(\nabla W,\nabla U) + \Big(f(\tilde{u}+U,\tilde{v}+V)-\tilde{f},\,U\Big)\\
&\quad +\Big(\eta_1(\tilde{u}+U,\tilde{v}+V,\tilde{w}+W)-\tilde{\eta}_1,\,U\Big)\\
&\le -\frac{d_1}{2}\|\nabla U\|^2 + \frac{\tilde{\alpha}^2}{2d_1}\|\nabla W\|^2 + c_{10}(\|U\|+\|V\|+\|W\|)\|U\|; \qquad (47)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|V\|^2 &= -d_2\|\nabla V\|^2 + \Big(g(\tilde{u}+U,\tilde{v}+V)-\tilde{g},\,V\Big)\\
&\quad +\Big(\eta_2(\tilde{u}+U,\tilde{v}+V,\tilde{w}+W)-\tilde{\eta}_2,\,V\Big)\\
&\le -d_2\|\nabla V\|^2 + c_{11}(\|U\|+\|V\|+\|W\|)\|V\|. \qquad (48)
\end{aligned}
$$

Similarly, with the aid of (43), we can deduce from the third equation of (44) that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|W\|^2 &= -(d_1+\tilde{\alpha})\|\nabla W\|^2 + \frac{1}{\varepsilon}\Big(\kappa_w(\tilde{u}+U,\,\tilde{v}+V,\,\tilde{w}+\theta W)W,\,W\Big)\\
&\quad +\frac{1}{\varepsilon}\Big(\kappa(\tilde{u}+U,\tilde{v}+V,\tilde{w})-\tilde{\kappa},\,W\Big)\\
&\quad +\Big(\eta_3(\tilde{u}+U,\tilde{v}+V,\tilde{w}+W)-\tilde{w}_t+(d_1+\tilde{\alpha})\Delta\tilde{w},\,W\Big)\\
&\le -\frac{\kappa_0}{2\varepsilon}\|W\|^2 + c_{12}\varepsilon + \frac{c_{13}}{\varepsilon}(\|U\|^2+\|V\|^2), \qquad (49)
\end{aligned}
$$

where the function $\theta=\theta(t,x;\varepsilon)$ satisfies $\theta\in(0,1)$. Choose positive numbers γ_1, γ_2 and γ_3 satisfying

$$
\begin{aligned}
\gamma_1 &\le \frac{d_1\kappa_0}{\tilde{\alpha}},\\
\gamma_2 &\ge \frac{1}{d_2}\left\{\frac{(c_2+c_3)^2}{\gamma_1 d_1}+\frac{3c_2}{2}+c_9\tilde{\alpha}\right\},\\
\gamma_3 &\ge \frac{1+2(c_1+c_9\tilde{\alpha}+c_8\tilde{\alpha}\varepsilon_0)}{\kappa_0},
\end{aligned}
$$

and combine (45), (46), (47), (48) and (49). Then we find

$$\begin{aligned}
&\frac{d}{dt}\left\{E(t)+\frac{\gamma_1}{2}\|U\|^2+\frac{\gamma_2}{2}\|V\|^2+\frac{\varepsilon\gamma_3}{2}\|W\|^2+\frac{\varepsilon\tilde{\alpha}}{2}\|\nabla W\|^2\right\}\\
\le\;& -\frac{\tilde{\alpha}\kappa_0}{2}\|\nabla W\|^2-\frac{\gamma_1 d_1}{2}\|\nabla U\|^2-\gamma_2 d_2\|\nabla V\|^2-\frac{\gamma_3\kappa_0}{2}\|W\|^2\\
&+\frac{\gamma_1\tilde{\alpha}^2}{2d_1}\|\nabla W\|^2+c_2(\|U\|+\|V\|+\|\nabla U\|+\|\nabla V\|)\|\nabla V\|\\
&+c_3(\|U\|+\|V\|+\|\nabla V\|)\|\nabla U\|+c_9\tilde{\alpha}(\|\nabla V\|^2+\|W\|^2)\\
&+(c_1+\varepsilon c_8\tilde{\alpha})(\|U\|^2+\|V\|^2+\|W\|^2)\\
&+(\gamma_1 c_{10}\|U\|+\gamma_2 c_{11}\|V\|)(\|U\|+\|V\|+\|W\|)\\
&+\gamma_3 c_{13}(\|U\|^2+\|V\|^2)+\varepsilon^2(c_7\tilde{\alpha}+\gamma_3 c_{12})\\
\le\;& c_{14}(\|U\|^2+\|V\|^2)+c_{15}\varepsilon^2 \qquad (50)
\end{aligned}$$

for $\varepsilon \in (0, \varepsilon_0]$. On the other hand, it follows from (38) that

$$E(t) \ge -c_{16}\|U\|\|V\| - c_{17}\|V\|^2 \ge -\frac{\gamma_1}{4}\|U\|^2 - \left(c_{17}+\frac{c_{16}^2}{\gamma_1}\right)\|V\|^2.$$

Thus, by taking γ_2 so large as

$$\gamma_2 \ge 2\left(c_{17}+\frac{c_{16}^2}{\gamma_1}\right)+\frac{\gamma_1}{2}$$

(if necessary), we obtain

$$E(t)+\frac{\gamma_1}{2}\|U\|^2+\frac{\gamma_2}{2}\|V\|^2 \ge \frac{\gamma_1}{4}(\|U\|^2+\|V\|^2). \qquad (51)$$

Therefore, we can deduce from (50) and Gronwall's inequality that

$$E(t;\varepsilon)+\frac{\gamma_1}{2}\|U(t,\cdot;\varepsilon)\|^2+\frac{\gamma_2}{2}\|V(t,\cdot;\varepsilon)\|^2 \le c_{18}\varepsilon^2$$

for $t \in [0, T]$. This inequality and (51) imply that

$$\|U(t,\cdot;\varepsilon)\|^2+\|V(t,\cdot;\varepsilon)\|^2 \le \frac{4c_{18}}{\gamma_1}\varepsilon^2.$$

Applying this result to the right-hand side of (49), we find

$$\frac{d}{dt}\|W\|^2+\frac{\kappa_0}{\varepsilon}\|W\|^2 \le c_{19}\varepsilon.$$

Hence

$$\|W(t,\cdot;\varepsilon)\|^2 \le \frac{c_{19}}{\kappa_0}\varepsilon^2;$$

the proof is completed. □ *Proof of Theorem 1* It follows from (20) and (21) that

$$
\begin{aligned}
&(r_1 - a_1(v_1+v_2) - b_1v_3)v_1 + \frac{1}{\varepsilon}[h(v_3)v_2 - k(v_3)v_1] \geq 0 \\
&\qquad\qquad\qquad\qquad \text{if } v_1 = 0,\quad v_2 \geq 0,\quad 0 \leq v_3 \leq M_2; \\
&(r_1 - a_1(v_1+v_2) - b_1v_3)v_2 + \frac{1}{\varepsilon}[k(v_3)v_1 - h(v_3)v_2] \geq 0 \\
&\qquad\qquad\qquad\qquad \text{if } v_1 \geq 0,\quad v_2 = 0,\quad 0 \leq v_3 \leq M_2; \\
&(r_2 - b_2(v_1+v_2) - a_2v_3)v_3 \geq 0 \quad \text{if } v_1 \geq 0,\quad v_2 \geq 0,\quad v_3 = 0; \\
&(r_2 - b_2(v_1+v_2) - a_2v_3)v_3 \leq 0 \quad \text{if } v_1 \geq 0,\quad v_2 \geq 0,\quad v_3 = M_2.
\end{aligned}
$$

Hence the region $[0,\infty) \times [0,\infty) \times [0, M_2]$ for (v_1, v_2, v_3) is positively invariant in the reaction-diffusion system (15). In other words, (14) and (23) imply

$$v_1(t,x;\varepsilon) \geq 0, \quad v_2(t,x;\varepsilon) \geq 0, \quad 0 \leq v_3(t,x;\varepsilon) \leq M_2.$$

Rewrite (u_1, u_2) and $(v_1+v_2,\ v_3,\ v_2)$ as $(\tilde{u}, \tilde{v})$ and (u, v, w) respectively. Set $\tilde{\alpha} := \alpha M_2$ and

$$
\begin{aligned}
&\phi(s_2) := \frac{s_2}{M_2}, \\
&\eta_1(s_1,s_2,s_3) := f_1(s_1-s_3,\ s_3,\ s_2) + f_2(s_1-s_3,\ s_3,\ s_2) - f(s_1,s_2), \\
&\eta_2(s_1,s_2,s_3) := f_3(s_1-s_3,\ s_3,\ s_2) - g(s_1,s_2), \\
&\eta_3(s_1,s_2,s_3) := f_2(s_1-s_3,\ s_3,\ s_2), \\
&\kappa(s_1,s_2,s_3) := k(s_2)s_1 - \{h(s_2)+k(s_2)\}s_3
\end{aligned}
$$

for $(s_1,s_2,s_3) \in [0,\infty)^3$. Here f, g, f_1, f_2 and f_3 are the functions defined in §3. Due to (19), (21), (22) and (27), the assumptions of Theorem 2 are fulfilled. Therefore we obtain not only (25) but also

$$
\begin{aligned}
&\sup_{t\in[0,T]} \left\| v_1(t,\cdot;\varepsilon) - \left\{1 - \frac{u_2(t,\cdot)}{M_2}\right\} u_1(t,\cdot) \right\|_{L^2(\Omega)} \leq C\varepsilon, \\
&\sup_{t\in[0,T]} \left\| v_2(t,\cdot;\varepsilon) - \frac{u_2(t,\cdot)}{M_2} u_1(t,\cdot) \right\|_{L^2(\Omega)} \leq C\varepsilon
\end{aligned}
$$

for $\varepsilon \in (0, \varepsilon_0]$. □

5 Concluding remarks

In this paper, it is shown that the cross-diffusion system (8) can be approximated by the reaction-diffusion system (15) by introducing the active state and the less active one. This approximation also reveals the relationship between the cross-diffusion induced instability of (8) and Turing's instability of (15).

Finally we remark that Theorem 2 is applicable to more general systems, e.g.

$$\begin{cases} u_{1t} = \Delta[(d_1 + \tilde{\alpha}\phi(u_2))u_1] + (r_1 - a_1u_1 - b_1u_2)u_1, \\ u_{2t} = d_2\Delta u_2 + (r_2 - b_2u_1 - a_2u_2)u_2, \end{cases} \tag{52}$$

than (8) if we rewrite (u_1, u_2) as $(\tilde{u}, \tilde{v})$. We also remark that there are several choices of the reaction-diffusion systems (15) which converge to the specific cross-diffusion system (52). For an example of ϕ in (52), h and k in (15) are not uniquely determined but there are other choices. In fact, we can choose

$$\phi(u_2) := \frac{u_2}{M_2}, \quad h(v_3) := 1, \quad k(v_3) := \frac{v_3}{M_2 - v_3},$$

though we chose

$$h(v_3) := 1 - \frac{v_3}{M_2}, \quad k(v_3) := \frac{v_3}{M_2}$$

in §1. If we choose

$$h(v_3) := 1, \quad k(v_3) := \varphi(v_3)$$

or

$$h(v_3) := \frac{1}{1 + \varphi(v_3)}, \quad k(v_3) := \frac{\varphi(v_3)}{1 + \varphi(v_3)}$$

in (15), then the corresponding cross-diffusion system is formally (52) with

$$\phi(u_2) := \frac{\varphi(u_2)}{1 + \varphi(u_2)}.$$

This is justified by Theorem 2, as long as (14) with $M_2 > r_2/a_2$, (23) and

$$\varphi(s) \geq 0 \quad \text{for } s \in [0, M_2)$$

hold true. In (15), there are also several choices of replacements of f_1, f_2 and f_3 defined in §3.

Acknowledgments

The authors would like to express their thanks to Mr. Izuhara for all the figures in this paper.

References

[1] H. Amann, Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems, in "Function spaces, differential operators and nonlinear analysis," H. Schmeisser and H. Triebel eds., Teubner-Texte Math. **133,** (1993) 9–126.

[2] Y. S. Choi, R. Lui and Y. Yamada, Existence of global solutions for the Shigesada-Kawasaki-Teramoto model with weak cross-diffusion, *Discrete Contin. Dynam. Systems* **9** (2003), 1193–1200.

[3] Hirsch, M.W.,Differential equations and convergence almost everywhere of strongly monotone semiflows, *PAM Technical Report*, University of California, Berkeley (1982).

[4] M. Iida and H. Ninomiya, A reaction-diffusion approximation to a cross-diffusion system, to appear in "Recent Advances on Elliptic and Parabolic Issues" (M. Chipot and H. Ninomiya eds.), World Scientific.

[5] Y. Kan-on, Stability of singularly perturbed solutions to nonlinear diffusion systems arising in population dynamics, *Hiroshima Math. J.* **23** (1993), 509–536.

[6] P. Kareiva and G. Odell Swarms of predators exhibit "preytaxis" if individual predator use area-restricted search, *The American Naturalist* **130** (1987), 233–270.

[7] Kishimoto, K. and Weinberger, H.F., The spatial homogeneity of stable equilibria of some reaction-diffusion system on convex domains, *J. Diff. Eq.*, **58** (1985), 15–21 .

[8] Y. Lou and W.-M. Ni, Diffusion vs cross-diffusion: an elliptic approach, *J. Differential Equations* **154** (1999), 157–190.

[9] Y. Lou, W.-M. Ni and S. Yotsutani, On a limiting system in the Lotka-Volterra competition with cross-diffusion, *Discrete Contin. Dynam. Systems* **10** (2004), 435–458.

[10] Y. Lou, W.-M. Ni and Y. Wu, On the global existence of a cross-diffusion system, *Discrete Contin. Dynam. Systems* **4** (1998), 193–203.

[11] H. Matano and M. Mimura, Pattern formation in competition-diffusion systems in nonconvex domains, *Publ. Res. Inst. Math. Sci. Kyoto Univ.* **19** (1983), 1049–1079.

[12] M. Mimura and K. Kawasaki, Spatial segregation in competitive interaction-diffusion equations, *J. Math. Biol.* **9** (1980), 49–64.

[13] M. Mimura, Y. Nishiura, A. Tesei and T. Tsujikawa, Coexistence problem for two competing species models with density-dependent diffusion, *Hiroshima Math. J.* **14** (1984), 425–449.

[14] A. Okubo, "Diffusion and ecological problems: mathematical models", Biomathematics **10**, Springer-Verlag, 1980.

[15] N. Shigesada, K. Kawasaki and E. Teramoto, Spatial segregation of interacting species, *J. Theoret. Biol.* **79** (1979), 83–99.

[16] A. M. Turing, The chemical basis of morphogenesis, *Phil. Trans. Roy. Soc. London* **B237** (1952), 37–72.