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Resurgent functions and splitting problems

David Sauzin (CNRS–IMCCE, Paris)

デイビット・ソザン（天体力学研究所、パリ）

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Abstract

The present text is an introduction to Écalle's theory of resurgent functions and alien calculus, in connection with problems of exponentially small separatrix splitting. An outline of the resurgent treatment of Abel's equation for resonant dynamics in one complex variable is included. Some proofs and details are omitted. The emphasis is on examples of nonlinear difference equations, as a simple and natural way of introducing the concepts.

Contents

1 The algebra of resurgent functions

Our first purpose is to present a part of Écalle's theory of resurgent functions and alien calculus in a self-contained way. Our main sources are the series of books [Eca81] (mainly the first two volumes), a course taught by Jean Écalle at Paris-Sud university (Orsay) in 1996 and the book [CNP93].

1.1 Formal Borel transform

A resurgent function can be viewed as a special kind of power series, the radius of convergence of which is zero, but which can be given an analytical meaning through Borel-Laplace summation. It is convenient to deal with formal series "at infinity", *i.e.* with elements of $\mathbb{C}[[z^{-1}]]$. We denote by $z^{-1}\mathbb{C}[[z^{-1}]]$ the subset of formal series without constant term.

Definition 1 *The formal Borel transform is the linear operator*

$$\mathcal{B} : \tilde{\varphi}(z) = \sum_{n\geq 0} c_n z^{-n-1} \in z^{-1}\mathbb{C}[[z^{-1}]] \;\mapsto\; \hat{\varphi}(\zeta) = \sum_{n\geq 0} c_n \frac{\zeta^n}{n!} \in \mathbb{C}[[\zeta]]. \tag{1}$$

Observe that if $\tilde{\varphi}(z)$ has nonzero radius of convergence, say if $\tilde{\varphi}(z)$ converges for $|z^{-1}| < \rho$, then $\hat{\varphi}(\zeta)$ defines an entire function, of exponential type in every direction: if $\tau > \rho^{-1}$, then $|\hat{\varphi}(\zeta)| \leq \text{const}\; \mathrm{e}^{\tau|\zeta|}$ for all $\zeta \in \mathbb{C}$.

Definition 2 *For any $\theta \in \mathbb{R}$, we define the Laplace transform in the direction θ as the linear operator $\mathcal{L}^\theta$,*

$$\mathcal{L}^\theta \hat{\varphi}(z) = \int_0^{\mathrm{e}^{\mathrm{i}\theta}\infty} \hat{\varphi}(\zeta)\, \mathrm{e}^{-z\zeta}\, \mathrm{d}\zeta. \tag{2}$$

Here, $\hat{\varphi}$ is assumed to be a function such that $r \mapsto \hat{\varphi}(r\,\mathrm{e}^{\mathrm{i}\theta})$ is analytic on $\mathbb{R}^+$ and $|\hat{\varphi}(r\,\mathrm{e}^{\mathrm{i}\theta})| \leq$ const $\mathrm{e}^{\tau r}$. The function $\mathcal{L}^\theta\hat{\varphi}$ is thus analytic in the half-plane $\Re e(z\,\mathrm{e}^{\mathrm{i}\theta}) > \tau$ (see Figure 1).

Recall that $z^{-n-1} = \int_0^{+\infty} \frac{\zeta^n}{n!}\, \mathrm{e}^{-z\zeta}\, \mathrm{d}\zeta$ for $\Re e\, z > 0$, thus

$$z^{-n-1} = \mathcal{L}^\theta\left(\frac{\zeta^n}{n!}\right), \qquad \Re e(z\,\mathrm{e}^{\mathrm{i}\theta}) > 0. \tag{3}$$

(For that reason, $\mathcal{B}$ is sometimes called "formal inverse Laplace transform".) As a consequence, if $\hat{\varphi}$ is an entire function of exponential type in every direction, that is if $\hat{\varphi} = \mathcal{B}\tilde{\varphi}$ with $\tilde{\varphi}(z) \in z^{-1}\mathbb{C}\{z^{-1}\}$, we recover $\tilde{\varphi}$ from $\hat{\varphi}$ by applying the Laplace transform: it can be shown[1] that $\mathcal{L}^\theta\hat{\varphi}(z) = \tilde{\varphi}(z)$ for all z and θ such that $\Re e(z\,\mathrm{e}^{\mathrm{i}\theta})$ is large enough.

Fine Borel-Laplace summation

Suppose now that $\mathcal{B}\tilde{\varphi} = \hat{\varphi} \in \mathbb{C}\{\zeta\}$ but $\hat{\varphi}$ is not entire, *i.e.* $\hat{\varphi}$ has finite radius of convergence. The radius of convergence of $\tilde{\varphi}$ is then zero. Still, it may happen that $\hat{\varphi}(\zeta)$ extends analytically to a half-strip $\{\, \zeta \in \mathbb{C} \mid \text{dist}(\zeta, \mathrm{e}^{\mathrm{i}\theta}\,\mathbb{R}^+) \leq \rho \,\}$, with exponential type less than a $\tau \in \mathbb{R}$. In such a case, formula (2) makes sense and the formal series $\tilde{\varphi}$ appears as the asymptotic expansion

[1] Here, as sometimes in this text, we omit the details of the proof. See *e.g.* [Mal95] for the properties of the Laplace and Borel transforms.

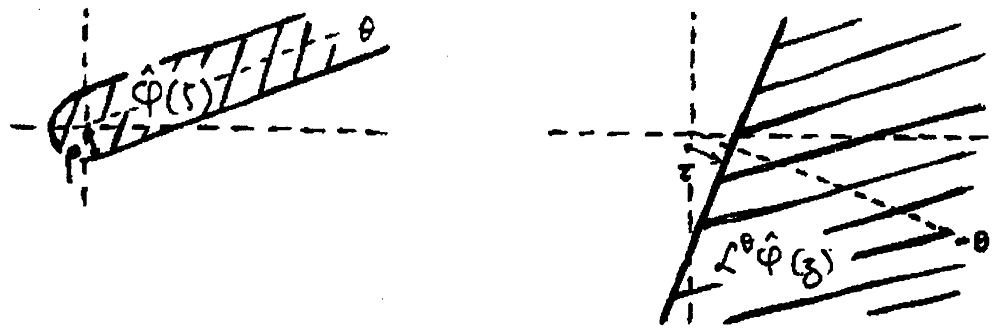


Figure 1: Laplace integral in the direction θ gives rise to functions analytic in the half-plane $\Re e(z\,e^{i\theta}) > \tau$.

of $\mathcal{L}^\theta\hat\varphi$ in the half-plane $\{\,\Re e(z\,e^{i\theta}) > \max(\tau,0)\,\}$ (as can be deduced from (3))[2]. This is more or less the classical definition of a "Borel-summable" formal series $\tilde\varphi$. One can consider the function $\mathcal{L}^\theta\mathcal{B}\tilde\varphi$ as a "sum" of $\tilde\varphi$, associated with the direction θ. This summation is called "fine" when $\hat\varphi$ is only known to extend to a half-strip in the direction θ, which is sufficient for recovering $\tilde\varphi$ as asymptotic expansion of $\mathcal{L}^\theta\hat\varphi$; more often, Borel-Laplace sums are associated with sectors.

Note: From the inversion of the Fourier transform, one can deduce a formula for the *integral Borel transform* which allows one to recover $\hat\varphi(\zeta)$ from $\mathcal{L}^\theta\hat\varphi(z)$. For instance, $\hat\varphi(\zeta) = \frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty}\mathcal{L}^0\hat\varphi(z)\,e^{z\zeta}\,dz$ for small $\zeta \geq 0$, with suitable $\rho > 0$.

Sectorial sums

Suppose that $\hat\varphi(\zeta)$ converges near the origin and extends analytically to a sector $\{\,\zeta\in\mathbb{C} \mid \theta_1 < \arg\zeta < \theta_2\,\}$ (where $\theta_1,\theta_2\in\mathbb{R}$, $|\theta_2-\theta_1| < 2\pi$), with exponential type less than τ, then we can move the direction of integration θ inside $]\theta_1,\theta_2[$. According to the Cauchy theorem, $\mathcal{L}^{\theta'}\hat\varphi$ is the analytic continuation of $\mathcal{L}^\theta\hat\varphi$ when $|\theta'-\theta| < \pi$, we can thus glue together these holomorphic functions and obtain a function $\mathcal{L}^{]\theta_1,\theta_2[}\hat\varphi$ analytic in the union of the half-planes $\{\,\Re e(z\,e^{i\theta}) > \tau\,\}$, which is a sectorial neighbourhood of infinity contained in $\{\,-\theta_2-\pi/2 < \arg z < -\theta_1+\pi/2\,\}$ (see Figure 2). Notice however that, if $\theta_2-\theta_1 > \pi$, the resulting function may be multivalued, *i.e.* one must consider the variable z as moving on the Riemann surface of the logarithm.

A frequent situation is the following: $\hat\varphi = \mathcal{B}\tilde\varphi$ converges and extends analytically to several infinite sectors, with bounded exponential type, but also has singularities at finite distance (in particular $\hat\varphi$ has finite radius of convergence and $\tilde\varphi$ is divergent). Then several "Borel-Laplace sums" are available on various domains, but are not the analytic continuations one of the other: the presence of singularities, which separate the sectors one from the other, prevents one from applying the Cauchy theorem. On the other hand, all these "sums" share the same asymptotic expansion: the mutual differences are exponentially small in the intersection of their domains of definition (see Figure 3).

[2]See footnote 1.

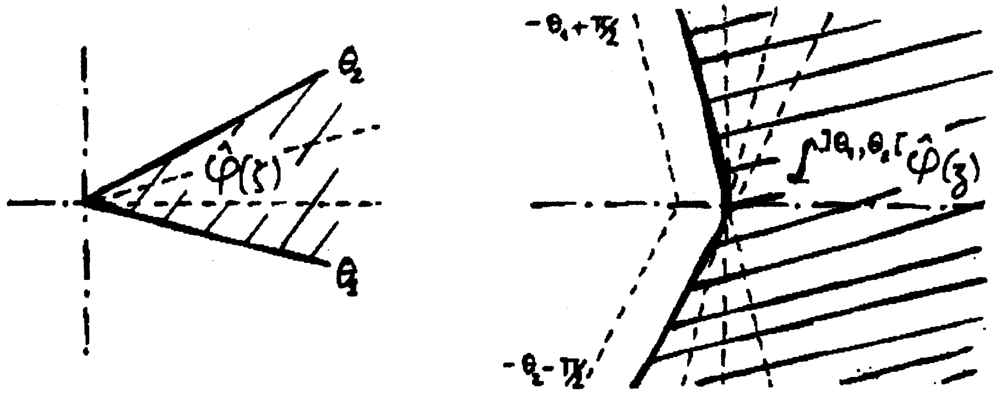


Figure 2: Sectorial sums.

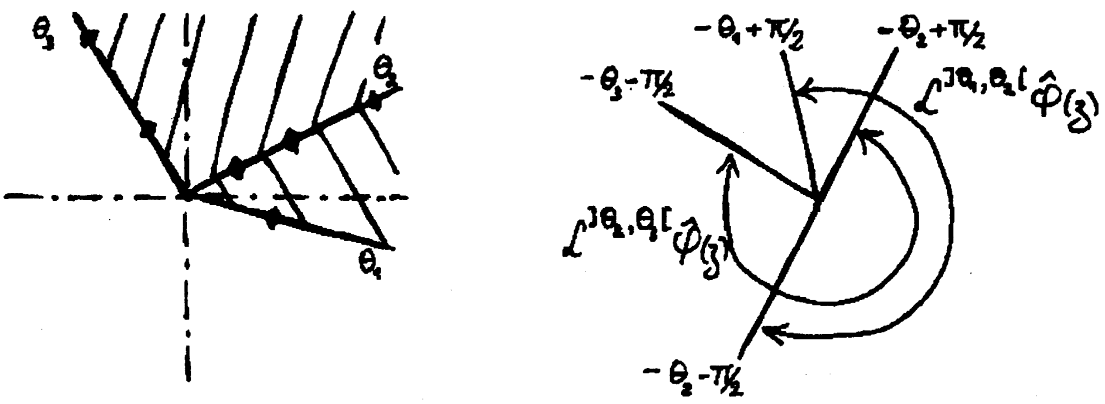


Figure 3: Several Borel-Laplace sums, analytic in different domains, may be attached to a single divergent series.

Resurgent functions

It is interesting to "measure" the singularities in the ζ-plane, since they can be considered as responsible for the divergence of the common asymptotic expansion $\tilde{\varphi}(z)$ and for the exponentially small differences between the various Borel-Laplace sums. The resurgent functions can be defined as a class of formal series $\tilde{\varphi}$ such that the analytic continuation of the formal Borel transform $\hat{\varphi}$ satisfies a certain condition regarding the possible singularities, which makes it possible to develop a kind of singularity calculus (named "alien calculus"). These notions were introduced in the late 70s by J. Écalle, who proved their relevance in a number of analytic problems [Eca81, Mal85]. We shall not try to expound the theory in its full generality, but shall rather content ourselves with explaining how it works in the case of certain difference equations.

Note: The formal Borel transform of a series $\tilde{\varphi}(z)$ has positive radius of convergence if and only if $\tilde{\varphi}(z)$ satisfies a "Gevrey-1" condition: $\hat{\varphi}(\zeta) \in \mathbb{C}\{\zeta\} \Leftrightarrow \tilde{\varphi}(z) \in z^{-1}\mathbb{C}[[z^{-1}]]_1$, where by definition

$$z^{-1}\mathbb{C}[[z^{-1}]]_1 = \left\{ \textstyle\sum_{n\geq 0} c_n z^{-n-1} \mid \exists\, \rho > 0 \text{ such that } |c_n| = \mathcal{O}(n!\rho^n) \right\}.$$

1.2 Linear and nonlinear difference equations

We shall be interested in formal series $\tilde{\varphi}$ solutions of certain equations involving the first-order difference operator $\tilde{\varphi}(z) \mapsto \tilde{\varphi}(z+1) - \tilde{\varphi}(z)$ (or second-order differences). This operator is well defined in $\mathbb{C}[[z^{-1}]]$, *e.g.* by way of the Taylor formula

$$\tilde{\varphi}(z+1) - \tilde{\varphi}(z) = \partial\tilde{\varphi}(z) + \frac{1}{2!}\partial^2\tilde{\varphi}(z) + \frac{1}{3!}\partial^3\tilde{\varphi}(z) + \cdots, \tag{4}$$

where $\partial = \frac{\mathrm{d}}{\mathrm{d}z}$ and the series is formally convergent because of increasing valuations (we say that the series $\sum \frac{1}{r!}\partial^r\tilde{\varphi}$ is formally convergent because the right-hand side of (4) is a well-defined formal series, each coefficient of which is given by a finite sum of terms; this is the notion of sequential convergence associated with the so-called Krull topology).

It is elementary to compute the counterpart of the differential and difference operators by $\mathcal{B}$:

$$\mathcal{B} \,:\, \partial\tilde{\varphi}(z) \mapsto -\zeta\,\hat{\varphi}(\zeta), \qquad \tilde{\varphi}(z+1) \mapsto \mathrm{e}^{-\zeta}\,\hat{\varphi}(\zeta).$$

When $\tilde{\varphi}(z)$ is obtained by solving an equation, a natural strategy is thus to study $\hat{\varphi}(\zeta)$ as solution of a transformed equation. If a Laplace transform $\mathcal{L}^\theta$ can be applied to $\hat{\varphi}$, one then recovers an analytic solution of the original equation, because $\mathcal{L}^\theta \circ \mathcal{B}$ commutes with the differential and difference operators.

Two linear equations

Let us illustrate this on two simple equations:

$$\tilde{\varphi}(z+1) - \tilde{\varphi}(z) = a(z), \qquad a(z) \in z^{-2}\mathbb{C}\{z^{-1}\} \text{ given}, \tag{5}$$

$$\tilde{\psi}(z+1) - 2\tilde{\psi}(z) + \tilde{\psi}(z-1) = b(z), \qquad b(z) \in z^{-3}\mathbb{C}\{z^{-1}\} \text{ given}. \tag{6}$$

The corresponding equations for the formal Borel transforms are

$$(\mathrm{e}^{-\zeta} - 1)\hat{\varphi}(\zeta) = \hat{a}(\zeta), \qquad \left(4\sinh^2\frac{\zeta}{2}\right)\hat{\psi}(\zeta) = \hat{b}(\zeta).$$

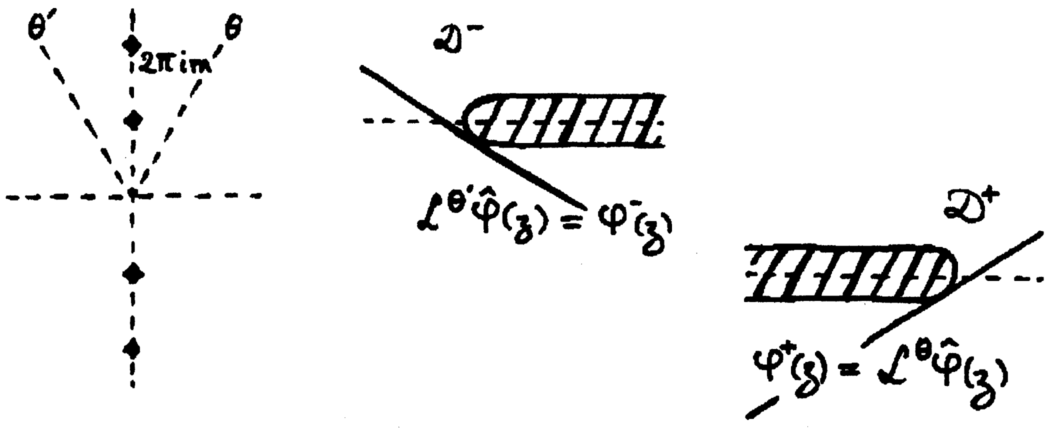


Figure 4: Borel-Laplace summation for the difference equation (5).

Here the power series $\hat{a}(\zeta)$ and $\hat{b}(\zeta)$ converge to entire functions of bounded exponential type in every direction, vanishing at the origin; moreover $\hat{b}'(0) = 0$. We thus get in $\mathbb{C}[[\zeta]]$ unique solutions $\hat{\varphi}(\zeta) = \hat{a}(\zeta)/(\mathrm{e}^{-\zeta} - 1)$ and $\hat{\psi}(\zeta) = \hat{b}(\zeta)/\left(4\sinh^2\frac{\zeta}{2}\right)$, which converge near the origin and define meromorphic functions, the possible poles being located in $2\pi\mathrm{i}\,\mathbb{Z}^*$.

The original equations thus admit unique solutions $\tilde{\varphi} = \mathcal{B}^{-1}\hat{\varphi}$ and $\tilde{\psi} = \mathcal{B}^{-1}\hat{\psi}$ in $z^{-1}\mathbb{C}[[z^{-1}]]$. For each of them, Borel-Laplace summation is possible and we get two natural sums, associated with two sectors:

$$\varphi^+(z) = \mathcal{L}^{\theta}\hat{\varphi}(z), \quad \theta \in \left]-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right[, \qquad \varphi^-(z) = \mathcal{L}^{\theta'}\hat{\varphi}(z), \quad \theta' \in \left]\tfrac{\pi}{2}, \tfrac{3\pi}{2}\right[,$$

and similarly $\hat{\psi}(\zeta)$ gives rise to $\psi^+(z)$ and $\psi^-(z)$.

The functions φ^+ and ψ^+ are solutions of (5) and (6), analytic in a domain of the form $\mathcal{D}^+ = \mathbb{C} \setminus \{\,\mathrm{dist}(z, \mathbb{R}^-) \le \tau\,\}$. The solutions φ^- and ψ^- are defined in a symmetric domain $\mathcal{D}^-$ (see Figure 4). The intersection $\mathcal{D}^+ \cap \mathcal{D}^-$ has two connected components, $\{\,\Im m\, z < -\tau\,\}$ and $\{\,\Im m\, z > \tau\,\}$. In the case of equation (5) for instance, the exponentially small difference $\varphi^+ - \varphi^-$ in the lower component is related to the singularities of $\hat{\varphi}$ in $2\pi\mathrm{i}\,\mathbb{N}^*$; it can be exactly computed by the resiuduum formula: the singularity at $\omega = 2\pi i m$ yields a contribution

$$A_\omega\, \mathrm{e}^{-\omega z}, \qquad \text{with } A_\omega = -2\pi\mathrm{i}\, \hat{a}(\omega)$$

(the modulus of which is $|A_\omega| \mathrm{e}^{2\pi m\, \Im m\, z}$, which is exponentially small for $\Im m\, z \to -\infty$); the difference $(\varphi^+ - \varphi^-)(z) = \int_{\mathrm{e}^{\mathrm{i}\theta'}\infty}^{\mathrm{e}^{\mathrm{i}\theta}\infty} \hat{\varphi}(\zeta)\, \mathrm{e}^{-z\zeta}\, \mathrm{d}\zeta$ is simply the sum of these contributions:

$$\varphi^+(z) - \varphi^-(z) = \sum_{\omega \in 2\pi\mathrm{i}\,\mathbb{N}^*} A_\omega\, \mathrm{e}^{-\omega z}, \qquad \Im m\, z < -\tau \tag{7}$$

as is easily seen by deforming the contour of integration (choose θ and θ' close enough to $\pi/2$ according to the precise location of z, and push the contour of integration upwards). Symmetrically, the difference in the upper component can be computed from the singularities in $-2\pi\mathrm{i}\,\mathbb{N}^*$.

Note: $\varphi^+(z)$ is the unique solution of (5) which tends to 0 when $\Re e\, z \to +\infty$ and can be written as $-\sum_{k\ge 0} a(z+k)$, and $\varphi^-(z) = \sum_{k \ge 1} a(z-k)$ is the unique solution which tends to 0 when

$\Re e\, z \to -\infty$; the difference defines two 1-periodic functions, the Fourier coefficients of which can be expressed in term of the Fourier transform of $a(\pm i\rho + z)$ (take $\rho > 0$ large enough). One recovers the previous formula for the difference by using the integral representation for the Borel transform to compute the numbers $\hat{a}(\omega)$.

Nonlinear equations

In the present text we shall show how one can deal with nonlinear difference equations like

$$\tilde{\varphi}(z+1) - \tilde{\varphi}(z) = a\big(z + \tilde{\varphi}(z)\big), \qquad a(z) \in z^{-2}\mathbb{C}\{z^{-1}\} \text{ given,} \tag{8}$$

which is related to Abel's equation and the classification of holomorphic germs in one complex variable, or

$$\tilde{\psi}(z+1) - 2\tilde{\psi}(z) + \tilde{\psi}(z-1) = b\big(\tilde{\psi}(z), \tilde{\psi}(z-1)\big), \tag{9}$$

with certain $b(x,y) \in \mathbb{C}\{x,y\}$, which is related to splitting problems in two complex variables.

Dealing with nonlinear equations will require the study of *convolution*, which is the subject of sections 1.3 and 1.4. The Borel transforms $\hat{\varphi}(\zeta)$ and $\hat{\psi}(\zeta)$ will still be holomorphic at the origin but no longer meromorphic in $\mathbb{C}$, as will be shown later; their analytic continuations have more complicated singularities than mere first- or second-order poles. We shall introduce *alien calculus* in Section 2 and a more general version of it in Section 3.3 to deal with these singularities.

1.3 The Riemann surface $\mathcal{R}$ and the analytic continuation of convolution

The first nonlinear operation to be studied is the multiplication of formal series.

Lemma 1 *Let $\hat{\varphi}$ and $\hat{\psi}$ denote the formal Borel transforms of $\tilde{\varphi}, \tilde{\psi} \in z^{-1}\mathbb{C}[[z^{-1}]]$ and consider the product series $\tilde{\chi} = \tilde{\varphi}\tilde{\psi}$. Then its formal Borel transform is given by the "convolution"*

$$(\mathcal{B}\tilde{\chi})(\zeta) = (\hat{\varphi} * \hat{\psi})(\zeta) = \int_0^\zeta \hat{\varphi}(\zeta_1)\hat{\psi}(\zeta - \zeta_1)\, \mathrm{d}\zeta_1. \tag{10}$$

The above formula must be interpreted termwise: $\int_0^\zeta \frac{\zeta_1^n}{n!} \frac{(\zeta-\zeta_1)^m}{m!}\, \mathrm{d}\zeta_1 = \frac{\zeta^{n+m+1}}{(n+m+1)!}$ (as can be checked *e.g.* by induction on n, which is sufficient to prove the lemma).

The problem of analytic continuation

The formula can be given an analytic meaning in the case of Gevrey-1 formal series: if $\hat{\varphi}, \hat{\psi} \in \mathbb{C}\{\zeta\}$, their convolution is convergent in the intersection of the discs of convergence of $\hat{\varphi}$ and $\hat{\psi}$ and defines a new holomorphic germ $\hat{\varphi} * \hat{\psi}$ at the origin; formula (10) then holds as a relation between holomorphic functions, but only for $|\zeta|$ small enough (smaller than the radii of convergence of $\hat{\varphi}$ and $\hat{\psi}$). What about the analytic continuation of $\hat{\varphi} * \hat{\psi}$ when $\hat{\varphi}$ and $\hat{\psi}$ themselves admit an analytic continuation beyond their discs of convergence? What about the case when $\hat{\varphi}$ and $\hat{\psi}$ extend to meromorphic functions for instance?

A preliminary answer is that $\hat{\varphi} * \hat{\psi}$ always admit an analytic continuation in the intersection of the "holomorphic stars" of $\hat{\varphi}$ and $\hat{\psi}$. We define the holomorphic star of a germ as the union of all the open sets U containing the origin in which it admits analytic continuation and which are star-shaped with respect to the origin (*i.e.* $\forall\zeta \in U$, $[0,\zeta] \subset U$). And it is indeed clear that if $\hat\varphi$ and $\hat\psi$ are holomorphic in such a U, formula (10) makes sense for all $\zeta \in U$ and provides the analytic continuation of $\hat\varphi * \hat\psi$. With a view to further use we notice that, if $|\hat\varphi(\zeta)| \le \Phi(|\zeta|)$ and $|\hat\psi(\zeta)| \le \Psi(|\zeta|)$ for all $\zeta \in U$, then

$$|\hat\varphi * \hat\psi(\zeta)| \le \Phi * \Psi(|\zeta|), \qquad \zeta \in U. \tag{11}$$

The next step is to study what happens on singular rays, behind singular points. The idea is that convolution of poles generates ramification ("multivaluedness") but is easy to continue analytically. For example, since

$$1 * \hat\varphi(\zeta) = \int_0^\zeta \hat\varphi(\zeta_1)\,\mathrm{d}\zeta_1,$$

we see that when $\hat\varphi$ is a meromorphic function with poles in a set $\Omega \subset \mathbb{C}^*$, $1 * \hat\varphi$ admits an analytic continuation along any path issuing from the origin and avoiding Ω; in other words, it defines a function holomorphic on the universal cover[3] of $\mathbb{C}\setminus\Omega$, with logarithmic singularities at the poles of $\hat\varphi$.

But convolution may also create new singular points. For instance, if $\hat\varphi(\zeta) = \frac{1}{\zeta-\omega'}$ and $\hat\psi(\zeta) = \frac{1}{\zeta-\omega''}$ with $\omega', \omega'' \in \mathbb{C}^*$, one gets

$$\hat\varphi * \hat\psi(\zeta) = \frac{1}{\zeta-\omega}\left(\int_0^\zeta \frac{\mathrm{d}\zeta_1}{\zeta_1-\omega'} + \int_0^\zeta \frac{\mathrm{d}\zeta_1}{\zeta_1-\omega''}\right), \qquad \omega = \omega' + \omega''.$$

We thus have logarithmic singularities at ω' and ω'', but also a pole at ω, the residuum of which is an integer multiple of $2\pi\mathrm{i}$ which depends on the path chosen to approach ω. In other words, $\hat\varphi * \hat\psi$ extends meromorphically to the universal cover of $\mathbb{C}\setminus\{\omega',\omega''\}$, with a pole lying over ω (the residuum of which depends on the sheet[4] of the Riemann surface which is considered; in particular it vanishes for the principal sheet[5] if $\arg\omega' \ne \arg\omega''$, which is consistent with what was previously said on the holomorphic star).

The Riemann surface $\mathcal{R}$

With a view to the difference equations we are interested in and to the expected behaviour of the Borel transforms, we define a Riemann surface which is obtained by adding a point to the universal cover of $\mathbb{C}\setminus 2\pi\mathrm{i}\,\mathbb{Z}$.

[3] Here it is understood that the base-point is at the origin. If Ω is a closed subset of $\mathbb{C}$ with $\mathbb{C}\setminus\Omega$ connected and $\zeta_0 \in \mathbb{C}\setminus\Omega$, the universal cover of $\mathbb{C}\setminus\Omega$ with base-point ζ_0 can be defined as the set of homotopy classes of paths issuing from ζ_0 and lying in $\mathbb{C}\setminus\Omega$ (equivalence classes for homotopy with fixed extremities). We denote it $(\widetilde{\mathbb{C}\setminus\Omega,\zeta_0})$. There is a covering map $\pi : (\widetilde{\mathbb{C}\setminus\Omega,\zeta_0}) \to \mathbb{C}\setminus\Omega$, which associates with any class c the extremity $\gamma(1)$ of any path $\gamma : [0,1] \to \mathbb{C}\setminus\Omega$ which represents c, and which allows one to define a Riemann surface structure on $(\widetilde{\mathbb{C}\setminus\Omega,\zeta_0})$ by pulling back the complex structure of $\mathbb{C}\setminus\Omega$ (see [CNP93, pp. 81–89 and 105–112]). For example, the Riemann surface of the logarithm is $(\widetilde{\mathbb{C}\setminus\{0\},1})$, the points of which can be written "$r\,\mathrm{e}^{\mathrm{i}\theta}$" with $r > 0$ and $\theta \in \mathbb{R}$. We often use the letter ζ for points of a universal cover, and then denote by $\dot\zeta = \pi(\zeta)$ their projection.

[4] Again we can take the base-point at the origin to define the universal cover of $\mathbb{C}\setminus\Omega$, here with $\Omega = \{\omega',\omega''\}$. The word "sheets" usually refers to the various lifts in the cover of an open subset U of the base space which is star-shaped with respect to one of its points, *i.e.* to the various connected components of $\pi^{-1}(U)$.

[5] In the case of a universal cover $(\widetilde{\mathbb{C}\setminus\Omega,\zeta_0})$, the "principal sheet" $\tilde U$ is obtained by considering the maximal open subset U of $\mathbb{C}\setminus\Omega$ which is star-shaped with respect to ζ_0 and lifting it by means of rectilinear segments: $\tilde U$ is the set of all the classes of segments $[\zeta_0,\zeta]$, $\zeta \in U$.

Definition 3 *Let $\mathcal{R}$ be the set of all homotopy classes of paths issuing from the origin and lying inside $\mathbb{C} \setminus 2\pi i\mathbb{Z}$ (except for their initial point), and let $\pi : \mathcal{R} \to \mathbb{C} \setminus 2\pi i\mathbb{Z}^*$ be the covering map, which associates with any class c the extremity $\gamma(1)$ of any path $\gamma : [0,1] \to \mathbb{C}$ which represents c. We consider $\mathcal{R}$ as a Riemann surface by pulling back by π the complex structure of $\mathbb{C} \setminus 2\pi i\mathbb{Z}^*$.*

Observe that $\pi^{-1}(0)$ consists of only one point (the homotopy class of the constant path), which we may call the origin of $\mathcal{R}$. Let U be the complex plane deprived from the half-lines $+2\pi i\,[1,+\infty[$ and $-2\pi i\,[1,+\infty[$. We define the "principal sheet" of $\mathcal{R}$ as the set of all the classes of segments $[0,\zeta]$, $\zeta \in U$; equivalently, it is the connected component of $\pi^{-1}(U)$ which contains the origin. We define the "half-sheets" of $\mathcal{R}$ as the various connected components of $\pi^{-1}(\{\Re e\,\zeta > 0\})$ or of $\pi^{-1}(\{\Re e\,\zeta < 0\})$.

A holomorphic function of $\mathcal{R}$ can be viewed as a germ of holomorphic function at the origin of $\mathbb{C}$ which admits analytic continuation along any path avoiding $2\pi i\mathbb{Z}$; we then say that this germ "extends holomorphically to $\mathcal{R}$". This definition a priori does not authorize analytic continuation along a path which leads to the origin, unless this path stays in the principal sheet[6]. More precisely, one can prove

Lemma 2 *If Φ is holomorphic in $\mathcal{R}$, then its restriction to the principal sheet defines a holomorphic function φ of U which extends analytically along any path γ issuing from 0 and lying in $\mathbb{C} \setminus 2\pi i\mathbb{Z}$. The analytic continuation is given by $\varphi(\gamma(t)) = \Phi(\Gamma(t))$, where Γ is the lift of γ which starts at the origin of $\mathcal{R}$.*

Conversely, given $\varphi \in \mathbb{C}\{\zeta\}$, if any $c \in \mathcal{R}$ can be represented by a path of analytic continuation for φ, then the value of φ at the extremity $\gamma(1)$ of this path depends only on c and the formula $\Phi(c) = \varphi(\gamma(1))$ defines a holomorphic function of $\mathcal{R}$.

The absence of singularity at the origin on the principal sheet is the only difference between $\mathcal{R}$ and the universal cover of $\mathbb{C} \setminus 2\pi i\mathbb{Z}$ with base-point at 1. For instance, among the two series

$$\sum_{m\in\mathbb{Z}^*} \frac{1}{\zeta}\,e^{-|m|} \int_1^\zeta \frac{d\zeta_1}{\zeta_1 - 2\pi i m}, \quad \sum_{m\in\mathbb{Z}^*} \frac{1}{\zeta}\,e^{-|m|} \int_0^\zeta \frac{d\zeta_1}{\zeta_1 - 2\pi i m},$$

the first one defines a function which is holomorphic in the universal cover of $\mathbb{C} \setminus 2\pi i\mathbb{Z}$ but not in $\mathcal{R}$, whereas the second one defines a holomorphic function of $\mathcal{R}$.

Analytic continuation of convolution in $\mathcal{R}$

The main result of this section is

Theorem 1 *If two germs at the origin extend holomorphically to $\mathcal{R}$, so does their convolution product.*

Idea of the proof. Let $\hat\varphi$ and $\hat\psi$ be holomorphic germs at the origin of $\mathbb{C}$ which admit analytic continuation along any path avoiding $2\pi i\mathbb{Z}$; we denote by the same symbols the corresponding

[6] That is, unless it lies in $U = \mathbb{C} \setminus \pm 2\pi i\,[1,+\infty[$. We shall often identify the paths issuing from 0 in $\mathbb{C} \setminus 2\pi i\mathbb{Z}$ and their lifts starting at the origin of $\mathcal{R}$. Sometimes, we shall even identify a point of $\mathcal{R}$ with its projection by π (the path which leads to this point being understood), which amounts to treating a holomorphic function of $\mathcal{R}$ as a multivalued function on $\mathbb{C} \setminus 2\pi i\mathbb{Z}$.

holomorphic functions of $\mathcal{R}$. One could be tempted to think that, if a point ζ of $\mathcal{R}$ is defined by a path γ, the integral

$$\hat{\chi}(\zeta) = \int_\gamma \hat{\varphi}(\zeta')\hat{\psi}(\zeta - \zeta')\,\mathrm{d}\zeta' \tag{12}$$

would give the value of the analytic continuation of $\hat{\varphi} * \hat{\psi}$ at ζ. However, this formula does not always make sense, since one must worry about the path γ' followed by $\zeta - \zeta'$ when ζ' follows γ: is $\hat{\psi}$ defined on this path? In fact, even if γ' lies in $\mathbb{C} \setminus 2\pi\mathrm{i}\,\mathbb{Z}$ (and thus $\hat{\psi}(\zeta - \zeta')$ makes sense), even if γ' coincides with γ, it may happen that this integral does not give the analytic continuation of $\hat{\varphi} * \hat{\psi}$ at ζ (usually, the value of this integral does not depend only on ζ but also on the path γ).[7]

The construction of the desired analytic continuation relies on the idea of "symmetrically contractile" paths. A path γ issuing from 0 is said to be *$\mathcal{R}$-symmetric* if it lies in $\mathbb{C} \setminus 2\pi\mathrm{i}\,\mathbb{Z}$ (except for its starting point) and is symmetric with respect to its midpoint: the paths $t \in [0,1] \mapsto \gamma(1) - \gamma(t)$ and $t \in [0,1] \mapsto \gamma(1-t)$ coincide up to reparametrisation. A path is said to be *$\mathcal{R}$-symmetrically contractile* if it is $\mathcal{R}$-symmetric and can be continuously deformed and shrunk to $\{0\}$ within the class of $\mathcal{R}$-symmetric paths. The main point is that any point of $\mathcal{R}$ can be defined by an $\mathcal{R}$-symmetrically contractile path. More precisely:

Lemma 3 *Let γ be a path issuing from 0 and lying in $\mathbb{C} \setminus 2\pi\mathrm{i}\,\mathbb{Z}$ (except for its starting point). Then there exists an $\mathcal{R}$-symmetrically contractile path Γ which is homotopic to γ. Moreover, one can construct Γ so that there is a continuous map $(s,t) \mapsto H(s,t) = H_s(t)$ satisfying*

i) $H_0(t) \equiv 0$ and $H_1(t) \equiv \Gamma(t)$,

ii) each H_s is an $\mathcal{R}$-symmetric path with $H_s(0) = 0$ and $H_s(1) = \gamma(s)$.

We shall not try to write a formal proof of this lemma, but it is easy to visualize a way of constructing H. Let a point $\zeta = \gamma(s)$ move along γ (as s varies from 0 to 1) and remain connected to 0 by an extensible thread, with moving nails pointing downwards at each point of $\zeta - 2\pi\mathrm{i}\,\mathbb{Z}$, while fixed nails point upwards at each point of $2\pi\mathrm{i}\,\mathbb{Z}$ (imagine for instance that the first nails are fastened to a moving rule and the last ones to a fixed rule). As s varies, the thread is progressively stretched but it has to meander between the nails. The path Γ is given by the thread in its final form, when ζ has reached the extremity of γ; the paths H_s correspond to the thread at intermediary stages[8] (see Figure 5).

It is now easy to end the proof of Theorem 1. Given $\hat{\varphi}$, $\hat{\psi}$ as above and γ a path of $\mathcal{R}$ along which we wish to follow the analytic continuation of $\hat{\varphi} * \hat{\psi}$, we take H as in Lemma 3 and let the reader convince himself that the formula

$$\hat{\chi}(\zeta) = \int_{H_s} \hat{\varphi}(\zeta')\hat{\psi}(\zeta - \zeta')\,\mathrm{d}\zeta', \qquad \zeta = \gamma(s), \tag{13}$$

defines the analytic continuation $\hat{\chi}$ of $\hat{\varphi} * \hat{\psi}$ along γ (in this formula, ζ' and $\zeta - \zeta'$ move on the same path H_s which avoids $2\pi\mathrm{i}\,\mathbb{Z}$, by $\mathcal{R}$-symmetry). See [Eca81, Vol. 1], [CNP93], [GS01] for more details. □

[7] However, if $\hat{\psi}$ is entire, it is true that the integral (12) does provide the analytic continuation of $\hat{\varphi} * \hat{\psi}$ along γ.

[8] Note that the mere existence of a continuous H satisfying conditions *i)* and *ii)* implies that γ and Γ are homotopic, as is visually clear (the formula

$$h_\lambda(t) = H\Big(\lambda + (1-\lambda)t, \frac{t}{\lambda + (1-\lambda)t}\Big), \qquad 0 \le \lambda \le 1$$

yields an explicit homotopy).

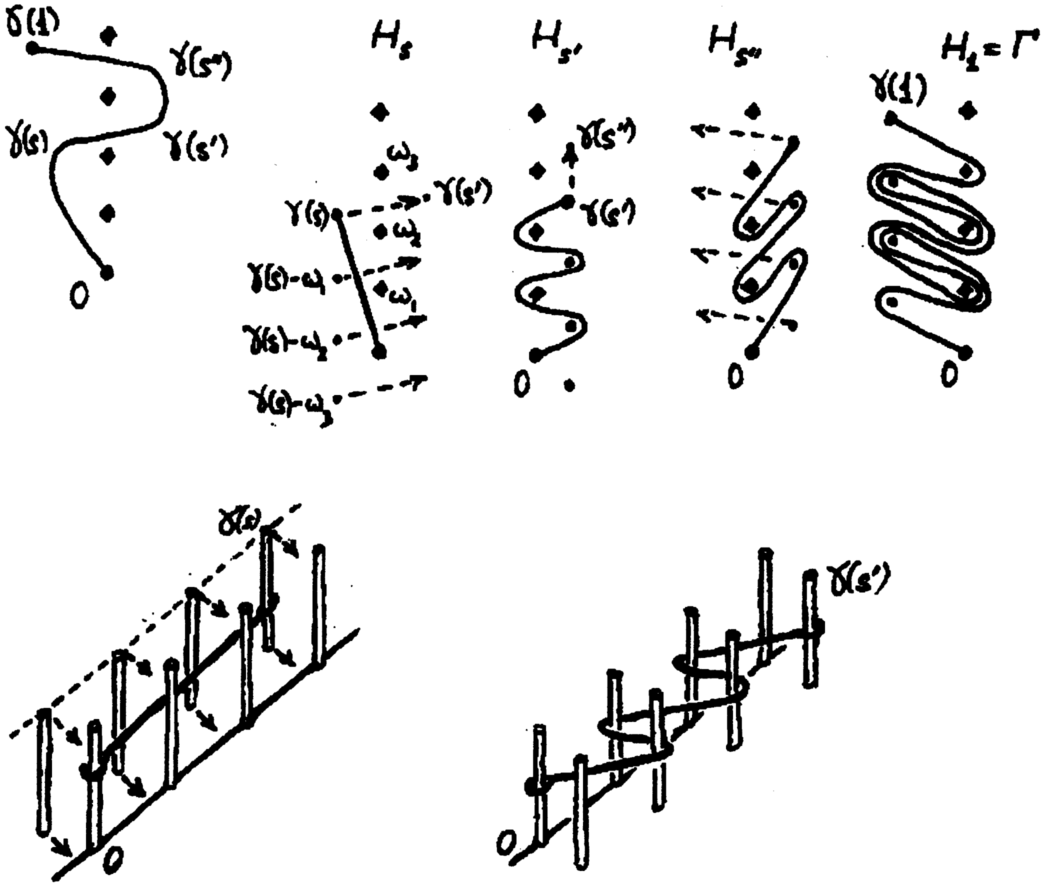


Figure 5: Construction of an $\mathcal{R}$-symmetrically contractile path Γ homotopic to γ.

Of course, if the path γ mentioned in the last part of the proof stays in the principal sheet of $\mathcal{R}$, the analytic continuation is simply given by formula (10). In particular, if $\hat{\varphi}$ and $\hat{\psi}$ have bounded exponential type in a direction $\arg\zeta = \theta$, $\theta \notin \frac{\pi}{2} + \pi\mathbb{Z}$, it follows from inequality (11) that $\hat{\varphi} * \hat{\psi}$ has the same property.

1.4 Formal and convolutive models of the algebra of resurgent functions, $\tilde{\mathcal{H}}$ and $\widehat{\mathcal{H}}(\mathcal{R})$

In view of Theorem 1, the convolution of germs induces an internal law on the space of holomorphic functions of $\mathcal{R}$, which is commutative and associative (being the counterpart of multiplication of formal series, by Lemma 1). In fact, we have a commutative algebra (without unit), which can be viewed as a subalgebra of the convolution algebra $\mathbb{C}\{\zeta\}$, and which corresponds via $\mathcal{B}$ to a subalgebra (for the ordinary product of formal series) of $z^{-1}\mathbb{C}[[z^{-1}]]$.

Definition 4 *The space $\widehat{\mathcal{H}}(\mathcal{R})$ of all holomorphic functions of $\mathcal{R}$, equipped with the convolution product, is an algebra called the convolutive model of the algebra of resurgent functions. The subalgebra $\tilde{\mathcal{H}} = \mathcal{B}^{-1}(\widehat{\mathcal{H}}(\mathcal{R}))$ of $z^{-1}\mathbb{C}[[z^{-1}]]$ is called the multiplicative model of the algebra of resurgent functions.*

The formal series in $\tilde{\mathcal{H}}$ (most of which have zero radius of convergence) are called "resurgent functions". These definitions will in fact be extended to more general objects in the following (see Section 3 on "singularities").

There is no unit for the convolution in $\widehat{\mathcal{H}}(\mathcal{R})$. Introducing a new symbol $\delta = \mathcal{B}1$, we extend the formal Borel transform:

$$\mathcal{B} \,:\, \tilde{\chi}(z) = c_0 + \sum_{n\geq 0} c_n z^{-n-1} \in \mathbb{C}[[z^{-1}]] \;\mapsto\; \hat{\chi}(\zeta) = c_0\,\delta + \sum_{n\geq 0} c_n \frac{\zeta^n}{n!} \in \mathbb{C}\,\delta \oplus \mathbb{C}[[\zeta]],$$

and also extend convolution from $\mathbb{C}[[\zeta]]$ to $\mathbb{C}\,\delta \oplus \mathbb{C}[[\zeta]]$ linearly, by treating δ as a unit (*i.e.* so as to keep $\mathcal{B}$ a morphism of algebras). This way, $\mathbb{C}\,\delta \oplus \widehat{\mathcal{H}}(\mathcal{R})$ is an algebra for the convolution, which is isomorphic via $\mathcal{B}$ to the algebra $\mathbb{C} \oplus \tilde{\mathcal{H}}$. Observe that

$$\mathbb{C}\{z^{-1}\} \subset \mathbb{C} \oplus \tilde{\mathcal{H}} \subset \mathbb{C}[[z^{-1}]]_1.$$

Having dealt with multiplication of formal series, we can study *composition* and its image in $\mathbb{C}\,\delta \oplus \widehat{\mathcal{H}}(\mathcal{R})$:

Proposition 1 *Let $\tilde{\chi} \in \mathbb{C} \oplus \tilde{\mathcal{H}}$. Then composition by $z \mapsto z + \tilde{\chi}(z)$ defines a linear operator of $\mathbb{C} \oplus \tilde{\mathcal{H}}$ into itself, and for any $\tilde{\psi} \in \tilde{\mathcal{H}}$ the Borel transform of $\tilde{\alpha}(z) = \tilde{\psi}(z + \tilde{\chi}(z)) = \sum_{r\geq 0} \frac{1}{r!}\partial^r\tilde{\psi}(z)\tilde{\chi}^r(z)$ is given by the series of functions*

$$\hat{\alpha}(\zeta) = \sum_{r\geq 0} \frac{1}{r!}\left((-\zeta)^r\hat{\psi}(\zeta)\right) * \hat{\chi}^{*r}(\zeta) \tag{14}$$

(where $\hat{\chi} = \mathcal{B}\tilde{\chi}$ and $\hat{\psi} = \mathcal{B}\tilde{\psi}$), which is uniformly convergent in every compact subset of $\mathcal{R}$.

The convergence of the series stems from the regularizing character of convolution (the convergence in the principal sheet of $\mathcal{R}$ can be proved by use of (11); see [Eca81, Vol. 1] or [CNP93] for the convergence in the whole Riemann surface).

The notation $\hat{\alpha} = \hat{\psi} \circledast (\delta' + \hat{\chi})$ and the name "composition-convolution" are used in [Mal95], with a symbol $\delta' = \mathcal{B}z$ which must be considered as the derivative of δ. The symbols δ and δ' will be interpreted as elementary singularities in Section 3.

In Proposition 1, the operator of composition by $z \mapsto z + \tilde{\chi}(z)$ is invertible; in fact, $\mathrm{Id} + \tilde{\chi}$ has a well-defined inverse for composition in $\mathrm{Id} + \mathbb{C}[[z^{-1}]]$, which turns out to be also resurgent:

Proposition 2 *If $\tilde{\chi} \in \mathbb{C} \oplus \tilde{\mathcal{H}}$, the formal transformation $\mathrm{Id} + \tilde{\chi}$ has an inverse (for composition) of the form $\mathrm{Id} + \tilde{\varphi}$ with $\tilde{\varphi} \in \tilde{\mathcal{H}}$.*

This can be proven by the same arguments as Proposition 1, since the Lagrange inversion formula allows one to write

$$\tilde{\varphi} = \sum_{k \geq 1} \frac{(-1)^k}{k!} \partial^{k-1}(\tilde{\chi}^k), \quad \text{hence } \hat{\varphi} = -\sum_{k \geq 1} \frac{\zeta^{k-1}}{k!} \hat{\chi}^{*k}. \tag{15}$$

One can thus think of $z \mapsto z + \tilde{\chi}(z)$ as of a "resurgent change of variable".

Similarly, *substitution* of a resurgent function without constant term into a convergent series is possible:

Proposition 3 *If $C(w) = \sum_{n \geq 0} C_n w^n \in \mathbb{C}\{w\}$ and $\tilde{\psi} \in \tilde{\mathcal{H}}$, then the formal series $C \circ \tilde{\psi}(z) = \sum_{n \geq 0} C_n \tilde{\psi}^n(z)$ belongs to $\mathbb{C} \oplus \tilde{\mathcal{H}}$.*

The proof consists in verifying the convergence of the series $\mathcal{B}(C \circ \tilde{\psi}) = \sum_{n \geq 0} C_n \hat{\psi}^{*n}$.

As a consequence, any resurgent function with nonzero constant term has a resurgent multiplicative inverse: $1/(c + \tilde{\psi}) = \sum_{n \geq 0} (-1)^n c^{-n-1} \tilde{\psi}^n \in \mathbb{C} \oplus \tilde{\mathcal{H}}$. The exponential of a resurgent function $\tilde{\psi}$ is also a resurgent function, the Borel transform of which is the *convolutive exponential*

$$\exp_*(\hat{\psi}) = \delta + \hat{\psi} + \frac{1}{2!} \hat{\psi} * \hat{\psi} + \frac{1}{3!} \hat{\psi} * \hat{\psi} * \hat{\psi} + \ldots$$

(in this case the substitution is well-defined even if $\tilde{\psi}(z)$ has a constant term).

We end this section by remarking that the role of the lattice $2\pi i \mathbb{Z}$ in the definition of $\mathcal{R}$ is not essential in the theory of resurgent functions. See Section 3.3 for a more general definition of the space of resurgent functions (in which the location of singular points is not a priori restricted to $2\pi i \mathbb{Z}$), with a property of stability by convolution as in Theorem 1, and with alien derivations more general than the ones to be defined in Section 2.3.

2 Alien calculus and Abel's equation

We now turn to the resurgent treatment of the nonlinear first-order difference equation (8), beginning with a few words of motivation.

2.1 Abel's equation and tangent-to-identity holomorphic germs of $(\mathbb{C}, 0)$

One of the origins of Écalle's work on Resurgence theory is the problem of the classification of holomorphic germs F of $(\mathbb{C}, 0)$ in the "resonant" case. This is the question, important for one-dimensional complex dynamics, of describing the conjugacy classes of the group $\mathbb{G}$ of local analytic transformations $w \mapsto F(w)$ which are locally invertible, *i.e.* of the form $F(w) = \lambda w + \mathcal{O}(w^2) \in \mathbb{C}\{w\}$ with $\lambda \in \mathbb{C}^*$. It is well-known that, if the multiplier $\lambda = F'(0)$ has modulus $\neq 1$, then F is holomorphically linearizable: there exists $H \in \mathbb{G}$ such that $H^{-1} \circ F \circ H(w) = \lambda w$. Resurgence comes into play when we consider the resonant case, *i.e.* when $F'(0)$ is a root of unity (the so-called "small divisor problems", which appear when $F'(0)$ has modulus 1 but is not a root of unity, are of different nature—see S. Marmi's lecture in this volume).

The references for this part of the text are: [Eca81, Vol. 2], [Eca84], [Mal85] (and Example 1 of [Eca05] p. 235). For non-resurgent approaches of the same problem, see [MR83], [DH84], [Shi98], [Shi00], [Mil99], [Lor06].

Non-degenerate parabolic germs

Here, for simplicity, we limit ourselves to $F'(0) = 1$, *i.e.* to germs F which are tangent to identity, with the further requirement that $F''(0) \neq 0$, a condition which is easily seen to be invariant by conjugacy. Rescaling the variable w if necessary, one can suppose $F''(0) = 2$. It will be more convenient to work "near infinity", *i.e.* to use the variable $z = -1/w$.

Definition 5 *We call "non-degenerate parabolic germ at the origin" any $F(w) \in \mathbb{C}\{w\}$ of the form*

$$F(w) = w + w^2 + \mathcal{O}(w^3).$$

We call "non-degenerate parabolic germ at infinity" a transformation $z \mapsto f(z)$ which is conjugated by $z = -1/w$ to a non-degenerate parabolic germ F at the origin:

$$f(z) = -1/\big(F(-1/z)\big),$$

i.e. any $f(z) = z + 1 + a(z)$ with $a(z) \in z^{-1}\mathbb{C}\{z^{-1}\}$.

Let $\mathbb{G}_1$ denote the subgroup of tangent-to-identity germs. One can easily check that, if $F, G \in \mathbb{G}_1$ and $H \in \mathbb{G}$, then $G = H^{-1} \circ F \circ H$ implies $G''(0) = H'(0)F''(0)$. In order to work with non-degenerate parabolic germs only, we can thus restrict ourselves to tangent-to-identity conjugating transformations H, *i.e.* we can content ourselves with studying the adjoint action of $\mathbb{G}_1$.

It turns out that formal transformations also play a role. Let $\tilde{\mathbb{G}}_1$ denote the group (for composition) of formal series of the form $\tilde{H}(w) = w + \mathcal{O}(w^2) \in \mathbb{C}[[w]]$. It may happen that two parabolic germs F and G are conjugated by such a formal series $\tilde{H}$, *i.e.* $G = H^{-1} \circ F \circ H$ in $\tilde{\mathbb{G}}_1$, without being conjugated by any convergent series: the $\mathbb{G}_1$-conjugacy classes we are interested in form a finer partition than the "formal conjugacy classes".

In fact, the formal conjugacy classes are easy to describe. One can check that, for any two non-degenerate parabolic germs $F(w), G(w) = w + w^2 + \mathcal{O}(w^3)$, there exists $\tilde{H} \in \tilde{\mathbb{G}}_1$ such that $G = H^{-1} \circ F \circ H$ if and only if the coefficient of w^3 is the same in $F(w)$ and $G(w)$. In the following, this coefficient will usually be denoted α.

Let us rephrase the problem at infinity, using the variable $z = -1/w$, and thus dealing with transformations belonging to $\mathrm{Id} + \mathbb{C}[[z^{-1}]]$. The formula $h(z) = -1/H(-1/z)$ puts in correspondence the conjugating transformations H of $\mathbb{G}_1$ or $\tilde{\mathbb{G}}_1$ and the series of the form

$$h(z) = z + b(z), \qquad b(z) \in \mathbb{C}\{z^{-1}\} \text{ or } b(z) \in \mathbb{C}[[z^{-1}]]. \tag{16}$$

Given a non-degenerate parabolic germ at infinity $f(z) = -1/\big(F(-1/z)\big)$, the coefficient α of w^3 in $F(w)$ shows up in the coefficient of z^{-1} in $f(z)$:

$$f(z) = z + 1 + a(z), \qquad a(z) = (1-\alpha)z^{-1} + \mathcal{O}(z^{-2}) \in \mathbb{C}\{z^{-1}\}. \tag{17}$$

The coefficient $\rho = \alpha - 1$ is called "résidu itératif" in Écalle's work, or "resiter" for short. Thus *any two germs of the form* (17) *are conjugated by a formal transformation of the form* (16) *if and only if they have the same resiter.*

The related difference equations

The simplest formal conjugacy class is the one corresponding to $\rho = 0$. Any non-degenerate parabolic germ f or F with vanishing resiter is conjugated by a formal h or H to $z \mapsto f_0(z) = z + 1$ or $w \mapsto F_0(w) = \frac{w}{1-w}$. We can be slightly more specific:

Proposition 4 *Let $f(z) = z + 1 + a(z)$ be a non-degenerate parabolic germ at infinity with vanishing resiter, i.e. $a(z) \in z^{-2}\mathbb{C}\{z^{-1}\}$. Then there exists a unique $\tilde{\varphi}(z) \in z^{-1}\mathbb{C}[[z^{-1}]]$ such that the formal transformation $\tilde{u} = \mathrm{Id} + \tilde{\varphi}$ is solution of*

$$u^{-1} \circ f \circ u(z) = z + 1. \tag{18}$$

The inverse formal transformation $\tilde{v} = \tilde{u}^{-1}$ is the unique transformation of the form $\tilde{v} = z + \tilde{\psi}(z)$, with $\tilde{\psi}(z) \in z^{-1}\mathbb{C}[[z^{-1}]]$, solution of

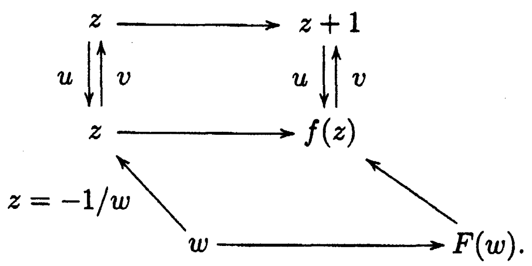
$$v\big(f(z)\big) = v(z) + 1. \tag{19}$$

All the other formal solutions of equations (18) *and* (19) *can be deduced from $\tilde{u}$ and $\tilde{v}$: they are the series*

$$u(z) = z + c + \tilde{\varphi}(z + c), \quad v(z) = z - c + \tilde{\psi}(z), \tag{20}$$

where c is an arbitrary complex number.

We omit the proof of this proposition, which can be done by substitution of an indetermined series $u \in \mathrm{Id} + \mathbb{C}[[z^{-1}]]$ in (18). Setting $v = u^{-1}$, the u-equation then translates into equation (19), as illustrated on the commutative diagram



Notice that, under the change of unknown $u(z) = z + \varphi(z)$, the conjugacy equation (18) is equivalent to the equation

$$\varphi(z) + a(z + \varphi(z)) = \varphi(z+1),$$

i.e. to the difference equation (8) with $a(z) = f(z) - z - 1$. Equation (19) is called *Abel's equation*[9].

The formal solutions $\tilde{u}$ and $\tilde{v}$ mentioned in Proposition 4 are generically divergent. It turns out that they are always resurgent. Before trying to explain this, let us mention that the case of a general resiter ρ can be handled by studying the same equations (18) and (19): if $\rho \neq 0$, there is no solution in $\mathrm{Id} + \mathbb{C}[[z^{-1}]]$, but one finds a unique formal solution of Abel's equation of the form

$$\tilde{v}(z) = z + \tilde{\psi}(z), \qquad \tilde{\psi}(z) = \rho \log z + \sum_{n \geq 1} c_n z^{-n},$$

the inverse of which is of the form

$$\tilde{u}(z) = z + \tilde{\varphi}(z), \qquad \tilde{\varphi}(z) = -\rho \log z + \sum_{\substack{n,m \geq 0 \\ n+m \geq 1}} C_{n,m} z^{-n} (z^{-1} \log z)^m,$$

and these series $\tilde{\psi}$ and $\tilde{\varphi}$ can be treated by Resurgence almost as easily as the corresponding series in the case $\rho = 0$. In Écalle's work, the formal solution $\tilde{v}$ of Abel's equation is called the *iterator* (itérateur, in French) of f and its inverse $\tilde{u}$ is called the *inverse iterator* because of their role in iteration theory (which we shall not develop in this text—see however footnote 16).

Resurgence in the case $\rho = 0$

From now on we focus on the case $\rho = 0$, thus with "formal normal forms" $f_0(z) = z + 1$ at infinity or $F_0(w) = \frac{w}{1-w}$ at the origin. We do not intend to give the complete resurgent solution of the classification problem, but only to convey some of the ideas used in Écalle's approach.

Theorem 2 *In the case $\rho = 0$ (vanishing resiter), the formal series $\tilde{\varphi}(z), \tilde{\psi}(z) \in z^{-1}\mathbb{C}[[z^{-1}]]$ in terms of which the solutions of equations* (18) *and* (19) *can be expressed as in* (20) *have formal Borel transforms $\hat{\varphi}(\zeta)$ and $\hat{\psi}(\zeta)$ which converge near the origin and extend holomorphically to $\mathcal{R}$, with at most exponential growth in the directions $\arg\zeta = \theta$, $\theta \notin \frac{\pi}{2} + \pi\mathbb{Z}$ (for every $\theta_0 \in \left]0, \frac{\pi}{2}\right[$, there exists $\tau > 0$ such that $\hat{\varphi}$ and $\hat{\psi}$ have exponential type $\leq \tau$ in the sectors $\{-\theta_0 + n\pi \leq \arg\zeta \leq \theta_0 + n\pi\}$, $n = 0$ or 1).*

In other words, Abel's equation gives rise to resurgent functions and it is possible to apply the Borel-Laplace summation process to $\tilde{\varphi}$ and $\tilde{\psi}$.

Idea of the proof. Equation (19) for $v(z) = z + \psi(z)$ translates into

$$\psi\big(z + 1 + a(z)\big) - \psi(z) = -a(z). \tag{21}$$

The proof indicated in [Eca81, Vol. 2] or [Mal85] relies on the expression of the unique solution in $z^{-1}\mathbb{C}[[z^{-1}]]$ as an infinite sum of iterated operators applied to $a(z)$; the formal Borel transform

[9] In fact, this name usually refers to the equation $V \circ F = V + 1$, for $V(w) = v(-1/w)$, which expresses the conjugacy by $w \mapsto V(w) = -1/w + \mathcal{O}(w)$ between the given germ F at the origin and the normal form f_0 at infinity.

then yields a sum of holomorphic functions which is uniformly convergent on every compact subset of $\mathcal{R}$. One can prove in this way that $\hat{\psi} \in \widehat{\mathcal{H}}(\mathcal{R})$ with at most exponential growth at infinity, and then deduce from Proposition 2 and formula (15) that $\hat{\varphi}$ has the same property.

Let us outline an alternative proof, which makes use of equation (18) to prove that $\hat{\varphi} \in \widehat{\mathcal{H}}(\mathcal{R})$. As already mentioned, the change of unkwnown $u = \mathrm{Id} + \varphi$ leads to equation (8) with $a(z) \in z^{-2}\mathbb{C}\{z^{-1}\}$, which we now treat as a perturbation of equation (5): we write it as

$$\varphi(z+1) - \varphi(z) = a_0(z) + \sum_{r \geq 1} a_r(z)\varphi^r(z),$$

with $a_r = \frac{1}{r!}\partial^r a$. The unique formal solution without constant term, $\tilde{\varphi}$, has a formal Borel transform $\hat{\varphi}$ which thus satisfies

$$\hat{\varphi} = E\,\hat{a}_0 + E\sum_{r \geq 1} \hat{a}_r * \hat{\varphi}^{*r}, \tag{22}$$

where $E(\zeta) = \dfrac{1}{e^{-\zeta} - 1}$ and $\hat{a}_r(\zeta) = \frac{1}{r!}(-\zeta)^r \hat{a}(\zeta)$, $\hat{a} = \mathcal{B}\,a$.

The convergence of $\hat{\varphi}$ and its analytic extension to the principal sheet of $\mathcal{R}$ are easily obtained: we have $\hat{\varphi} = \sum_{n \geq 1} \hat{\varphi}_n$ with

$$\hat{\varphi}_1 = E\,\hat{a}_0, \qquad \hat{\varphi}_n = E \sum_{\substack{r \geq 1 \\ n_1 + \cdots + n_r = n-1}} \hat{a}_r * \hat{\varphi}_{n_1} * \cdots * \hat{\varphi}_{n_r}, \qquad n \geq 2 \tag{23}$$

(more generally $\tilde{u}(z) = z + \sum_{n \geq 1} \varepsilon^n \tilde{\varphi}_n$ is the solution corresponding to $f(z) = z + 1 + \varepsilon a(z)$). Observe that this series is well-defined and formally convergent, because $\hat{a} \in \zeta\mathbb{C}[[\zeta]]$ and $E \in \zeta^{-1}\mathbb{C}[[\zeta]]$ imply $\hat{\varphi}_n \in \zeta^{2(n-1)}\mathbb{C}[[\zeta]]$, and that each $\hat{\varphi}_n$ is convergent and extends holomorphically to $\mathcal{R}$ (by virtue of Theorem 1, because $\hat{a}$ converges to an entire function and E is meromorphic with poles in $2\pi\mathrm{i}\,\mathbb{Z}$); we shall check that the series of functions $\sum \hat{\varphi}_n$ is uniformly convergent in every compact subset of the principal sheet. Since $a(z) \in z^{-2}\mathbb{C}\{z^{-1}\}$, we can find positive constants C and τ such that

$$|\hat{a}(\zeta)| \leq C \min(1, |\zeta|)\, e^{\tau|\zeta|}, \qquad \zeta \in \mathbb{C}.$$

Identifying the principal sheet of $\mathcal{R}$ with the cut plane $\mathbb{C} \setminus \pm 2\pi\mathrm{i}\,[1, +\infty[$, we can write it as the union over $c > 0$ of the sets $\mathcal{R}_c^{(0)} = \{\, \zeta \in \mathbb{C} \mid \mathrm{dist}([0,\zeta], \pm 2\pi\mathrm{i}) \geq c \,\}$ (with $c < 2\pi$). For each $c > 0$, we can find $\lambda = \lambda(c) > 0$ such that

$$|E(\zeta)| \leq \lambda(1 + |\zeta|^{-1}), \qquad \zeta \in \mathcal{R}_c^{(0)}.$$

We deduce that $|\hat{\varphi}_1(\zeta)| \leq 2\lambda C\, e^{\tau|\zeta|}$ in $\mathcal{R}_c^{(0)}$, and the fact that $\mathcal{R}_c^{(0)}$ is star-shaped with respect to the origin allows us to construct majorants by inductive use of (11):

$$|\hat{\varphi}_n(\zeta)| \leq \hat{\Phi}_n(|\zeta|)\, e^{\tau|\zeta|}, \qquad \zeta \in \mathcal{R}_c^{(0)},$$

with

$$\hat{\Phi}_1(\xi) = 2\lambda C, \quad \hat{\Phi}_n = 2\lambda C \sum_{\substack{r \geq 1 \\ n_1 + \cdots + n_r = n-1}} \frac{\xi^r}{r!} * \hat{\Phi}_{n_1} * \cdots * \hat{\Phi}_{n_r}$$

(we also used the fact that $|\hat\alpha(\zeta)| \le A(|\zeta|)\,\mathrm{e}^{a|\zeta|}$ and $|\hat\beta(\zeta)| \le B(|\zeta|)\,\mathrm{e}^{b|\zeta|}$ imply $|\hat\alpha * \hat\beta(\zeta)| \le (A*B)(|\zeta|)\,\mathrm{e}^{\max(a,b)|\zeta|}$, and that $\frac{1}{\xi}((\xi A)*B) \le A*B$ for $\xi \ge 0$). The generating series $\hat\Phi = \sum \varepsilon^n \hat\Phi_n$ is the formal Borel transform of the solution $\tilde\Phi = \sum \varepsilon^n \tilde\Phi_n$ of the equation $\tilde\Phi = 2\varepsilon\lambda C z^{-1} + 2\varepsilon\lambda C \sum z^{-r-1}\tilde\Phi^r$. We get $\tilde\Phi = \frac{1-(1-8\varepsilon\lambda C z^{-2})^{1/2}}{2z^{-1}}$ by solving this algebraic equation of degree 2, hence $\tilde\Phi_n(z) = \gamma_n z^{-2n+1}$ with $0 < \gamma_n \le \Gamma^n$ (with an explicit $\Gamma > 0$ depending on λC), and finally $\hat\Phi_n(|\zeta|) \le \Gamma^n \frac{|\zeta|^{2(n-1)}}{(2(n-1))!}$. Therefore the series $\sum \hat\varphi_n$ converges in $\mathcal{R}_c^{(0)}$ and $\hat\varphi$ extends to the principal sheet of $\mathcal{R}$ with at most exponential growth.

The analytic continuation to the rest of $\mathcal{R}$ is more difficult. A natural idea would be to try to extend the previous method of majorants to every half-sheet of $\mathcal{R}$, but the problem is to find a suitable generalisation of inequality (11). As shown in [GS01] or [OSS03], this can be done in the union $\mathcal{R}^{(1)}$ of the half-sheets which are contiguous to the principal sheet, *i.e.* the ones which are reached after crossing the imaginary axis exactly once; indeed, the symmetrically contractile paths Γ constructed in Lemma 3 can be described quite simply for the points ζ belonging to these half-sheets and it is possible to define an analogue $\mathcal{R}_c^{(1)}$ of $\mathcal{R}_c^{(0)}$. But it is not so for the general half-sheets of $\mathcal{R}$, because of the complexity of the symmetrically contractile paths which are needed. The remedy employed in [GS01] and [OSS03] consists in performing the resurgent analysis, *i.e.* describing the action of the alien derivations Δ_ω to be defined in Section 2.3, gradually: the possibility of following the analytic continuation of $\hat\varphi$ in $\mathcal{R}^{(1)}$ is sufficient to define $\Delta_{2\pi\mathrm{i}}\tilde\varphi$ and $\Delta_{-2\pi\mathrm{i}}\tilde\varphi$, which amounts to computing the difference between the principal branch of $\hat\varphi$ at a given point ζ and the branch $\hat\varphi^{\pm}(\zeta)$ of $\hat\varphi$ obtained by turning once around $\pm 2\pi\mathrm{i}$ and coming back at ζ; one then discovers that this difference $\hat\varphi^{\pm}(\zeta) - \hat\varphi(\zeta)$ is proportional to $\hat\varphi(\zeta \mp 2\pi\mathrm{i})$ (we shall try to make clear the reason of this phenomenon in Section 2.4); $\hat\varphi^{\pm}$ is thus a function continuable along paths which cross the imaginary axis once (as the sum of such functions), which means that $\hat\varphi$ is continuable to a set $\mathcal{R}^{(2)}$ defined by paths which are authorized to cross two times (provided the first time is between $2\pi\mathrm{i}$ and $4\pi\mathrm{i}$ or between $-2\pi\mathrm{i}$ and $-4\pi\mathrm{i}$). The Riemann surface $\mathcal{R}$ can then be explored progressively, using more and more alien derivations, $\Delta_{\pm 4\pi\mathrm{i}}$ and $\Delta_{\pm 2\pi\mathrm{i}} \circ \Delta_{\pm 2\pi\mathrm{i}}$ to reach a set $\mathcal{R}^{(3)}$, etc. □

2.2 Sectorial normalisations (Fatou coordinates) and nonlinear Stokes phenomenon (horn maps)

We now apply the Borel-Laplace summation process and immediately get

Corollary 1 *With the hypothesis and notations of Theorem 2, for every $\theta_0 \in \left]0, \frac{\pi}{2}\right[$, there exists $\tau > 0$ such that the Borel-Laplace sums*

$$\varphi^+ = \mathcal{L}^\theta\hat\varphi, \quad \psi^+ = \mathcal{L}^\theta\hat\psi, \qquad -\theta_0 \le \theta \le \theta_0,$$
$$\varphi^- = \mathcal{L}^\theta\hat\varphi, \quad \psi^- = \mathcal{L}^\theta\hat\psi, \qquad \pi-\theta_0 \le \theta \le \pi+\theta_0$$

are analytic in $\mathcal{D}^+$, resp. $\mathcal{D}^-$, where

$$\mathcal{D}^+ = \bigcup_{-\theta_0 \le \theta \le \theta_0} \{\Re e(z\,\mathrm{e}^{\mathrm{i}\theta}) > \tau\}, \quad \mathcal{D}^- = \bigcup_{\pi-\theta_0 \le \theta \le \pi+\theta_0} \{\Re e(z\,\mathrm{e}^{\mathrm{i}\theta}) > \tau\},$$

and define transformations $u^\pm = \mathrm{Id} + \varphi^\pm$ and $v^\pm = \mathrm{Id} + \psi^\pm$ which satisfy

$$v^+ \circ f = v^+ + 1 \quad \text{and} \quad u^+ \circ v^+ = v^+ \circ u^+ = \mathrm{Id} \quad \text{on } \mathcal{D}^+,$$
$$v^- \circ f = v^- + 1 \quad \text{and} \quad u^- \circ v^- = v^- \circ u^- = \mathrm{Id} \quad \text{on } \mathcal{D}^-.$$

One can consider $z^+ = v^+(z) = z + \psi^+(z)$ and $z^- = v^-(z) = z + \psi^-(z)$ as normalising coordinates for F; they are sometimes called "Fatou coordinates".[10] When expressed in these coordinates, the germ F simply reads $z^\pm \mapsto z^\pm + 1$ (see Figure 7), the complexity of the dynamics being hidden in the fact that neither v^+ nor v^- is defined in a whole neighbourhood of infinity and that these transformations do not coincide on the two connected components of $\mathcal{D}^+ \cap \mathcal{D}^-$ (except of course if F and F_0 are analytically conjugated)—see the note at the end of this section for a more "dynamical" and quicker construction of $v^\pm$.

This complexity can be analysed through the change of chart $v^+ \circ u^- = \mathrm{Id} + \chi$, which is a priori defined on $\mathcal{E} = \mathcal{D}^- \cap (u^-)^{-1}(\mathcal{D}^+)$; this set has an "upper" and a "lower" connected components, $\mathcal{E}^{\mathrm{up}}$ and $\mathcal{E}^{\mathrm{low}}$ (because $(u^-)^{-1}(\mathcal{D}^+)$ is a sectorial neighbourhood of infinity of the same kind as $\mathcal{D}^+$), and we thus get two analytic functions χ^{up} and χ^{low} (this situation is reminiscent of the one described in Section 1.2). The conjugacy equations satisfied by u^- and v^+ yield $\chi(z+1) = \chi(z)$, hence both χ^{up} and χ^{low} are 1-periodic; moreover, we know that these functions tend to 0 as $\Im m\, z \to \pm\infty$ (faster than any power of z^{-1}, by composition of asymptotic expansions). We thus get two Fourier series

$$\chi^{\mathrm{low}}(z) = v^+ \circ u^-(z) - z = \sum_{m \geq 1} B_m \,\mathrm{e}^{-2\pi i m z}, \qquad \Im m\, z < -\tau_0, \tag{24}$$

$$\chi^{\mathrm{up}}(z) = v^+ \circ u^-(z) - z = \sum_{m \leq -1} B_m \,\mathrm{e}^{-2\pi i m z}, \qquad \Im m\, z > \tau_0, \tag{25}$$

which are convergent for $\tau_0 > 0$ large enough. It turns out that the classification problem can be solved this way: *two non-degenerate parabolic germs with vanishing resiter are analytically conjugate if and only if they define the same pair of Fourier series $(\chi^{\mathrm{up}}, \chi^{\mathrm{low}})$ up to a change of variable $z \mapsto z + c$; morevover, any pair of Fourier series of the type* (24)–(25) *can be obtained this way.*[11] The numbers B_m are said to be "analytic invariants" for the germ f or F. The functions $\mathrm{Id} + \chi^{\mathrm{low}}$ and $\mathrm{Id} + \chi^{\mathrm{up}}$ themselves are called "horn maps".[12]

[10] Observe that, when the parabolic germ at the origin $F(w) \in w\mathbb{C}\{w\}$ extends to an entire function, the function $U^-(z) = -1/u^-(z)$ also extends to an entire function (because the domain of analyticity $\mathcal{D}^-$ contains the half-plane $\Re e\, z < -\tau$ and the relation $U^-(z+1) = F(U^-(z))$ allows one to define the analytic continuation of U^- by $U^-(z) = F^n(U^-(z-n))$, with $n \geq 1$ large enough for a given z), which admits $-1/\tilde{u}(z) = -z^{-1}(1 + z^{-1}\tilde{\varphi}(z))^{-1} \in z^{-1}\mathbb{C}[[z^{-1}]]$ as asymptotic expansion in $\mathcal{D}^-$. In this case, the formal series $\tilde{\varphi}(z)$ must be divergent (if not, $-1/\tilde{u}(z)$ would be convergent, U^- would be its sum and this entire function would have to be constant), as well as $\tilde{\psi}(z)$, and the Fatou coordinates v^+ and v^- cannot be the analytic continuation one of the other. We have a similar situation when $F^{-1}(w) \in w\mathbb{C}\{w\}$ extends to an entire function, with $U^+(z) = -1/u^+(z)$ entire.

[11] For the first statement, consider f_1 and f_2 satisfying $\chi_2^{\mathrm{up}}(z) = \chi_1^{\mathrm{up}}(z+c)$ and $\chi_2^{\mathrm{low}}(z) = \chi_1^{\mathrm{low}}(z+c)$ with $c \in \mathbb{C}$, thus $v_2^+ \circ u_2^- = \tau^{-1} \circ v_1^+ \circ u_1^- \circ \tau$ in $\mathcal{E}^{\mathrm{up}}$ and $\mathcal{E}^{\mathrm{low}}$, with $\tau(z) = z + c$. Using $(\tilde{u}_1 \circ \tau, \tau^{-1} \circ \tilde{v}_1)$ instead of $(\tilde{u}_1, \tilde{v}_1)$, we see that a formal conjugacy between f_1 and f_2 is given by $\tilde{u}_2 \circ \tau^{-1} \circ \tilde{v}_1$; its Borel-Laplace sums $u_2^+ \circ \tau^{-1} \circ v_1^+$ and $u_2^- \circ \tau^{-1} \circ v_1^-$ can be glued together and give rise to an analytic conjugacy, since $u_2^- = u_2^+ \circ \tau^{-1} \circ v_1^+ \circ u_1^- \circ \tau$. Conversely, if there exists $h \in \mathrm{Id} + \mathbb{C}\{z^{-1}\}$ such that $f_2 \circ h = h \circ f_1$, we see that $h \circ \tilde{u}_1$ establishes a formal conjugacy between f_2 and $z \mapsto z+1$, Proposition 4 thus implies the existence of $c \in \mathbb{C}$ such that $\tilde{u}_2 = h \circ \tilde{u}_1 \circ \tau$ and $\tilde{v}_2 = \tau^{-1} \circ \tilde{v}_1 \circ h^{-1}$, with $\tau(z) = z + c$, whence $u_2^\pm = h \circ u_1^\pm \circ \tau$ and $v_2^\pm = \tau^{-1} \circ v_1^\pm \circ h^{-1}$, and $v_2^+ \circ u_2^- = \tau^{-1} \circ v_1^+ \circ u_1^- \circ \tau$, as desired. The proof of the second statement is beyond the scope of the present text.

[12] In fact, this name (which is of A. Douady's coinage) usually refers to the maps $\mathrm{Id} + \chi^{\mathrm{low}}$ expressed in the coordinate $w_- = e^{-2\pi i z}$, *i.e.* $w_- \mapsto w_- \exp\left(-2\pi i \sum_{m\geq 1} B_m w_-^m\right)$, and $\mathrm{Id} + \chi^{\mathrm{up}}$ expressed in the coordinate $w_+ = e^{2\pi i z}$, *i.e.* $w_+ \mapsto w_+ \exp\left(2\pi i \sum_{m \geq 1} B_{-m} w_+^m\right)$, which are holormophic for $|w_\pm| < \mathrm{e}^{-2\pi\tau_0}$ and can be viewed as return maps; the variables $w_\pm$ are natural coordinates at both ends of "Écalle's cylinder". See [MR83], [DH84], [Mil99], [Shi98], [Shi00], [Lor06].

Now, consider the relation (24) for instance. In the half-plane $\Pi^{\rm low} = \{\Im m\, z < -\tau_0\}$, $|\chi^{\rm low}(z)|$ is uniformly bounded by a constant which can be made arbitrarily small by choosing τ_0 large enough. We can use this to extend analytically φ^- beyond $\mathcal{D}^-$, in $\Pi^{\rm low}$: we can indeed write $u^-(z) = u^+\big(z+\chi^{\rm low}(z)\big)$ if $z \in \mathcal{E}^{\rm low}$, $\Re e\, z > 0$ and $\Im m\, z < -\tau_0$, and the right-hand side in this identity is holomorphic on $\Pi^{\rm low} \cap \{\Re e\, z > 0\}$ (because the image of this domain by $\mathrm{Id} + \chi^{\rm low}$ is included in $\mathcal{D}^+$). However, this implies $\varphi^-(z) = \chi^{\rm low}(z) + \varphi^+\big(z+\chi^{\rm low}(z)\big) = \chi^{\rm low}(z) + \mathcal{O}(1/z)$ for z tending to infinity in $\Pi^{\rm low} \cap \{\Re e\, z > 0\}$, thus $\varphi^-(z)$ is no longer asymptotic to $\tilde\varphi$ there, an oscillating term shows up when z moves along any horizontal half-line $-\mathrm{i}\, s + \mathbb{R}^+$, $s > \tau_0$. Similarly, $\varphi^-(z) = \chi^{\rm up}(z) + \varphi^+\big(z + \chi^{\rm up}(z)\big)$ extends analytically to the half-plane $\{\Im m > \tau_0\}$ if τ_0 is large enough, with an oscillating asymptotic behaviour determined by $\chi^{\rm up}$. This can be considered as a nonlinear analogue of the classical Stokes phenomenon (well-known in the case of linear ODEs).

In the next sections, we shall outline the resurgent approach, which consists in extracting information from the singularities of $\hat\varphi$ or $\hat\psi$ in order to construct a set of analytic invariants $\{A_{2\pi i m},\ m \in \mathbb{Z}^*\}$, and its relation with $\{B_m,\ m \in \mathbb{Z}^*\}$. Before explaining this, let us mention an interpretation of the non-coincidence of v^+ and v^- as a "splitting problem", following [Gel98].

Splitting of the invariant foliation

The dynamical behaviour of F_0 is easily visualized: the invariant foliation by horizontal lines for f_0 gives rise to an invariant foliation by circles for F_0, as shown on Figure 6 (notice that, for a global understanding of the dynamics of F_0, one should let w vary on the Riemann sphere, including the point at infinity, which is the image of 1). For the given parabolic germ f or F, we can use the Fatou coordinates to define "stable" and "unstable" foliations: for each $s \in \mathbb{R}$, the line $L_s = \{t + \mathrm{i}s,\ t \in \mathbb{R}\}$ intersects $\mathcal{D}^+$ and $\mathcal{D}^-$ along half-lines $L_s^+ = \{t + \mathrm{i}s,\ t > -T_s\}$ and $L_s^- = \{t + \mathrm{i}s,\ t < T_s\}$ and we may set $Z_s^+(t) = u^+(t + \mathrm{i}s)$, $Z_s^-(t) = u^-(t + \mathrm{i}s)$ and $W_s^\pm(t) = -1/Z_s^\pm(t)$ for $t > -T_s$, resp. $t < T_s$. We have

$$f^n\big(Z_s^\pm(t)\big) = Z_s^\pm(t+n) = t + n + \mathrm{i}s + O\big((t+n)^{-1}\big), \qquad F^n\big(W_s^\pm(t)\big) = W_s^\pm(t+n) \xrightarrow[n\to\pm\infty]{} 0.$$

The invariant foliation $\{L_s\}_{s\in\mathbb{R}}$ of F_0 is so to say split, giving rise to two foliations $\{u^+(L_s^+)\}_{s\in\mathbb{R}}$ and $\{u^-(L_s^-)\}_{s\in\mathbb{R}}$ which in general do not coincide but can be compared for $|s| > \tau_0$ large enough (because T_s is then positive—see Figure 7; one can also use the analytic continuation of $u^\pm$ to $\Pi^{\rm low}$ and $\Pi^{\rm up}$ and consider $u^\pm(L_s)$ for $|s|$ large enough). It is proven in [Gel98] that, if $B_1 = \cdots = B_{n-1} = 0$ and $B_n \neq 0$, for $s < -\tau_0$ the curves $\{u^+(L_s^+)\}_{s\in\mathbb{R}}$ and $\{u^-(L_s^-)\}_{s\in\mathbb{R}}$ intersect along $2n$ orbits of f, and that, for each of these orbits, the intersection angle (which is the same for all the points of the orbit because f is conformal) is $2\pi n|B_n|\mathrm{e}^{-2\pi n|s|} + \mathcal{O}(\mathrm{e}^{-2\pi(n+1)|s|})$ (there is a symmetric result for $s > \tau_0$ in terms of $B_{-1}, B_{-2}, \ldots$).

Note on another construction of the Fatou coordinates

The result expressed in Corollary 1 can be obtained through formulas which are reminiscent of the Note at the end of Section 1.2, being analogous to the explicit formulas available for the linear case. Indeed, observe that, since $f = \mathrm{Id} + 1 + a$ with $\lim_{|z|\to\infty} |a(z)| = 0$, $f(\mathcal{D}^+) \subset \mathcal{D}^+$ and $f^{-1}(\mathcal{D}^-) \subset \mathcal{D}^-$ (provided these sets are defined using a large enough constant τ); one can thus iterate Abel's equation forward or backward in these domains: setting $v^\pm = \mathrm{Id} + \psi^\pm$, equation (21) yields

$$\psi^+\circ f^{k+1} - \psi^+\circ f^k = -a\circ f^k \text{ in } \mathcal{D}^+, \qquad \psi^-\circ f^{-(k+1)} - \psi^-\circ f^{-k} = a\circ f^{-(k+1)} \text{ in } \mathcal{D}^-, \qquad k \geq 0,$$

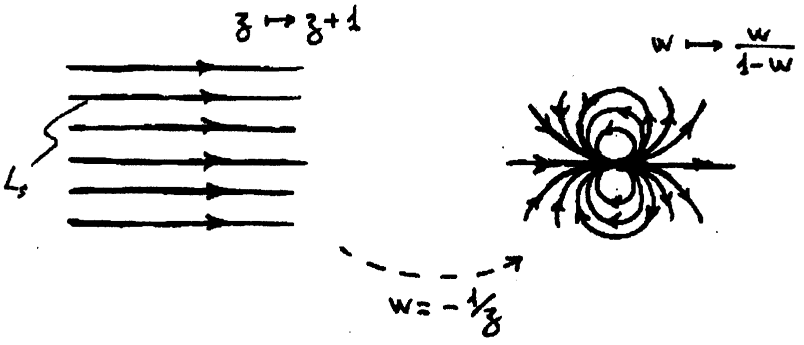


Figure 6: Dynamics of $w \mapsto F_0(w) = \frac{w}{1-w}$.

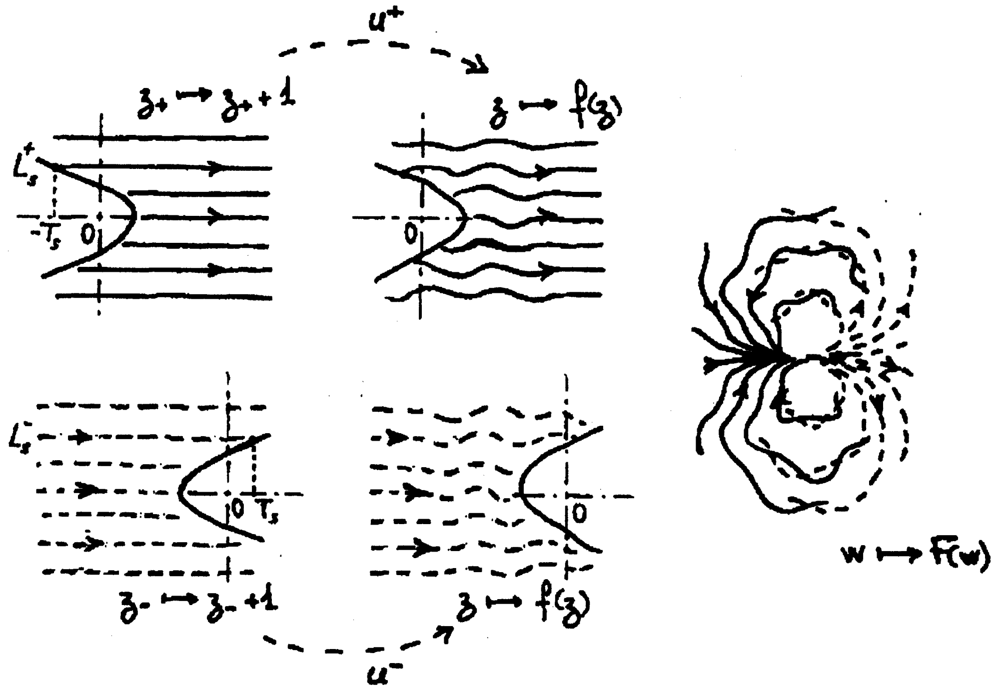


Figure 7: Stable and unstable foliations for f and F.

where f^n denotes the n^{th} power of iteration of f for any $n \in \mathbb{Z}$. The assumption $a(z) \in z^{-2}\mathbb{C}\{z^{-1}\}$ implies the existence and unicity of solutions which tend to 0 at infinity, which can be expressed as uniformly convergent series

$$\psi^+ = \sum_{k\geq 0} a \circ f^k, \qquad \psi^- = -\sum_{k\geq 1} a \circ f^{-k}. \tag{26}$$

(One can check that $f^n = \operatorname{Id} + n + \sum_{k=0}^{n-1} a \circ f^k$ by induction on $n \geq 1$, and similarly, that $f^{-1} = \operatorname{Id} - 1 - a \circ f^{-1}$, hence $f^{-n} = \operatorname{Id} - n - \sum_{k=1}^{n} a \circ f^{-k}$; formulas (26) are thus consistent with the formulas

$$v^{\pm}(z) = \lim_{n\to\pm\infty} \big(f^n(z) - n\big), \qquad u^{\pm}(z) = \lim_{n\to\pm\infty} f^{-n}(z+n)$$

which can be found in [Eca81, Vol. 2, p. 322] or [Lor06, p. 39].)

As a consequence, if we introduce the f-invariant functions β^{up} and β^{low} defined by

$$\beta = \sum_{k\in\mathbb{Z}} a \circ f^k$$

in the two connected components of $\mathcal{D}^+ \cap \mathcal{D}^-$ (they are first integrals of the dynamics: $\beta^{\text{low}} = \beta^{\text{low}} \circ f$, $\beta^{\text{up}} = \beta^{\text{up}} \circ f$), we get $v^+ - v^- = \psi^+ - \psi^- = \beta$. Since the horn maps $\operatorname{Id} + \chi^{\text{low}}$ and $\operatorname{Id} + \chi^{\text{up}}$ are defined by $v^+ = (\operatorname{Id} + \chi) \circ v^-$, writing $v^- = \operatorname{Id} + \psi^-$ and $v^+ = \operatorname{Id} + \psi^- + \beta$ we finally get

$$\chi^{\text{up}} = \beta^{\text{up}} \circ (\operatorname{Id} + \psi^-)^{-1}, \qquad \chi^{\text{low}} = \beta^{\text{low}} \circ (\operatorname{Id} + \psi^-)^{-1}.$$

2.3 Alien calculus for simple resurgent fuctions

In Section 2.1, we saw how solving a difference equation could lead to Borel transforms $\hat{\varphi}(\zeta)$ or $\hat{\psi}(\zeta)$ which are holomorphic in $\mathcal{R}$ and likely to develop singularities at the integer multiples of $2\pi i$. The purpose of "alien calculus" is to give an efficient way of encoding these singularities and of obtaining information on them. We shall describe the general formalism of singularities and give the definition of alien derivations in Section 3, but we begin here with a class of resurgent functions for which the definitions are simpler and which is sufficient to deal with Abel's equation.

Simple resurgent functions

Definition 6 *Let $\omega \in \mathbb{C}$. We say that a function $\hat{\varphi}$, which is holomorphic in an open disc $D \subset \mathbb{C}$ to which ω is adherent, "has a simple singularity at ω" if there exist $C \in \mathbb{C}$ and $\hat{\Phi}(\zeta), \operatorname{reg}(\zeta) \in \mathbb{C}\{\zeta\}$ such that*

$$\hat{\varphi}(\zeta) = \frac{C}{2\pi i(\zeta - \omega)} + \frac{1}{2\pi i}\hat{\Phi}(\zeta - \omega)\log(\zeta - \omega) + \operatorname{reg}(\zeta - \omega) \tag{27}$$

for all $\zeta \in D$ with $|\zeta - \omega|$ small enough. We then use the notation

$$\operatorname{sing}_\omega \hat{\varphi} = C\,\delta + \hat{\Phi} \in \mathbb{C}\,\delta \oplus \mathbb{C}\{\zeta\}.$$

Obviously, a change of branch of the logarithm in (27) only results in the replacement of $\operatorname{reg}(\zeta)$ by another regular germ; the interesting part of the formula is the singularity encoded by the "residuum" C and the "variation" $\hat{\Phi}$ which are unambiguously determined. For instance, the variation (also called "the minor of the singularity of $\hat{\varphi}$ at ω") can be written

$$\hat{\Phi}(\zeta) = \hat{\varphi}(\omega + \zeta) - \hat{\varphi}(\omega + \zeta\, \mathrm{e}^{-2\pi \mathrm{i}}),$$

where it is understood that considering $\omega + \zeta\, \mathrm{e}^{-2\pi \mathrm{i}}$ means following the analytic continuation of $\hat{\varphi}$ along the circular path $t \in [0,1] \mapsto \omega + \zeta\, \mathrm{e}^{-2\pi \mathrm{i} t}$ (the analytic continuation exists when $|\zeta|$ is small enough, since $\hat{\Phi}$ and reg are regular near the origin).

Any analytic function $\check{\Phi}(\zeta)$ which differ from $\hat{\varphi}(\omega+\zeta)$ by a regular germ is called a "major" of the singularity $\operatorname{sing}_\omega \hat{\varphi}$. Any major thus satisfies

$$\operatorname{sing}_0 \check{\Phi} = \operatorname{sing}_\omega \hat{\varphi}, \qquad \hat{\Phi}(\zeta) = \check{\Phi}(\zeta) - \check{\Phi}(\zeta\, \mathrm{e}^{-2\pi \mathrm{i}})$$

(the minor is the variation of any major). In fact, the singularity can be identified with an equivalence class of majors modulo $\mathbb{C}\{\zeta\}$. This will be used as a way of generalising the previous definition to deal with more complicated singularities in Section 3.

For any path γ issuing from 0 and lying in $\mathbb{C} \setminus 2\pi \mathrm{i}\,\mathbb{Z}$, and for any $\hat{\varphi} \in \widehat{\mathcal{H}}(\mathcal{R})$, we denote by $\operatorname{cont}_\gamma \hat{\varphi}$ the branch of $\hat{\varphi}$ obtained by following the analytic continuation of $\hat{\varphi}$ along γ, which is a function holomorphic at least in any open disc containing the extremity $\gamma(1)$ of γ and avoiding $2\pi \mathrm{i}\,\mathbb{Z}$. In particular, if $\omega \in 2\pi \mathrm{i}\,\mathbb{Z}$ satisfies $|\omega - \gamma(1)| < \pi$, there is a disc D avoiding $2\pi \mathrm{i}\,\mathbb{Z}$ which contains $\gamma(1)$ and to which ω is adherent.

Definition 7 *A "simple resurgent function" is any $\hat{\chi} = c\,\delta + \hat{\varphi}(\zeta) \in \mathbb{C}\,\delta \oplus \widehat{\mathcal{H}}(\mathcal{R})$ such that, for each $\omega \in 2\pi \mathrm{i}\,\mathbb{Z}$ and for each path γ which starts from 0, lies in $\mathbb{C} \setminus 2\pi \mathrm{i}\,\mathbb{Z}$ and has its extremity in the disc of radius π centred at ω, the branch $\operatorname{cont}_\gamma \hat{\varphi}$ has a simple singularity at ω.*

One can check that the corresponding subspace of $\mathbb{C}\,\delta \oplus \widehat{\mathcal{H}}(\mathcal{R})$ is stable by convolution:

Proposition 5 *The subspace $\mathrm{RES}^{\mathrm{simp}}$ consisting of all simple resurgent functions is a subalgebra of the convolution algebra $\mathbb{C}\,\delta \oplus \widehat{\mathcal{H}}(\mathcal{R})$.*

(This can be done with the help of the symmetrically contractile paths of Lemma 3; see the arguments given in the proof of Lemma 4 below.)

As a consequence, *the Borel transform $\hat{\varphi}$ of the solution of equation* (8)*, which belongs to $\widehat{\mathcal{H}}(\mathcal{R})$ according to Theorem 2, must be a simple resurgent function.* Indeed, as indicated in the proof of this theorem, $\hat{\varphi}$ can be expressed as a uniformly convergent series $\sum_{n\geq 1} \hat{\varphi}_n$, where the functions $\hat{\varphi}_n \in \widehat{\mathcal{H}}(\mathcal{R})$ are defined inductively by (23). It is essentially sufficient to check that each $\hat{\varphi}_n$ belongs to $\mathrm{RES}^{\mathrm{simp}}$, and this is easily done by induction ($\hat{\varphi}_1$ is meromorphic with simple poles because E is; since $\hat{a}_r \in \zeta^2\mathbb{C}\{\zeta\}$ is entire, one can write $\hat{a}_r = 1 * 1 * \hat{a}''_r$ with $\hat{a}''_r \in \mathrm{RES}^{\mathrm{simp}}$, hence $\hat{\varphi}_n = E\,(1 * 1 * \hat{A}_n)$ with $\hat{A}_n \in \mathrm{RES}^{\mathrm{simp}}$ by the inductive hypothesis; one concludes by observing that the singularities of $1 * 1 * \hat{A}_n$ have no residuum and that their variations have valuation at least 1 at the origin, while $E(\zeta) = -\frac{1}{\zeta-\omega} + \operatorname{reg}(\zeta - \omega)$).

The Borel transform $\hat{\psi}$ of the solution of equation (21) *is also a simple resurgent function,* because the space $\mathrm{RES}^{\mathrm{simp}}$ enjoys stability properties similar to the properties of $\mathbb{C}\,\delta \oplus \widehat{\mathcal{H}}(\mathcal{R})$ indicated in Propositions 1 and 2 of Section 1.4.

What we just defined is the "convolutive model of the algebra of simple resurgent functions". The "formal model of the algebra of simple resurgent functions" $\widetilde{\mathrm{RES}}^{\mathrm{simp}}$ is defined as the subalgebra of $\mathbb{C} \oplus \tilde{\mathcal{H}}$ obtained by pulling back $\mathrm{RES}^{\mathrm{simp}}$ by $\mathcal{B}$. The formal solution $\tilde{v} = \mathrm{Id} + \tilde{\psi}$ of Abel's equation can thus be viewed as a simple resurgent change of variable which normalises f, with $\tilde{u} = \mathrm{Id} + \tilde{\varphi} \in \mathrm{Id} + \widetilde{\mathrm{RES}}^{\mathrm{simp}}$ as inverse transformation.

Alien derivations

Let ω and γ be as in Definition 7. For $c\,\delta + \hat{\varphi} \in \mathrm{RES}^{\mathrm{simp}}$, let C_γ and $\hat{\Phi}_\gamma$ denote the residuum and the minor (variation) of the singularity of $\mathrm{cont}_\gamma\,\hat{\varphi}$ at ω. It is clear that $\hat{\Phi}_\gamma$, which is holomorphic at the origin by assumption, must extend holomorphically to $\mathcal{R}$ and possess itself only simple singularities. The formula

$$c\,\delta + \hat{\varphi} \;\mapsto\; C_\gamma\,\delta + \hat{\Phi}_\gamma = \mathrm{sing}_\omega\,(\mathrm{cont}_\gamma\,\hat{\varphi})$$

thus defines a linear operator of $\mathrm{RES}^{\mathrm{simp}}$ to itself.

Definition 8 *For each $\omega = \pm 2\pi\mathrm{i}\, m$, $m \in \mathbb{N}^*$, we define a linear operator Δ_ω from $\mathrm{RES}^{\mathrm{simp}}$ to itself by using 2^{m-1} particular paths γ:*

$$\Delta_\omega(c\,\delta + \hat{\varphi}) = \sum_{\varepsilon_1,\dots,\varepsilon_{m-1} \in \{+,-\}} \frac{p(\varepsilon)!q(\varepsilon)!}{m!} \left(C_{\gamma(\varepsilon)}\delta + \hat{\Phi}_{\gamma(\varepsilon)} \right), \tag{28}$$

where $p(\varepsilon)$ and $q(\varepsilon) = m - 1 - p(\varepsilon)$ denote the numbers of signs '+' and of signs '−' in the sequence ε, and the path $\gamma(\varepsilon)$ connects $\left]0, \frac{1}{m}\omega\right[$ and $\left]\frac{m-1}{m}\omega, \omega\right[$, following the segment $]0,\omega[$ but circumventing the intermediary singular points $\frac{r}{m}\omega$ to the right if $\varepsilon_r = +$ and to the left if $\varepsilon_r = -$ (see Figure 8). We also define a linear operator Δ^+_ω from $\mathrm{RES}^{\mathrm{simp}}$ to itself by setting

$$\Delta^+_\omega(c\,\delta + \hat{\varphi}) = C_{\gamma_\omega}\delta + \hat{\Phi}_{\gamma_\omega}, \qquad \gamma_\omega = \gamma(+,\dots,+). \tag{29}$$

Via $\mathcal{B}$, the operators Δ_ω or Δ^+_ω of the convolutive model give rise to operators of the formal model, which we denote by the same symbols:

$$\begin{array}{ccc} \mathrm{RES}^{\mathrm{simp}} & \xrightarrow{\Delta_\omega} & \mathrm{RES}^{\mathrm{simp}} \\ \mathcal{B}\uparrow & & \uparrow\mathcal{B} \\ \widetilde{\mathrm{RES}}^{\mathrm{simp}} & \xrightarrow{\Delta_\omega} & \widetilde{\mathrm{RES}}^{\mathrm{simp}} \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{RES}^{\mathrm{simp}} & \xrightarrow{\Delta^+_\omega} & \mathrm{RES}^{\mathrm{simp}} \\ \mathcal{B}\uparrow & & \uparrow\mathcal{B} \\ \widetilde{\mathrm{RES}}^{\mathrm{simp}} & \xrightarrow{\Delta^+_\omega} & \widetilde{\mathrm{RES}}^{\mathrm{simp}} \end{array}$$

Proposition 6 *The operator Δ_ω is a derivation, i.e. the Leibniz rule holds in the convolutive model:*

$$\Delta_\omega(\hat{\chi}_1 * \hat{\chi}_2) = (\Delta_\omega\hat{\chi}_1) * \hat{\chi}_2 + \hat{\chi}_1 * (\Delta_\omega\hat{\chi}_2), \qquad \hat{\chi}_1, \hat{\chi}_2 \in \mathrm{RES}^{\mathrm{simp}}. \tag{30}$$

Equivalently, we have in the formal model

$$\Delta_\omega(\tilde{\chi}_1\tilde{\chi}_2) = (\Delta_\omega\tilde{\chi}_1)\tilde{\chi}_2 + \tilde{\chi}_1(\Delta_\omega\tilde{\chi}_2), \qquad \tilde{\chi}_1, \tilde{\chi}_2 \in \widetilde{\mathrm{RES}}^{\mathrm{simp}}. \tag{31}$$

Moreover, for any $\tilde{\chi}, \tilde{\chi}_1, \tilde{\chi}_2 \in \widetilde{\mathrm{RES}}^{\mathrm{simp}}$,

$$\Delta_\omega\partial\tilde{\chi} = \partial\Delta_\omega\tilde{\chi} - \omega\Delta_\omega\tilde{\chi}, \tag{32}$$

$$\Delta_\omega\big(\tilde{\chi}_1 \circ (\mathrm{Id} + \tilde{\chi}_2)\big) = \mathrm{e}^{-\omega\tilde{\chi}_2}\big((\Delta_\omega\tilde{\chi}_1) \circ (\mathrm{Id} + \tilde{\chi}_2)\big) + \big((\partial\tilde{\chi}_1) \circ (\mathrm{Id} + \tilde{\chi}_2)\big)\Delta_\omega\tilde{\chi}_2. \tag{33}$$

In particular Δ_ω commutes with $\tilde{\chi}(z) \mapsto \tilde{\chi}(z+1)$ for each $\omega \in 2\pi\mathrm{i}\,\mathbb{Z}^*$.

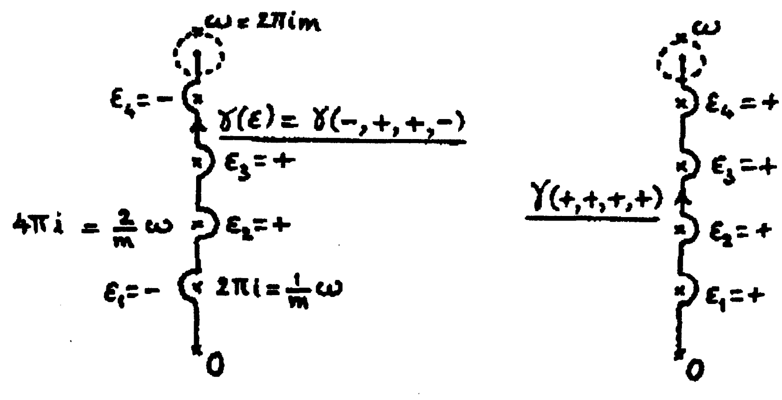


Figure 8: Paths for the definition of $\Delta_\omega \hat{\chi}$ or $\Delta^+_\omega \hat{\chi}$.

Because of (30) or (31), the operators Δ_ω were called "alien derivations" by Écalle, by contrast with the natural derivation $\partial = \frac{d}{dz}$. Some formulas get simpler when introducing the "dotted alien derivations" $\dot{\Delta}_\omega : \tilde{\chi}(z) \mapsto e^{-\omega z} \Delta_\omega \tilde{\chi}(z)$ (where $e^{-\omega z}$ is understood as a symbol external to $\widetilde{\text{RES}}^{\text{simp}}$, obeying the usual rules with respect to multiplication and differentiation): the dotted alien derivations commute with ∂ and satisfy

$$\dot{\Delta}_\omega(\tilde{\chi}_1 \circ (\text{Id} + \tilde{\chi}_2)) = (\dot{\Delta}_\omega \tilde{\chi}_1) \circ (\text{Id} + \tilde{\chi}_2) + ((\partial \tilde{\chi}_1) \circ (\text{Id} + \tilde{\chi}_2)) \, \dot{\Delta}_\omega \tilde{\chi}_2. \tag{34}$$

There is no relation between the operators Δ_ω: they generate a free Lie algebra.[13] They provide a way of encoding the whole singular behaviour of a minor $\hat{\varphi} \in \text{RES}^{\text{simp}}$: given a sequence $\omega_1, \ldots, \omega_r \in 2\pi i \mathbb{Z}^*$, the evaluation of the composition $\Delta_{\omega_r} \cdots \Delta_{\omega_1}(c\delta + \hat{\varphi})$ is a combination of singularities at $\omega_1 + \cdots + \omega_r$ for various branchs of $\hat{\varphi}$. Conversely, any singularity of any branch of $\hat{\varphi}$ can be computed if the collection of these objects for all sequences $(\omega_1, \ldots, \omega_r)$ is known.[14]

Before proving Proposition 6, we turn to the operators Δ^+_ω. Their definition is simpler (*cf.* formula (29)), but they are not derivations (except for $\omega = \pm 2\pi i$, since we have then $\Delta^+_\omega = \Delta_\omega$). Here is the way they act on products:

Lemma 4 *For $\omega \in 2\pi i \mathbb{Z}^*$ and $\hat{\chi}_1, \hat{\chi}_2 \in \text{RES}^{\text{simp}}$,*

$$\Delta^+_\omega(\hat{\chi}_1 * \hat{\chi}_2) = (\Delta^+_\omega \hat{\chi}_1) * \hat{\chi}_2 + \sum (\Delta^+_{\omega_1} \hat{\chi}_1) * (\Delta^+_{\omega_2} \hat{\chi}_2) + \hat{\chi}_1 * (\Delta^+_\omega \hat{\chi}_2), \tag{35}$$

where the sum extends to all (ω_1, ω_2) such that $\omega_1 + \omega_2 = \omega$ and $\omega_j \in]0, \omega[\cap 2\pi i \mathbb{Z}^*$.

[13]We mean that, for any $N \geq 1$ and for any collection of non-zero simple resurgent functions $(\tilde{\chi}_{\omega_1 \cdots \omega_r})$ indexed by finitely many words $\omega_1 \cdots \omega_r$ of any length $r \in \{0, \ldots, N\}$, the operator

$$\sum_{r=0}^{N} \sum_{\omega_1 \cdots \omega_r} \tilde{\chi}_{\omega_1 \cdots \omega_r} \Delta_{\omega_r} \cdots \Delta_{\omega_1}$$

(with the convention $\tilde{\chi}_{\omega_1 \cdots \omega_r} \Delta_{\omega_r} \cdots \Delta_{\omega_1} = \tilde{\chi}_\emptyset$ if $r = 0$) is not identically zero on $\widetilde{\text{RES}}^{\text{simp}}$. We omit the proof.

[14] One must not limit oneself to $r = 1$. For instance, $\Delta_\omega \hat{\chi} = 0$ does not mean that there is no singularity at ω for any branch of the minor; consider for example $\omega = \omega_1 + \omega_2$ and $\hat{\chi} = \hat{\varphi} * \hat{\psi}$ with $\hat{\varphi} = 1/(\zeta - \omega_1)$, $\hat{\psi} = 1/(\zeta - \omega_2)$ and $\omega_1 \neq \omega_2$: $\Delta_{\omega_1} \hat{\varphi} = \Delta_{\omega_2} \hat{\psi} = 2\pi i \, \delta$ and $\Delta_\omega \hat{\chi} = 0$, but $\Delta_{\omega_1} \hat{\chi} = 2\pi i \hat{\psi}$ and $\Delta_{\omega_2} \hat{\chi} = 2\pi i \hat{\varphi}$ imply $\Delta_{\omega_2} \Delta_{\omega_1} \hat{\chi} = \Delta_{\omega_1} \Delta_{\omega_2} \hat{\chi} = -4\pi^2 \delta$, which reveals the presence of a singularity at ω at least for some branchs of $\hat{\chi}$.

Proof of Lemma 4. Formula (35) results from a kind of combinatorics of symmetrically contractile paths. We begin with a proof in the simplest case, when $\omega = 2\pi\mathrm{i}$ (the case of $-2\pi\mathrm{i}$ is similar), and then sketch the case of $\omega = 2\pi\mathrm{i}\, m$ (the case of $-2\pi\mathrm{i}\, m$ would be similar).

Let $\omega = 2\pi\mathrm{i}$. By linearity, observing that Δ^+_ω annihilates the multiples of δ, it is sufficient to consider simple resurgent functions of the form $\hat\chi_1 = \hat\varphi$ and $\hat\chi_2 = \hat\psi$ (*i.e.* without any multiple of δ). We must prove

$$\Delta^+_{2\pi\mathrm{i}}(\hat\varphi * \hat\psi) = (\Delta^+_{2\pi\mathrm{i}}\hat\varphi) * \hat\psi + \hat\varphi * (\Delta^+_{2\pi\mathrm{i}}\hat\psi).$$

Let $\Delta^+_\omega\hat\varphi = a\,\delta + \hat\Phi$ and $\Delta^+_\omega\hat\psi = b\,\delta + \hat\Psi$, and consider the formula

$$\hat\varphi * \hat\psi(\xi) = \int_0^\xi \hat\varphi(\xi_1)\hat\psi(\xi - \xi_1)\,\mathrm{d}\xi_1, \tag{36}$$

which holds for ξ close to ω provided ξ lies in the principal sheet of $\mathcal{R}$, *i.e.* the segment $\ell = [0,\xi]$ avoids $2\pi\mathrm{i}$ and $-2\pi\mathrm{i}$. Writing $\xi = \omega + \zeta$, we have

$$\hat\varphi(\omega+\zeta) = \frac{1}{2\pi\mathrm{i}}\left(\frac{a}{\zeta} + \hat\Phi(\zeta)\log\zeta + \mathrm{reg}(\zeta)\right), \quad \hat\psi(\omega+\zeta) = \frac{1}{2\pi\mathrm{i}}\left(\frac{b}{\zeta} + \hat\Psi(\zeta)\log\zeta + \mathrm{reg}(\zeta)\right). \tag{37}$$

It can be seen that the residuum of $\hat\varphi * \hat\psi$ at ω is zero: in the path of integration of (36), the singularity for $\xi \to \omega$ stems from the extremities, ξ (because of $\hat\varphi(\xi_1)$, which is multiplied by a function holomorphic for ξ_1 close to ω) and 0 (because of $\hat\psi(\xi - \xi_1)$, which is multiplied by a function holomorphic for ξ_1 close to 0); the singularity is thus obtained by *integrating* simple poles and logarithmic singularities.[15]

Hence, to show that $\Delta^+_\omega(\hat\varphi * \hat\psi) = (a\,\delta + \hat\Phi) * \hat\psi + \hat\varphi * (b\,\delta + \hat\Psi)$, we just need to check that

$$\hat\varphi * \hat\psi(\omega + \zeta) - \hat\varphi * \hat\psi(\omega + \zeta\,\mathrm{e}^{-2\pi\mathrm{i}}) = a\hat\psi(\zeta) + \hat\Phi * \hat\psi(\zeta) + b\hat\varphi(\zeta) + \hat\varphi * \hat\Psi(\zeta). \tag{38}$$

For $|\zeta| \le \pi$, the first term in the left-hand side is $\mathrm{cont}_\ell\,\hat\varphi * \hat\psi(\xi)$, given by (36) with $\xi = \omega + \zeta$, while the second term is $\mathrm{cont}_\Gamma\,\hat\varphi * \hat\psi(\xi) = \int_\Gamma \hat\varphi(\xi_1)\hat\psi(\xi - \xi_1)\,\mathrm{d}\xi_1$, with a symmetrically contractile path Γ as shown on Figure 9. The difference is thus $(\int_\gamma - \int_{\gamma'})\hat\varphi(\xi_1)\hat\psi(\xi-\xi_1)\,\mathrm{d}\xi_1$, where the path γ reaches ξ, turning once anticlockwise around ω, having started from ξ (or rather from $\omega+\zeta\,\mathrm{e}^{-2\pi\mathrm{i}}$), and where $\gamma' = \zeta - \gamma$. With the change of variable $\xi_1 \mapsto \xi - \xi_1$ in the second integral, we get

$$\hat\varphi * \hat\psi(\omega + \zeta) - \hat\varphi * \hat\psi(\omega + \zeta\,\mathrm{e}^{-2\pi\mathrm{i}}) = \int_\gamma \hat\varphi(\xi_1)\hat\psi(\xi - \xi_1)\,\mathrm{d}\xi_1 + \int_\gamma \hat\psi(\xi_1)\hat\varphi(\xi - \xi_1)\,\mathrm{d}\xi_1$$

and the identity (38) will follow if we prove that $\int_\gamma \hat\varphi(\xi_1)\hat\psi(\xi - \xi_1)\,\mathrm{d}\xi_1 = a\hat\psi(\zeta) + \hat\Phi * \hat\psi(\zeta)$ (with a symmetric formula for the second integral).

With the change of variable $\xi_1 \mapsto \zeta_1 = \xi_1 - \omega$, this integral can be written

$$\frac{1}{2\pi\mathrm{i}}\int_{-\omega+\gamma}\left(\frac{a}{\zeta_1} + \hat\Phi(\zeta_1)\log\zeta_1 + \mathrm{reg}(\zeta_1)\right)\hat\psi(\zeta - \zeta_1)\,\mathrm{d}\zeta_1.$$

The conclusion follows, since ζ_1 and $\zeta - \zeta_1$ stay in a neighbourhood of the origin where $\hat\Phi$, reg and $\hat\psi$ are holomorphic: the residuum formula takes care of the simple pole and the Cauchy theorem cancels the contribution of $\mathrm{reg}(\zeta_1)$, while the contribution of the logarithm can be computed by collapsing the path $-\omega + \gamma$ onto the segment $[\zeta\,\mathrm{e}^{-2\pi\mathrm{i}}, 0]$ followed by $[0, \zeta]$.

[15] This argument can be avoided by arguing as at the end of the case $m \ge 2$, writing $\hat\varphi * \hat\psi = (\frac{\mathrm{d}}{\mathrm{d}\zeta})^2(1 * \hat\varphi * 1 * \hat\psi)$.

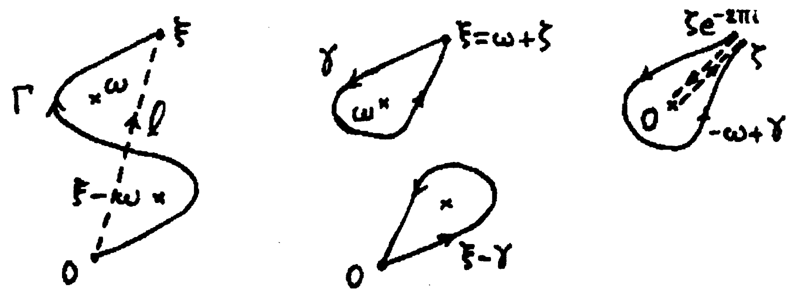


Figure 9: Derivation of formula (38).

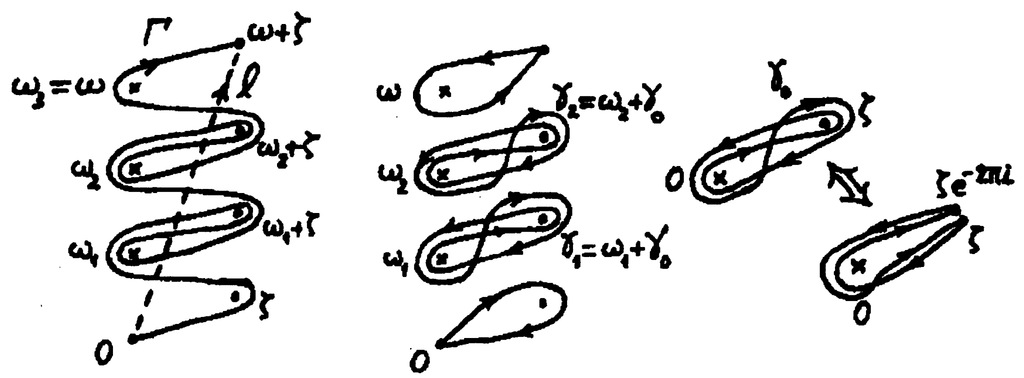


Figure 10: Derivation of formula (39).

Now let $m \geq 2$ and consider $\omega = 2\pi\mathrm{i}\,m$. Assume $\hat{\chi}_1 = \hat{\varphi}$ and $\hat{\chi}_2 = \hat{\psi}$ with $\Delta^+_{\omega_j}\hat{\varphi} = a_j\,\delta + \hat{\Phi}_j$ and $\Delta^+_{\omega_j}\hat{\psi} = b_j\,\delta + \hat{\Psi}_j$ for $\omega_j = 2\pi\mathrm{i}\,j$, $j \in \{1,\dots,m\}$. We must prove

$$\Delta^+_{2\pi\mathrm{i}\,m}(\hat{\varphi} * \hat{\psi}) = (\Delta^+_{2\pi\mathrm{i}\,m}\hat{\varphi}) * \hat{\psi} + \sum_{m_1+m_2=m} (\Delta^+_{2\pi\mathrm{i}\,m_1}\hat{\varphi}) * (\Delta^+_{2\pi\mathrm{i}\,m_2}\hat{\psi}) + \hat{\varphi} * (\Delta^+_{2\pi\mathrm{i}\,m}\hat{\psi}).$$

This time, to simplify the computations, we begin with the case where all the constants a_j and b_j vanish. This means that, considering $\xi = \omega + \zeta$ such that $|\zeta| < \pi$ and $\ell = [0,\xi] \subset \mathbb{C} \setminus 2\pi\mathrm{i}\,\mathbb{Z}^*$, instead of (38) we now must show

$$\hat{\varphi} * \hat{\psi}(\omega + \zeta) - \hat{\varphi} * \hat{\psi}(\omega + \zeta\,\mathrm{e}^{-2\pi\mathrm{i}}) = \hat{\Phi}_m * \hat{\psi}(\zeta) + \sum_{m_1+m_2=m} \hat{\Phi}_{m_1} * \hat{\Psi}_{m_2} + \hat{\varphi} * \hat{\Psi}_m(\zeta), \quad (39)$$

where the first term in the left-hand side is $\mathrm{cont}_\ell\,\hat{\varphi}*\hat{\psi}(\xi)$, still given by (36), and the second term is $\mathrm{cont}_\Gamma\,\hat{\varphi} * \hat{\psi}(\xi) = \int_\Gamma \hat{\varphi}(\xi_1)\hat{\psi}(\xi - \xi_1)\,\mathrm{d}\xi_1$, with a more complicated symmetrically contractile path Γ, as shown on Figure 10.

The difference can thus be decomposed as the sum of $m+1$ terms, with the same extreme terms as in the case $m = 1$, which thus yield $\hat{\Phi}_m * \hat{\psi}$ and $\hat{\varphi} * \hat{\Psi}_m$ as in the first part of the proof, and with intermediary terms $\int_{\gamma_{m_1}} \hat{\varphi}(\xi_1)\hat{\psi}(\xi - \xi_1)\,\mathrm{d}\xi_1$ for $m_1 \in \{1,\dots,m-1\}$, with paths $\gamma_{m_1} = \omega_{m_1} + \gamma_0$ shown on Figure 10. Each intermediary term can be written

$$\frac{1}{2\pi\mathrm{i}} \int_{\gamma_0} \hat{\varphi}(\omega_{m_1} + \zeta_1) \left(\hat{\Psi}_{m_2}(\zeta - \zeta_1)\log(\zeta - \zeta_1) + \mathrm{reg}(\zeta - \zeta_1)\right) \mathrm{d}\zeta_1$$

with $m_2 = m - m_1$. Collapsing the path γ_0 as indicated on Figure 10, we see that this term is nothing but $\int_{\tilde\gamma_0} \hat\varphi(\omega_{m_1} + \zeta_1)\hat\Psi_{m_2}(\zeta - \zeta_1)\,\mathrm{d}\zeta_1$, where the path $\tilde\gamma_0$ is identical to the path $-\omega + \gamma$ of Figure 9; such an integral was already computed in the first part of the proof: it is $\hat\Phi_{m_1} * \hat\Psi_{m_2}(\zeta)$, which yields formula (39).

We end with the general case, with arbitrary constants a_j and b_j. As alluded to in footnote 15, it is sufficient to write $\hat\varphi * \hat\psi$ as the second derivative of $(1 * \hat\varphi) * (1 * \hat\psi)$ and to observe that $(1 * \hat\varphi)(\omega_j + \zeta) = \frac{1}{2\pi \mathrm{i}}\Big(\big(a_j + (1 * \hat\Phi_j)(\zeta)\big)\log\zeta + \mathrm{reg}(\zeta)\Big)$, and similarly for $(1 * \hat\psi)(\omega_j + \zeta)$, by integrating (37) (we call indifferently $\mathrm{reg}(\zeta)$ all the regular germs that appear and that do not affect the final result; we used the fact that, in the integration by parts, the primitives of $(1 * \hat\Phi_j)(\zeta)\frac{\mathrm{d}}{\mathrm{d}\zeta}(\log\zeta)$ are regular). Of course, convolution is commutative and $1 * 1 = \zeta$, the formula for the case of vanishing residua thus yields

$$\zeta * \hat\varphi * \hat\psi(\omega + \zeta) = \frac{1}{2\pi \mathrm{i}} \sum_{m_1 + m_2 = m} \Big(\big(a_{m_1} b_{m_2}\zeta + a_{m_1}\zeta * \hat\Psi_{m_2}(\zeta) + b_{m_2}\zeta * \hat\Phi_{m_1}(\zeta) + \zeta * \hat\Phi_{m_1} * \hat\Psi_{m_2}(\zeta)\big)\log\zeta + \mathrm{reg}(\zeta)\Big),$$

with a sum extending to $(0, m)$ and $(m, 0)$, with the convention $a_0 = b_0 = 0$. The conclusion follows by differentiating twice (observing that $(1 * \hat A)(\zeta)/\zeta$ is regular for whatever regular germ $\hat A$). □

It remains to prove Proposition 6. For the moment, we content ourselves with indicating a relation between the operators Δ_ω and Δ^+_ω (which will follow from the choice of the weights $\frac{p(\epsilon)!q(\epsilon)!}{m!}$), the idea being that according to Lemma 4 the operators $\Delta^+_{\pm 2\pi \mathrm{i} m}$ are the homogeneous components of two formal automorphisms of graded algebra and that, according to the next lemma, the operators $\Delta_{\pm 2\pi \mathrm{i} m}$ are the homogeneous components of the logarithms of these automorphims.

Lemma 5 *For each $m \in \mathbb{N}^*$ and $\omega = \pm 2\pi \mathrm{i}\, m$, we have*

$$\Delta_\omega = \sum_{1 \le r \le m} \frac{(-1)^{r-1}}{r} \sum_{\substack{m_1, \dots, m_r \ge 1 \\ m_1 + \dots + m_r = m}} \Delta^+_{\omega_{m_1}} \cdots \Delta^+_{\omega_{m_r}} \tag{40}$$

with the notation $\omega_j = \frac{j}{m}\omega$.

We thus have

$$\Delta_{\omega_1} = \Delta^+_{\omega_1}$$

$$\Delta_{\omega_2} = \Delta^+_{\omega_2} - \tfrac{1}{2}\Delta^+_{\omega_1}\Delta^+_{\omega_1}$$

$$\Delta_{\omega_3} = \Delta^+_{\omega_3} - \tfrac{1}{2}\left(\Delta^+_{\omega_2}\Delta^+_{\omega_1} + \Delta^+_{\omega_1}\Delta^+_{\omega_2}\right) + \tfrac{1}{3}\Delta^+_{\omega_1}\Delta^+_{\omega_1}\Delta^+_{\omega_1}$$

$$\vdots$$

with $\omega_m = 2\pi \mathrm{i}\, m$ for all $m \ge 1$ or $\omega_m = -2\pi \mathrm{i}\, m$ for all $m \ge 1$.

The proof of Lemma 5 and the way it implies formula (30) *of Proposition 6 are deferred to the end of the next section.* Formula (31) follows by passage to the formal model. The rest of

Proposition 6 is easy: formula (32) is best seen in the convolutive model; formula (33) follows from (34), which can be checked in the convolutive model, with the help of a Taylor expansion analogous to (14).

We end this section by mentioning that substitution of a simple resurgent function $\tilde{\psi}$ without constant term into a convergent series $C(w) \in \mathbb{C}\{w\}$ gives rise to a simple resurgent function $C \circ \tilde{\psi}$, the alien derivatives of which are given by

$$\Delta_\omega(C \circ \tilde{\psi}) = (C' \circ \tilde{\psi})\Delta_\omega\tilde{\psi}. \tag{41}$$

2.4 Bridge equation for non-degenerate parabolic germs

We can now state Écalle's result for tangent-to-identity holomorphic germs, which is at the origin of the name "Resurgence".

Theorem 3 *Let f be a non-degenerate parabolic germ at infinity with vanishing resiter ($\rho = 0$), and let $\tilde{v}(z) = z + \tilde{\psi}(z)$ and $\tilde{u}(z) = z + \tilde{\varphi}(z)$ be the formal solutions of equations* (18) *and* (19) *with $\tilde{\varphi}, \tilde{\psi} \in z^{-1}\mathbb{C}[[z^{-1}]]$. Then $\tilde{\varphi}, \tilde{\psi} \in \widetilde{\mathrm{RES}}^{\mathrm{simp}}$, and there exists a sequence of complex numbers $\{A_\omega\}_{\omega \in 2\pi\mathrm{i}\,\mathbb{Z}^*}$ such that, for each $\omega \in 2\pi\mathrm{i}\,\mathbb{Z}^*$,*

$$\Delta_\omega\tilde{u} = A_\omega\partial\tilde{u} = A_\omega\big(1 + \partial\tilde{\varphi}(z)\big), \quad \Delta_\omega\tilde{v} = -A_\omega\,\mathrm{e}^{-\omega(\tilde{v}(z)-z)} = -A_\omega\,\mathrm{e}^{-\omega\tilde{\psi}(z)}. \tag{42}$$

Here the notation was slightly extended with respect to Definition 8: $\Delta_\omega\tilde{u}$ is to be understood as being equal to $\Delta_\omega\tilde{\varphi}$ (since $\tilde{u}(z) - \tilde{\varphi}(z) = z$ is convergent: the difference offers no singularity to be measured by any alien derivation). In the convolutive model, this amounts to setting $\Delta_\omega\delta' = 0$ (we already had $\Delta_\omega\delta = 0$). Similarly, $\Delta_\omega\tilde{v} = \Delta_\omega\tilde{\psi}$. The translation of (42) in the convolutive model is thus

$$\Delta_\omega\hat{\varphi} = A_\omega\,\delta - A_\omega\zeta\hat{\varphi}(\zeta), \quad \Delta_\omega\hat{\psi} = -A_\omega\left(\delta - \omega\hat{\psi} + \tfrac{1}{2!}\omega^2\hat{\psi}^{*2} - \tfrac{1}{3!}\omega^3\hat{\psi}^{*3} + \cdots\right). \tag{43}$$

The existence of such relations is the resurgent phenomenon: $\hat{\varphi}(\zeta)$ or $\hat{\psi}(\zeta)$, which are a holomorphic germs at the origin, reappear in a disguised form at the singularities of their analytic continuation, when singularities are measured in an appropriate way.

Equation (42) is called the "Bridge equation", because the first equation may be viewed as a bridge between alien calculus (Δ_ω) and ordinary calculus ($\partial = \frac{\mathrm{d}}{\mathrm{d}z}$) in the case of $\tilde{u}$. Notice that it is possible to iterate these equations to compute the successive alien derivatives $\Delta_{\omega_r}\cdots\Delta_{\omega_1}\tilde{u}$ or $\Delta_{\omega_r}\cdots\Delta_{\omega_1}\tilde{v}$, since we know how the alien derivations interact with ∂ (see formula (32)) and with exponentiation (using (41)). The computation is simpler with dotted alien derivations: we get

$$\dot{\Delta}_{\omega_r}\cdots\dot{\Delta}_{\omega_1}\tilde{u} = D_{\omega_1}\cdots D_{\omega_r}\tilde{u}, \qquad D_\omega = A_\omega\,\mathrm{e}^{-\omega z}\,\partial \tag{44}$$

(beware of the non-commutation of ∂ and multiplication by $\mathrm{e}^{-\omega z}$: the vector fields D_{ω_j} do not commute one with the other, but they do commute with the dotted alien derivations, hence the reversal of order) and

$$\dot{\Delta}_{\omega_r}\cdots\dot{\Delta}_{\omega_1}\tilde{v} = -A_{\omega_1}\cdots A_{\omega_r}\,\omega_1(\omega_1+\omega_2)\cdots(\omega_1+\cdots+\omega_{r-1})\,\mathrm{e}^{-(\omega_1+\cdots+\omega_r)\tilde{v}}. \tag{45}$$

The collection of $\{A_\omega,\ \omega \in 2\pi\mathrm{i}\,\mathbb{Z}^*\}$ is thus sufficient to describe the whole singular behaviour of $\hat{\varphi}$ and $\hat{\psi}$.

Proof of Theorem 3. The fact that $\hat{\varphi}$ and $\hat{\psi}$ are simple resurgent functions was already alluded to in the previous section (after Proposition 5). With our extension of Δ_ω to the space $\mathrm{Id} + \widetilde{\mathrm{RES}}^{\mathrm{simp}}$ (which contains $f = \mathrm{Id} + 1 + a$ and $\tilde{u} = \mathrm{Id} + \tilde{\varphi}$), equation (33) yields

$$\Delta_\omega(f \circ \tilde{u}) = \mathrm{e}^{-\omega\tilde{\varphi}}\big((\Delta_\omega f) \circ \tilde{u}\big) + (\partial f \circ \tilde{u})\Delta_\omega \tilde{u}.$$

But $\Delta_\omega f = 0$ because a is a convergent power series (its Borel transform has no singularity); on the other hand, $\Delta_\omega(f \circ \tilde{u}) = \Delta_\omega(\tilde{u}(z+1)) = (\Delta_\omega \tilde{u})(z+1)$, hence

$$(\Delta_\omega \tilde{u})(z+1) = (\partial f \circ \tilde{u})(z)\Delta_\omega \tilde{u}(z).$$

The equation $Y(z+1) = (\partial f \circ \tilde{u})(z) Y(z)$ is nothing but the linearization of the equation $u(z+1) = f \circ u(z)$ around the solution $\tilde{u}$, and we know a solution of this linear difference equation, namely $\partial \tilde{u}$ (because ∂ is also a derivation which commutes with $\tilde{\chi}(z) \mapsto \tilde{\chi}(z+1)$); moreover $\partial \tilde{u} \in 1 + z^{-1}\mathbb{C}[[z^{-1}]]$ admits a multiplicative inverse. As a consequence, the formal series $A_\omega = \frac{1}{\partial \tilde{u}} \Delta_\omega \tilde{u}$ must be invariant by $z \mapsto z+1$, hence constant.

The extension of equation (34) to $\mathrm{Id} + \widetilde{\mathrm{RES}}^{\mathrm{simp}}$ yields

$$\dot{\Delta}_\omega(\tilde{u} \circ \tilde{v}) = \big(\dot{\Delta}_\omega \tilde{u}\big) \circ \tilde{v} + (\partial \tilde{u} \circ \tilde{v})\, \dot{\Delta}_\omega \tilde{v}$$

and this expression must vanish, since $\tilde{u} \circ \tilde{v}(z) = z$, hence

$$\big(\dot{\Delta}_\omega \tilde{v}\big) \circ \tilde{v}^{-1} = -\frac{1}{\partial \tilde{u}}\, \dot{\Delta}_\omega \tilde{u} = -A_\omega\, \mathrm{e}^{-\omega z},$$

which implies $\dot{\Delta}_\omega \tilde{v} = -A_\omega\, \mathrm{e}^{-\omega \tilde{v}}$. □

Écalle's analytic invariants

The coefficients A_ω in the Bridge equation are called "Écalle's analytic invariants". Observe that A_ω can be defined as the coefficient of δ in $\Delta_\omega \hat{\varphi}$ (or of $-\Delta_\omega \hat{\psi}$), which is an average of the residua of 2^{m-1} branchs of $\hat{\varphi}$ (or of $-\hat{\psi}$) if $\omega = \pm 2\pi\mathrm{i}\, m$. These coefficients form a complete system of analytic invariants in the following sense:

Proposition 7 *Two non-degenerate parabolic germs f_1 and f_2 with vanishing resiter are analytically conjugate if and only if there exists $c \in \mathbb{C}$ such that $A_\omega^{(2)} = \mathrm{e}^{-\omega c} A_\omega^{(1)}$ for all $\omega \in 2\pi\mathrm{i}\,\mathbb{Z}^*$.*

The proof of one implication is easy. Suppose that $f_2 = h^{-1} \circ f_1 \circ h$ with $h \in \mathrm{Id} + \mathbb{C}\{z^{-1}\}$; we shall prove that $A_\omega^{(2)} = \mathrm{e}^{-\omega c} A_\omega^{(1)}$, where c is defined by $h(z) = z + c + \mathcal{O}(z^{-1})$. By the same argument as in footnote 11, since $\tilde{v}_1 \circ h(z) = z + c + \mathcal{O}(z^{-1})$ is a formal solution of Abel's equation for f_2, Proposition 4 implies the existence of $c' \in \mathbb{C}$ such that $\tilde{v}_1 \circ h = \tilde{v}_2 + c'$; we get $c' = c$ since $\tilde{v}_2(z) = z + \mathcal{O}(z^{-1})$. Using the chain rule for alien derivations, since h is convergent, we find

$$\dot{\Delta}_\omega \tilde{v}_2 = \big(\dot{\Delta}_\omega \tilde{v}_1\big) \circ h = -A_\omega^{(1)}\, \mathrm{e}^{-\omega\, \tilde{v}_1 \circ h} = -A_\omega^{(1)}\, \mathrm{e}^{-\omega c}\, \mathrm{e}^{-\omega \tilde{v}_2}$$

hence $A_\omega^{(2)} = A_\omega^{(1)}\, \mathrm{e}^{-\omega c}$. (We could have expoited the relation $h \circ \tilde{u}_2(z) = \tilde{u}_1(z+c)$ equally).

The direct verification of the other implication requires an extra work and we shall not give all the details. Suppose that $A_\omega^{(2)} = \mathrm{e}^{-\omega c} A_\omega^{(1)}$ for all $\omega \in 2\pi\mathrm{i}\,\mathbb{Z}^*$ and consider $\tilde{h} = \tilde{u}_1 \circ (c + \tilde{v}_2) \in$

$\mathrm{Id} + \widetilde{\mathrm{RES}}^{\mathrm{simp}}$, which establishes a formal conjugacy between f_1 and f_2. A computation similar to the above yields

$$\Delta_\omega \tilde{h} = \left(A_\omega^{(1)} \,\mathrm{e}^{-\omega c} - A_\omega^{(2)} \right) \mathrm{e}^{-\omega \tilde{v}_2}\, \partial \tilde{u}_1 \circ (c + \tilde{v}_2) = 0$$

for all ω, thus $\hat{h} = \mathcal{B}(\tilde{h} - \mathrm{Id})$ has no singularity at all, and one can deduce from Theorem 2 that this entire function has at most exponential growth in the non-vertical directions. However, an extra argument is needed to make sure that $\hat{h}$ has at most exponential growth with bounded type in all directions, including the imaginary axis, which is sufficient to conclude that $\tilde{h}$ is convergent.[16]

The extra argument that we have not given is related to the existence of constraints on the growth of $\hat{\varphi}$ and $\hat{\psi}$ along the lines which are parallel to the imaginary axis, which imply constraints on the growth of the numbers $|A_{\pm 2\pi \mathrm{i} m}|$ as $m \to \infty$. This can be obtained by a fine analysis in the Borel plane. Another approach consists in relating Écalle's analytic invariants and the horn maps: there is a one-to-one correspondence between $\{A_\omega,\ \omega \in 2\pi\mathrm{i}\,\mathbb{N}^*\}$ and $\{B_m,\ m \in \mathbb{N}^*\}$ on the one hand, and between $\{A_\omega,\ \omega \in -2\pi\mathrm{i}\,\mathbb{N}^*\}$ and $\{B_{-m},\ m \in \mathbb{N}^*\}$ on the other hand. This will be the subject of the next paragraphs. We shall see that the growth contraint on the $|A_\omega|$ amounts exactly to the convergence of the Fourier series (24)–(25) with some $\tau_0 > 0$.

Relation with the horn maps

Let us give the recipe before trying to justify it: if we work in the graded algebras

$$\widetilde{\mathrm{RES}}^{\mathrm{simp}}[[\mathrm{e}^{-2\pi\mathrm{i}\, z}]] = \bigoplus_{\omega \in 2\pi\mathrm{i}\,\mathbb{N}} \mathrm{e}^{-\omega z}\, \widetilde{\mathrm{RES}}^{\mathrm{simp}}, \qquad \text{resp.} \quad \widetilde{\mathrm{RES}}^{\mathrm{simp}}[[\mathrm{e}^{2\pi\mathrm{i}\, z}]] = \bigoplus_{\omega \in -2\pi\mathrm{i}\,\mathbb{N}} \mathrm{e}^{-\omega z}\, \widetilde{\mathrm{RES}}^{\mathrm{simp}} \tag{46}$$

(so as to give a meaning to the $\overset{\bullet}{\Delta}_\omega$'s as internal operators, which commute with the multiplication by $\mathrm{e}^{-\omega_0 z}$ for any ω_0 and are ω-homogeneous[17]) and define the "directional alien derivations"

[16] Here is another interesting property of the A_ω's: since the normal form $f_0(z) = z+1$ is the time-1 map of $\frac{\partial}{\partial z}$, we can define its t^{th} power of iteration by $f_0^{[t]}(z) = z + t$ for any $t \in \mathbb{C}$, and by formal conjugacy we retrieve the "t^{th} power of iteration" of f: $\tilde{f}^{[t]} = \tilde{u} \circ f_0^{[t]} \circ \tilde{v} \in \mathrm{Id} + \widetilde{\mathrm{RES}}^{\mathrm{simp}}$, which is the unique $\tilde{f}^{[t]}(z) \in z + t + z^{-2}\mathbb{C}[[z^{-1}]]$ such that $\tilde{f}^{(t)} \circ f = f \circ \tilde{f}^{[t]}$ (in other words, we embed f in a formal one-parameter group $\{\tilde{f}^{[t]},\ t \in \mathbb{C}\}$, which is generated by the formal vector field $\partial\tilde{u} \circ \tilde{v}(z) \frac{\partial}{\partial z} = \frac{1}{\partial \tilde{v}(z)} \frac{\partial}{\partial z}$). For a given $t \in \mathbb{C}$, $\tilde{f}^{[t]}$ is always resurgent but usually divergent, unless $t \in \mathbb{Z}$, however it may happen that some values of t give rise to a convergent trnasformation (and this does not imply the convergence of the formal infinitesimal generator $\frac{1}{\partial \tilde{v}(z)} \frac{\partial}{\partial z}$). *The set of all $t \in \mathbb{C}$ such that $\tilde{f}^{[t]}$ is convergent is the group $\frac{1}{q}\mathbb{Z}$, where $q \in \mathbb{N}^*$ is determined by the condition* $\omega \notin 2\pi\mathrm{i}\, q\mathbb{Z}^* \Rightarrow A_\omega = 0$, which is consistent with the relations $\overset{\bullet}{\Delta}_\omega \tilde{f}^{[t]} = A_\omega \left(\mathrm{e}^{-\omega t} - 1 \right) \mathrm{e}^{-\omega \tilde{v}} \frac{1}{\partial \tilde{v}} \partial \tilde{f}^{[t]}$ which follow from computations analogous to the previous ones. This corresponds to the fact that the 1-periodic functions $\chi^{\mathrm{low,up}}$ encoding the horn maps may admit a period q larger than 1.

[17] We simply mean that, if $\omega = 2\pi\mathrm{i}\, m$ with $m \geq 1$ for instance, there is a unique linear operator

$$\overset{\bullet}{\Delta}_\omega : \widetilde{\mathrm{RES}}^{\mathrm{simp}}[[\mathrm{e}^{-2\pi\mathrm{i}\, z}]] \to \widetilde{\mathrm{RES}}^{\mathrm{simp}}[[\mathrm{e}^{-2\pi\mathrm{i}\, z}]]$$

which extends the operator $\overset{\bullet}{\Delta}_\omega$ previously defined in $\widetilde{\mathrm{RES}}^{\mathrm{simp}}$ only and which commutes with multiplication by $\mathrm{e}^{-\omega_0 z}$ for any ω_0. This operator is ω-homogeneous in the sense that it sends the space of ω_0-homogeneous elements in the set of $(\omega_0 + \omega)$-homogeneous elements:

$$\overset{\bullet}{\Delta}_\omega \big(\mathrm{e}^{-\omega_0 z}\, \widetilde{\mathrm{RES}}^{\mathrm{simp}} \big) \subset \mathrm{e}^{-(\omega_0 + \omega) z}\, \widetilde{\mathrm{RES}}^{\mathrm{simp}}.$$

as the two operators

$$\Delta_{\mathrm{i}\mathbb{R}^+} = \sum_{\omega\in 2\pi\mathrm{i}\,\mathbb{N}^*} \dot{\Delta}_\omega, \qquad \text{resp.} \quad \Delta_{-\mathrm{i}\mathbb{R}^+} = \sum_{\omega\in -2\pi\mathrm{i}\,\mathbb{N}^*} \dot{\Delta}_\omega,$$

then the Borel-Laplace summation in a direction θ slightly smaller than $\pm\pi/2$ is equivalent to the composition of the Borel-Laplace summation in a direction θ' slightly larger than $\pm\pi/2$ with the exponential of the directional alien derivation associated with $\mathrm{e}^{\pm\mathrm{i}\pi/2}\mathbb{R}^+$:

$$\mathcal{L}^\theta \circ \mathcal{B} \sim \mathcal{L}^{\theta'} \circ \mathcal{B} \circ \exp\left(\Delta_{\pm\mathrm{i}\mathbb{R}^+}\right) = \mathcal{L}^{\theta'} \circ \mathcal{B} \circ \left(\sum_{r\geq 0} \frac{1}{r!} \Delta^r_{\pm\mathrm{i}\mathbb{R}^+} \right).$$

The symbol $\sim$ here means that the operators on both sides should coincide when applied to formal sums ("transseries") $\varphi = \sum \mathrm{e}^{-\omega z}\,\tilde{\varphi}_\omega(z)$ and yield

$$\sum \mathrm{e}^{-\omega z}\,\mathcal{L}^\theta \hat{\varphi}_\omega = \sum \mathrm{e}^{-\omega z}\,\mathcal{L}^{\theta'} \hat{\psi}_\omega, \qquad \tilde{\psi}_\omega = \sum_{r\geq 0} \frac{1}{r!} \sum_{\omega_0+\omega_1+\cdots+\omega_r=\omega} \Delta_{\omega_r} \cdots \Delta_{\omega_1} \tilde{\varphi}_{\omega_0}, \tag{47}$$

whenever both sides of the first equation in (47) have an analytical meaning (with all indices ω_j in $2\pi\mathrm{i}\,\mathbb{N}^*$, or all ω_j in $-2\pi\mathrm{i}\,\mathbb{N}^*$, with the exception of ω_0 and ω which may vanish; observe that each $\tilde{\psi}_\omega$ is defined by a finite sum).

In our case, the recipe yields

$$u^+ = \mathcal{L}^{\theta'}\mathcal{B}\left[\exp\left(\Delta_{\mathrm{i}\mathbb{R}^+}\right)\tilde{u}\right], \qquad v^+ = \mathcal{L}^{\theta'}\mathcal{B}\left[\exp\left(\Delta_{\mathrm{i}\mathbb{R}^+}\right)\tilde{v}\right], \qquad \Im m\, z < -\tau_0,$$
$$u^- = \mathcal{L}^{\pi+\theta'}\mathcal{B}\left[\exp\left(\Delta_{-\mathrm{i}\mathbb{R}^+}\right)\tilde{u}\right]; \quad v^- = \mathcal{L}^{\pi+\theta'}\mathcal{B}\left[\exp\left(\Delta_{-\mathrm{i}\mathbb{R}^+}\right)\tilde{v}\right], \qquad \Im m\, z > \tau_0,$$

with θ' slightly larger than $\pi/2$. To interpret and understand this, one can remember the computation which was made in Section 1.2 in the case of the linear difference equation (5): we had $\tilde{\varphi}(z+1) - \tilde{\varphi}(z) = a(z) \in z^{-2}\mathbb{C}\{z^{-1}\}$, hence $\Delta_\omega\tilde{\varphi} = A_\omega$ had to be constant (indeed, $A_\omega = -2\pi\mathrm{i}\,\hat{a}(\omega)$), thus $\Delta_{\mathrm{i}\mathbb{R}^+}\tilde{\varphi} = \sum_{\omega\in 2\pi\mathrm{i}\,\mathbb{N}^*} A_\omega\,\mathrm{e}^{-\omega z}$ and $\Delta^r_{\mathrm{i}\mathbb{R}^+}\tilde{\varphi} = 0$ for $r \geq 2$, formula (7) (which was nothing but the residuum formula) can thus be interpreted as

$$\mathcal{L}^\theta\mathcal{B}\tilde{\varphi} = \varphi^+ = \varphi^- + \mathcal{L}^{\theta'}\mathcal{B}\,\Delta_{\mathrm{i}\mathbb{R}^+}\tilde{\varphi} = \mathcal{L}^{\theta'}\mathcal{B}\exp\left(\Delta_{\mathrm{i}\mathbb{R}^+}\right)\tilde{\varphi} \quad \text{in } \{\Im m\, z < -\tau\}.$$

In the case of $\tilde{u}$ or $\tilde{v}$, we need to take into account the action of $\Delta^r_{\pm\mathrm{i}\mathbb{R}^+}$ for $r \geq 2$, but formulas (44) and (45) allow us to perform the calculation.

Let us begin with $\tilde{u}$: we have $\exp\left(\Delta_{\pm\mathrm{i}\mathbb{R}^+}\right)\tilde{u} = \exp\left(D_{\pm\mathrm{i}\mathbb{R}^+}\right)\tilde{u}$, with

$$D_{\mathrm{i}\mathbb{R}^+} = \sum_{\omega\in 2\pi\mathrm{i}\,\mathbb{N}^*} A_\omega\,\mathrm{e}^{-\omega z}\frac{\mathrm{d}}{\mathrm{d}z}, \qquad D_{-\mathrm{i}\mathbb{R}^+} = \sum_{\omega\in -2\pi\mathrm{i}\,\mathbb{N}^*} A_\omega\,\mathrm{e}^{-\omega z}\frac{\mathrm{d}}{\mathrm{d}z}.$$

The operator $D_{\pm\mathrm{i}\mathbb{R}^+}$ is a derivation of our graded algebra, *i.e.* a (formal) vector field; its exponential is thus an automorphism, which can be represented as a substitution operator:

$$\exp\left(D_{\pm\mathrm{i}\mathbb{R}^+}\right)\tilde{u} = \tilde{u} \circ P_{\pm\mathrm{i}\mathbb{R}^+}, \qquad P_{\pm\mathrm{i}\mathbb{R}^+} = \exp\left(D_{\pm\mathrm{i}\mathbb{R}^+}\right)\mathrm{Id}.$$

In fact, $P_{\pm\mathrm{i}\mathbb{R}^+}$ is the time-1 map of $D_{\pm\mathrm{i}\mathbb{R}^+}$. Writing this formal vector field in the coordinate $w_\pm = \mathrm{e}^{\pm 2\pi\mathrm{i}z}$, as $D_\pm = \pm 2\pi\mathrm{i}\sum A_{\pm 2\pi\mathrm{i}m} w^{m+1}\frac{\mathrm{d}}{\mathrm{d}w}$, we get a tangent-to-identity transformation, which we can write as

$$\exp(D_\pm) \;:\; w \mapsto w\,\mathrm{e}^{\pm 2\pi\mathrm{i}\sum P_{\pm 2\pi\mathrm{i}m} w^m}.$$

This relation determines the coefficients P_ω in terms of the A_ω's, so that

$$P_{\pm i\mathbb{R}^+}(z) = z + \sum_{\omega\in\pm 2\pi i\,\mathbb{N}^*} P_\omega\, e^{-\omega z}.$$

The final result is

$$u^+ = u^- \circ P_{i\mathbb{R}^+} \quad \text{in } \{\Im m\, z < -\tau_0\}, \qquad u^- = u^+ \circ P_{-i\mathbb{R}^+} \quad \text{in } \{\Im m\, z > \tau_0\}. \tag{48}$$

The computation with $\tilde v$ is more direct: formula (45) shows that

$$\exp(\Delta_{\pm i\mathbb{R}^+})\,\tilde v = \tilde v - \sum_{r\geq 1}\frac{1}{r!}\sum_{\omega_1,\ldots,\omega_r\in\pm 2\pi i\,N^*} A_{\omega_1}\cdots A_{\omega_r}\,\Gamma_{\omega_1\cdots\omega_r}\, e^{-(\omega_1+\cdots+\omega_r)\tilde v} = Q_{\pm i\mathbb{R}^+}\circ\tilde v$$

with the notation $\Gamma_{\omega_1} = 1$, $\Gamma_{\omega_1\cdots\omega_r} = \omega_1(\omega+\omega_2)\cdots(\omega_1+\omega_2+\cdots+\omega_{r-1})$, and

$$Q_{\pm i\mathbb{R}^+}(z) = z + \sum_{\omega\in\pm 2\pi i\,\mathbb{N}^*} Q_\omega\, e^{-\omega z}, \qquad Q_\omega = -\sum_{r\geq 1}\frac{1}{r!}\sum_{\substack{\omega_1,\ldots,\omega_r\in\pm 2\pi i\,N^*\\ \omega_1+\cdots+\omega_r=\omega}} \Gamma_{\omega_1\cdots\omega_r}A_{\omega_1}\cdots A_{\omega_r}. \tag{49}$$

The upshot is

$$v^+ = Q_{i\mathbb{R}^+}\circ v^- \quad \text{in } \{\Im m\, z < -\tau_0\}, \qquad v^- = Q_{-i\mathbb{R}^+}\circ v^+ \quad \text{in } \{\Im m\, z > \tau_0\}. \tag{50}$$

The comparison of (48) and (50) shows that $Q_{\pm i\mathbb{R}^+} = P^{-1}_{\pm i\mathbb{R}^+} = v^+\circ u^-$ or $v^-\circ u^+$ (according to the half-plane under consideration); $Q_{\pm i\mathbb{R}^+}$ can thus be obtained as the time-1 map of $-D_{\pm i\mathbb{R}^+}$, and the relation relating the $-A_\omega$'s to the Q_ω's in (49) can thus be paralleled by a relation relating the A_ω's to the P_ω's. We arrive at

Theorem 4 *We have*

$$v^+\circ u^- = Q_{i\mathbb{R}^+} \quad \textit{with inverse} \quad v^-\circ u^+ = P_{i\mathbb{R}^+} \quad \textit{in } \{\Im m\, z < -\tau_0\}, \tag{51}$$

$$v^+\circ u^- = P_{-i\mathbb{R}^+} \quad \textit{with inverse} \quad v^-\circ u^+ = Q_{-i\mathbb{R}^+} \quad \textit{in } \{\Im m\, z > \tau_0\}, \tag{52}$$

where

$$Q_{\pm i\mathbb{R}^+} = \mathrm{Id} + \sum_{\omega\in\pm 2\pi i\,\mathbb{N}^*} Q_\omega\, e_\omega, \qquad P_{\pm i\mathbb{R}^+} = \mathrm{Id} + \sum_{\omega\in\pm 2\pi i\,\mathbb{N}^*} P_\omega\, e_\omega,$$

with the notation $e_\omega(z) = e^{-\omega z}$, and the coefficients Q_ω, P_ω depend on the coefficients of the Bridge Equation according to the formulas

$$Q_\omega = -\sum_{r\geq 1}\frac{1}{r!}\sum_{\substack{\omega_1,\ldots,\omega_r\in\pm 2\pi i\,N^*\\ \omega_1+\cdots+\omega_r=\omega}} \Gamma_{\omega_1\cdots\omega_r}A_{\omega_1}\cdots A_{\omega_r},$$

$$P_\omega = \sum_{r\geq 1}\frac{(-1)^{r-1}}{r!}\sum_{\substack{\omega_1,\ldots,\omega_r\in\pm 2\pi i\,N^*\\ \omega_1+\cdots+\omega_r=\omega}} \Gamma_{\omega_1\cdots\omega_r}A_{\omega_1}\cdots A_{\omega_r},$$

using the notation $\Gamma_{\omega_1} = 1$, $\Gamma_{\omega_1\cdots\omega_r} = \omega_1(\omega+\omega_2)\cdots(\omega_1+\omega_2+\cdots+\omega_{r-1})$.

The comparison with (24)–(25) now shows that $Q_{\mathrm{i}\mathbb{R}^+} = \mathrm{Id} + \chi^{\mathrm{low}}$ and $P_{-\mathrm{i}\mathbb{R}^+} = \mathrm{Id} + \chi^{\mathrm{up}}$, hence

$$B_m = Q_{2\pi \mathrm{i} m}, \quad B_{-m} = P_{-2\pi \mathrm{i} m}, \qquad m \geq 1.$$

The fact that the A_ω's constitute a complete set of analytic invariants can be found again this way.

Observe that $P_{\pm\mathrm{i}\mathbb{R}^+}$ or $Q_{\pm\mathrm{i}\mathbb{R}^+}$ was obtained as a formal Fourier series, being the time-1 map of the formal vector field $D_{\pm\mathrm{i}\mathbb{R}^+}$ or $-D_{\pm\mathrm{i}\mathbb{R}^+}$. However, when identified with $v^+ \circ u^-$ or $v^- \circ u^+$, these expansions prove to be convergent. This is the growth constraint we were alluding to: the A_ω's must be such that $Q_{\pm\mathrm{i}\mathbb{R}^+}$ defined by (49) be convergent.

Alien derivations as components of the logarithm of the Stokes automorphism

Let

$$\mathcal{S}_{\mathrm{i}\mathbb{R}^+} = \mathrm{Id} + \sum_{\omega \in 2\pi\mathrm{i}\,\mathbb{N}^*} \mathrm{e}^{-\omega z} \Delta^+_\omega, \quad \mathcal{S}_{-\mathrm{i}\mathbb{R}^+} = \mathrm{Id} + \sum_{\omega \in -2\pi\mathrm{i}\,\mathbb{N}^*} \mathrm{e}^{-\omega z} \Delta^+_\omega.$$

Lemma 4 can be rephrased[18] by saying that $\mathcal{S}_{\pm\mathrm{i}\mathbb{R}^+}$ is an automorphism of the graded algebra $\widetilde{\mathrm{RES}}^{\mathrm{simp}}[[\mathrm{e}^{\mp 2\pi\mathrm{i}\,z}]]$. We close this chapter with two things.

i) We shall indicate the proof of Lemma 5, the content of which can be rephrased[19] as identities between operators of $\widetilde{\mathrm{RES}}^{\mathrm{simp}}[[\mathrm{e}^{\mp 2\pi\mathrm{i}\,z}]]$: *the directional alien derivations satisfy*

$$\Delta_{\mathrm{i}\mathbb{R}^+} = \log \mathcal{S}_{\mathrm{i}\mathbb{R}^+}, \qquad \Delta_{-\mathrm{i}\mathbb{R}^+} = \log \mathcal{S}_{-\mathrm{i}\mathbb{R}^+}.$$

This was the only step missing in the proof of Proposition 6 (the fact that the Δ_ω's are derivations follows).

ii) We shall interpret the operators $\mathcal{S}_{\mathrm{i}\mathbb{R}^+}$ and $\mathcal{S}_{-\mathrm{i}\mathbb{R}^+}$ as "Stokes automorphisms" (or "passage automorphisms", [CNP93]): *they correspond to composing the Laplace transform in a direction with the inverse Laplace transform in another direction.* This will serve as a justification of the recipe which led us to Theorem 4 in the previous section, since the exponential of $\Delta_{\pm\mathrm{i}\mathbb{R}^+}$ will then appear as the link between Borel-Laplace summations in different directions.

Let us focus on the singular direction $\mathrm{i}\mathbb{R}^+$ (the case of $-\mathrm{i}\mathbb{R}^+$ is analogous) and introduce a graded algebra which corresponds to $\widetilde{\mathrm{RES}}^{\mathrm{simp}}[[\mathrm{e}^{-2\pi\mathrm{i}\,z}]]$ via formal Borel transform. For each $\omega \in 2\pi\mathrm{i}\,\mathbb{N}$, we define the translation operator

$$\tau_\omega : \; c\,\delta + \hat{\varphi} \in \mathrm{RES}^{\mathrm{simp}} \mapsto c\,\delta_\omega + \hat{\varphi}_\omega, \qquad \hat{\varphi}_\omega(\zeta) = \hat{\varphi}(\zeta - \omega),$$

[18]Here, as in footnote 17, we extend these operators from $\widetilde{\mathrm{RES}}^{\mathrm{simp}}$ to $\widetilde{\mathrm{RES}}^{\mathrm{simp}}[[\mathrm{e}^{\mp 2\pi\mathrm{i}\,z}]]$ by declaring that they commute with multiplication by $\mathrm{e}^{-\omega_0 z}$ for any ω_0. The Borel counterparts of the relations (35) are obtained by projecting the automorphism property

$$\mathcal{S}_{\pm\mathrm{i}\mathbb{R}^+}(\tilde{\chi}_1\tilde{\chi}_2) = (\mathcal{S}_{\pm\mathrm{i}\mathbb{R}^+}\tilde{\chi}_1)(\mathcal{S}_{\pm\mathrm{i}\mathbb{R}^+}\tilde{\chi}_2), \qquad \tilde{\chi}_1, \tilde{\chi}_2 \in \widetilde{\mathrm{RES}}^{\mathrm{simp}}[[\mathrm{e}^{\mp 2\pi\mathrm{i}\,z}]]$$

onto the spaces $\mathrm{e}^{-\omega_0 z}\,\widetilde{\mathrm{RES}}^{\mathrm{simp}}$, $\omega_0 \in \pm 2\pi\mathrm{i}\,\mathbb{N}$.

[19]Formula (40) of Lemma 5 simply expresses the fact that the ω-homogeneous component of $\Delta_{\pm\mathrm{i}\mathbb{R}^+}$ coincides with that of

$$\log \mathcal{S}_{\pm\mathrm{i}\mathbb{R}^+} = \sum_{r \geq 1} \frac{(-1)^{r-1}}{r} \Big(\sum_{m \geq 1} \mathrm{e}^{-2\pi\mathrm{i} m z} \Delta^+_{\pm 2\pi\mathrm{i} m} \Big)^r.$$

where δ_ω is a symbol to be identified with $\mathcal{B}(\mathrm{e}^{-\omega z})$ and $\hat{\varphi}_\omega$ is to be thought of as a holomorphic function based at ω (well-defined on $]\omega - 2\pi\mathrm{i}, \omega + 2\pi\mathrm{i}[$, with multivalued analytic continuation on the rest of the singular direction $\mathrm{i}\mathbb{R}^+$); the range of τ_ω will be the space $\hat{\mathcal{R}}^\omega$ of ω-homogeneous elements. We consider

$$\hat{\mathcal{R}} = \underset{\omega\in 2\pi\mathrm{i}\,\mathbb{N}}{\oplus} \hat{\mathcal{R}}^\omega, \qquad \hat{\mathcal{R}}^\omega = \tau_\omega\left(\mathrm{RES}^{\mathrm{simp}}\right),$$

as a graded algebra by defining the (convolution) product of two homogeneous elements to be

$$\tau_{\omega_1}\hat{\chi}_1 * \tau_{\omega_2}\hat{\chi}_2 = \tau_{\omega_1+\omega_2}(\hat{\chi}_1 * \hat{\chi}_2), \qquad \hat{\chi}_1, \hat{\chi}_2 \in \mathrm{RES}^{\mathrm{simp}},\ \omega_1, \omega_2 \in 2\pi\mathrm{i}\,\mathbb{N}.$$

The operators Δ^+_ω extend uniquely from $\hat{\mathcal{R}}^0 = \mathrm{RES}^{\mathrm{simp}}$ to $\hat{\mathcal{R}}$ by declaring that they commute with all translations τ_{ω_0}. The Borel counterpart of $\mathcal{S}_{\mathrm{i}\mathbb{R}^+}$ is

$$\Delta^+ = \sum_{\omega\in 2\pi\mathrm{i}\,\mathbb{N}} \dot{\Delta}^+_\omega : \hat{\mathcal{R}} \to \hat{\mathcal{R}}, \qquad \text{with } \dot{\Delta}^+_0 = \mathrm{Id} \text{ and } \dot{\Delta}^+_\omega = \tau_\omega \Delta^+_\omega \text{ for } \omega \neq 0.$$

Each $\dot{\Delta}^+_\omega$ is thus seen as the ω-homogeneous component of the operator Δ^+.

We now introduce elementary operators $\dot{\ell}_+$, $\dot{\ell}_-$, $\dot{A}$ and μ, which will allow us to rewrite the definition (29) as

$$\Delta^+_\omega = \tau_\omega^{-1}\left(\dot{A} + \dot{\ell}_+ - \dot{\ell}_-\right)\dot{\ell}_+^{m-1}\mu, \qquad \omega = 2\pi\mathrm{i}\,m. \tag{53}$$

The first two ones are the *lateral continuation operators* and act in $\check{\mathcal{R}} = \oplus\check{\mathcal{R}}^\omega$, where $\check{\mathcal{R}}^\omega$ is the space of all the functions $\check{\varphi}_\omega$ which are holomorphic on $]\omega, \omega + 2\pi\mathrm{i}[$, can be analytically continued along any path which avoids $2\pi\mathrm{i}\,\mathbb{Z}$ and admit at worse simple singularities. The functions of $\check{\mathcal{R}}^\omega$ are unambiguously determined on $]\omega, \omega + 2\pi\mathrm{i}[$, but their analytic continuation gives rise to various branchs. Let $\lambda \in]0, \pi[$, and let γ_+, resp. γ_-, be the semi-circular path which starts from $\omega + 2\pi\mathrm{i} - \mathrm{i}\lambda$ and ends at $\omega + 2\pi\mathrm{i} + \mathrm{i}\lambda$, circumventing $\omega + 2\pi\mathrm{i}$ to the right, resp. to the the left. We define $\dot{\ell}_+$ and $\dot{\ell}_-$ to be the operators of analytic continuation along γ_+ and γ_-:

$$\dot{\ell}_\pm : \check{\varphi}_\omega \in \check{\mathcal{R}}^\omega \mapsto \mathrm{cont}_{\gamma_\pm}\check{\varphi}_\omega \in \check{\mathcal{R}}^{\omega+2\pi\mathrm{i}}.$$

These operators induce $2\pi\mathrm{i}$-homogeneous operators of $\check{\mathcal{R}}$, the difference of which sends $\check{\mathcal{R}}$ in $\hat{\mathcal{R}}$: for any $\check{\varphi}_\omega \in \check{\mathcal{R}}^\omega$, $(\dot{\ell}_+ - \dot{\ell}_-)\check{\varphi}_\omega \in \hat{\mathcal{R}}^{\omega+2\pi\mathrm{i}}$ is simply the variation of $\mathrm{sing}_{\omega+2\pi\mathrm{i}}\check{\varphi}_\omega$ translated by $\tau_{\omega+2\pi\mathrm{i}}$. Denoting by $\alpha_{\omega+2\pi\mathrm{i}}(\check{\varphi}_\omega)$ the residuum of $\mathrm{sing}_{\omega+2\pi\mathrm{i}}\check{\varphi}_\omega$, we set

$$\dot{A} : \check{\varphi}_\omega \in \check{\mathcal{R}}^\omega \mapsto \alpha_{\omega+2\pi\mathrm{i}}(\check{\varphi})\,\delta_{\omega+2\pi\mathrm{i}} \in \hat{\mathcal{R}}^{\omega+2\pi\mathrm{i}}.$$

The whole singularity $\mathrm{sing}_{\omega+2\pi\mathrm{i}}\check{\varphi}_\omega$ is thus determined by

$$\tau_{\omega+2\pi\mathrm{i}}\,\mathrm{sing}_{\omega+2\pi\mathrm{i}}\check{\varphi}_\omega = \left(\dot{A} + \dot{\ell}_+ - \dot{\ell}_-\right)\check{\varphi}_\omega \in \hat{\mathcal{R}}^{\omega+2\pi\mathrm{i}}.$$

We define the last new elementary operator of the list by

$$\mu : \hat{\chi}_\omega = c\,\delta_\omega + \hat{\varphi}_\omega \in \hat{\mathcal{R}}^\omega \mapsto \hat{\varphi}_\omega \in \check{\mathcal{R}}^\omega,$$

i.e. we forget the multiple of δ_ω, retaining only $\hat{\varphi}_\omega$ but forgetting that this function is regular at ω: the result is considered as element of $\check{\mathcal{R}}^\omega$. Formula (53) is now an obvious translation of Definition 8. We thus have

$$\Delta^+ = \mathrm{Id} + \sum_{m\geq 1}\left(\dot{A} + \dot{\ell}_+ - \dot{\ell}_-\right)\dot{\ell}_+^{m-1}\mu. \tag{54}$$

In this framework, we can also rephrase the part of Definition 8 concerning Δ_ω, or rather the dotted version of the operator (still in the convolutive model):

$$\overset{\bullet}{\Delta}_\omega = \tau_\omega \Delta_\omega = \sum_{\varepsilon_1,\dots,\varepsilon_{m-1}\in\{+,-\}} \frac{p(\varepsilon)!q(\varepsilon)!}{m!} \left(\dot{A} + \dot{\ell}_+ - \dot{\ell}_-\right) \dot{\ell}_{\varepsilon_{m-1}} \cdots \dot{\ell}_{\varepsilon_1} \mu, \qquad \omega = 2\pi\mathrm{i}\, m.$$

We now give the proof of Lemma 5, which amounts to the fact that the (Borel counterpart of the) directional alien derivation

$$\Delta = \sum_{\omega\in 2\pi\mathrm{i}\,\mathbb{N}^*} \overset{\bullet}{\Delta}_\omega$$

is the logarithm of Δ^+. *We thus must show that, for each $\omega \in 2\pi\mathrm{i}\,\mathbb{N}^*$, $\overset{\bullet}{\Delta}_\omega$ is the ω-homogeneous component of the operator*

$$\log \Delta^+ = \sum_{r\geq 1} \frac{(-1)^{r-1}}{r} \left(\sum_{m\geq 0} \left(\dot{A} + \dot{\ell}_+ - \dot{\ell}_-\right) \dot{\ell}_+^m \mu \right)^r.$$

Using the obvious identity $\mu\left(\dot{A} + \dot{\ell}_+ - \dot{\ell}_-\right) = \dot{\ell}_+ - \dot{\ell}_-$, we can write

$$\log \Delta^+ = \sum_{\substack{m_1,\dots,m_r\geq 0\\ r\geq 1}} \frac{(-1)^{r-1}}{r} (\dot{A} + \dot{\ell}_+ - \dot{\ell}_-)\dot{\ell}_+^{m_1}\mu(\dot{A} + \dot{\ell}_+ - \dot{\ell}_-)\dot{\ell}_+^{m_2}\mu \cdots (\dot{A} + \dot{\ell}_+ - \dot{\ell}_-)\dot{\ell}_+^{m_r}\mu$$

as $\sum_{m\geq 1} (\dot{A} + \dot{\ell}_+ - \dot{\ell}_-) B_{m-1}\mu$, with $2\pi\mathrm{i}(m-1)$-homogeneous operators

$$B_{m-1} = \sum_{\substack{m_1+\dots+m_r+r=m\\ m_1,\dots,m_r\geq 0,\ r\geq 1}} \frac{(-1)^{r-1}}{r} \dot{\ell}_+^{m_1}(\dot{\ell}_+ - \dot{\ell}_-)\dot{\ell}_+^{m_2} \cdots (\dot{\ell}_+ - \dot{\ell}_-)\dot{\ell}_+^{m_r}.$$

The result follows from the following identity (which is an identity for polynomials in two non-commutative variables):

$$B_{m-1} = \sum_{\varepsilon_1,\dots,\varepsilon_{m-1}\in\{+,-\}} \frac{p(\varepsilon)!q(\varepsilon)!}{m!} \dot{\ell}_{\varepsilon_{m-1}} \cdots \dot{\ell}_{\varepsilon_1},$$

the proof of which is left to the reader.

Formula (30) of Proposition 6 follows, because the logarithm of an automorphism is a derivation, and the homogeneous components of a derivation are also derivations.

As promised, we end this section with the interpretation of Δ^+ as Stokes automorphism. Let us extend $\mathcal{L}^- = \mathcal{L}^{\theta'}$ for $\theta' \in \left]\frac{\pi}{2}, \pi\right[$ to the part $\hat{\mathcal{R}}_{\exp}$ of $\hat{\mathcal{R}}$ consisting of the formal sums $\sum(c_\omega\,\delta_\omega + \hat{\varphi}_\omega)$ in which each $\hat{\varphi}_\omega$ has at most exponential growth at infinity, by setting

$$\mathcal{L}^- \sum_{\omega\in 2\pi\mathrm{i}\,\mathbb{N}} (c_\omega\,\delta_\omega + \hat{\varphi}_\omega) = \sum_{\omega\in 2\pi\mathrm{i}\,\mathbb{N}} \left(c_\omega\,\mathrm{e}^{-\omega z} + \int_\omega^{\mathrm{e}^{\mathrm{i}\theta'}\infty} \mathrm{e}^{-z\zeta}\hat{\varphi}_\omega(\zeta)\,\mathrm{d}\zeta\right)$$

$$= \sum_{\omega\in 2\pi\mathrm{i}\,\mathbb{N}} \mathrm{e}^{-\omega z}\left(c_\omega + \int_0^{\mathrm{e}^{\mathrm{i}\theta'}\infty} \mathrm{e}^{-z\xi}\hat{\varphi}_\omega(\omega+\xi)\,\mathrm{d}\xi\right),$$

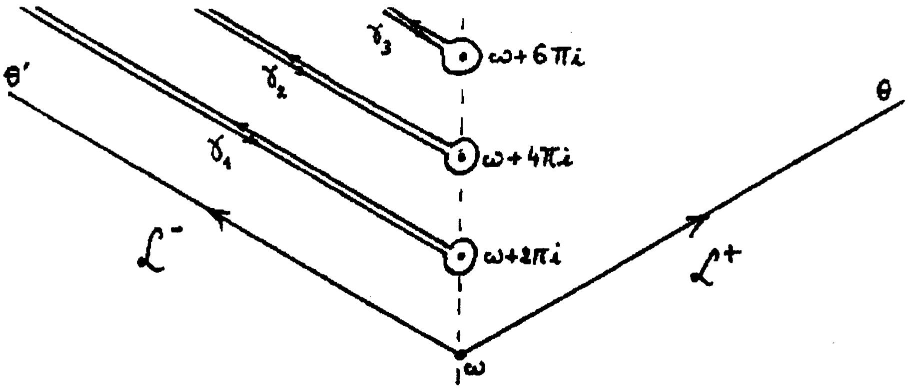


Figure 11: Illustration of the formula $\mathcal{L}^- \circ \Delta^+ = \mathcal{L}^+$.

where the right-hand side is to be considered as a formal series in $\mathrm{e}^{-2\pi i z}$ with coefficients holomorphic in a domain $\mathcal{D}^-$. We define $\mathcal{L}^+ = \mathcal{L}^\theta$ for $\theta \in \left]0, \frac{\pi}{2}\right[$ similarly on $\hat{\mathcal{R}}_{\exp}$. We have

$$\mathcal{L}^+\hat{\chi}_\omega = \mathcal{L}^-\Delta^+\hat{\chi}_\omega, \qquad \hat{\chi}_\omega \in \hat{\mathcal{R}}^\omega_{\exp}, \tag{55}$$

at least when the series $\sum \mathrm{e}^{-2\pi i m z}\mathcal{L}^-\Delta^+_{2\pi i m z}\hat{\chi}_\omega$ is convergent. Indeed, one can decompose the contour of integration for $\mathcal{L}^+$ as follows:

$$\mathcal{L}^+\hat{\chi}_\omega = c_\omega\,\mathrm{e}^{-\omega z} + \left(\int_\omega^{\mathrm{e}^{i\theta'}\infty} + \int_{\gamma_1} + \int_{\gamma_2} + \cdots\right)\mathrm{e}^{-z\zeta}\hat{\varphi}_\omega(\zeta)\,\mathrm{d}\zeta = \mathcal{L}^-\hat{\chi}_\omega + \sum_{m\geq 1}\int_{\gamma_m}\mathrm{e}^{-z\zeta}\hat{\varphi}_\omega(\zeta)\,\mathrm{d}\zeta,$$

where $\gamma_m = \omega + 2\pi i\, m + \gamma$ and γ is the path coming from $\mathrm{e}^{i\theta'}\infty$, turning anticlockwise around 0 and going back to $\mathrm{e}^{i\theta'}\infty$ (see Figure 11), and the contribution of γ_m is precisely (because of the residuum formula)

$$\alpha_{\omega+2\pi i m}(\ell_+^{m-1}\mu\hat{\chi}_\omega)\,\mathrm{e}^{-(\omega+2\pi i m)z} + \int_{\omega+2\pi i m}^{\mathrm{e}^{i\theta'}\infty}\mathrm{e}^{-z\zeta}\,(\dot{\ell}_+ - \dot{\ell}_-)\ell_+^{m-1}\mu\hat{\chi}_\omega(\zeta)\,\mathrm{d}\zeta =$$

$$\mathcal{L}^-\left(\dot{A} + \dot{\ell}_+ - \dot{\ell}_-\right)\ell_+^{m-1}\mu\hat{\chi}_\omega = \mathcal{L}^-\,\dot{\Delta}^+_{2\pi i m}\hat{\chi}_\omega.$$

Formula (55) may serve as a heuristic explanation of the fact that $\Delta^+ =$ "$(\mathcal{L}^-)^{-1} \circ \mathcal{L}^+$" is an automorphism, since both $\mathcal{L}^+$ and $\mathcal{L}^-$ transform convolution into multiplication. This formula is the expression of an "abstract Stokes phenomenon", without reference to any particular equation, which manifests itself in the Stokes phenomenon when specialized to the resurgent solution of an equation like (18) or (19). It was used in the form $\mathcal{L}^+ = \mathcal{L}^- \circ (\exp\Delta)$ in the previous section.

3 Formalism of singularities, general resurgent functions and alien derivations

Let us return to the convolution algebra $\widehat{\mathcal{H}}(\mathcal{R})$, consisting of all holomorphic germs at the origin which extend to the Riemann surface $\mathcal{R}$. In Section 2.3 we have focused on simple singularities and this led to the definition of $\mathrm{RES}^{\mathrm{simp}}$, but what about more complicated singularities than simple ones?

Even in the elementary situation described in Section 1.2 with the Borel transform $\hat{\psi}(\zeta)$ of the solution of the second linear equation (6), poles of order higher than 1 appear.

Or, as alluded to in Section 2.1 after Proposition 4, in the case of a nonzero resiter ρ Abel's equation has a solution of the form $\mathrm{Id} + \rho \log z + \tilde{\psi}(z)$ with $\tilde{\psi} \in z^{-1}\mathbb{C}[[z^{-1}]]$. One can prove that $\tilde{\psi} \in \tilde{\mathcal{H}}$, *i.e.* the Borel transform $\hat{\psi}$ is in $\widehat{\mathcal{H}}(\mathcal{R})$, but the singularities one finds in the Borel plane can be of the form $\zeta^{\alpha}\hat{\Phi}(\zeta) + \mathrm{reg}(\zeta)$ with $\alpha \in \mathbb{C}$ and $\hat{\Phi}(\zeta)$, $\mathrm{reg}(\zeta)$ regular at the origin (see [Eca81, Vol. 2]).

It may also happen that an equation give rise to formal solutions involving non-integer powers of z, or monomials of the form $z^{-n}(\log z)^m$.

All these issues are addressed satisfactorily by the formalism of singularities, as developed in [Eca81, Vol. 3], [Eca92] or [Eca93] (see also [CNP93] and [OSS03]).

3.1 General singularities. Majors and minors. Integrable singularities

Let $\underset{\bullet}{\mathbb{C}}$ denote the Riemann surface of the logarithm, *i.e.*

$$\underset{\bullet}{\mathbb{C}} = (\widetilde{\mathbb{C} \setminus \{0\}, 1}) = \{\zeta = r\,\mathrm{e}^{\mathrm{i}\theta} \mid r > 0,\ \theta \in \mathbb{R}\}$$

(*cf.* footnote 3 in Section 1.3). We denote by $\zeta \in \underset{\bullet}{\mathbb{C}} \mapsto \overset{\bullet}{\zeta} \in \mathbb{C} \setminus \{0\}$ the canonical projection (covering map).[20] We are interested in analytic functions which are potentially singular at the origin of $\mathbb{C}$, possibly with multivalued analytic continuation around the origin. We thus define ANA to be the space of the germs of functions analytic in a "spiralling neighbourhood of the origin", *i.e.* analytic in a domain of the form $\mathcal{V} = \{r\,\mathrm{e}^{\mathrm{i}\theta} \mid 0 < r < h(\theta),\ \theta \in \mathbb{R}\} \subset \underset{\bullet}{\mathbb{C}}$, with a continuous function $h : \mathbb{R} \to]0, +\infty[$. The space $\mathbb{C}\{\zeta\}$ of regular germs is obviously a subspace of ANA.

Definition 9 *Let* $\mathrm{SING} = \mathrm{ANA}/\mathbb{C}\{\zeta\}$. *The elements of this space are called "singularities".*[21] *The canonical projection is denoted* sing_0 *and we use the notation*

$$\mathrm{sing}_0 : \begin{cases} \mathrm{ANA} \to \mathrm{SING} \\ \overset{\vee}{\varphi} \ \mapsto \overset{\triangledown}{\varphi} = \mathrm{sing}_0(\overset{\vee}{\varphi}(\zeta)). \end{cases}$$

Any representative $\overset{\vee}{\varphi}$ *of a singularity* $\overset{\triangledown}{\varphi}$ *is called a "major" of this singularity.*

The map induced by the variation map $\overset{\vee}{\varphi}(\zeta) \mapsto \overset{\vee}{\varphi}(\zeta) - \overset{\vee}{\varphi}(\zeta\,\mathrm{e}^{-2\pi\mathrm{i}})$ *is denoted*

$$\mathrm{var} : \begin{cases} \mathrm{SING} \ \to \ \mathrm{ANA} \\ \overset{\triangledown}{\varphi} = \mathrm{sing}_0(\overset{\vee}{\varphi}) \mapsto \hat{\varphi}(\zeta) = \overset{\vee}{\varphi}(\zeta) - \overset{\vee}{\varphi}(\zeta\,\mathrm{e}^{-2\pi\mathrm{i}}). \end{cases}$$

The germ $\hat{\varphi} = \mathrm{var}\,\overset{\triangledown}{\varphi}$ *is called the "minor" of the singularity* $\overset{\triangledown}{\varphi}$.

[20] As a Riemann surface, $\underset{\bullet}{\mathbb{C}}$ is isomorphic to $\mathbb{C}$ (with a biholomorphism $\zeta \in \underset{\bullet}{\mathbb{C}} \mapsto \log\zeta \in \mathbb{C}$), but it is more significant for us to consider it as a universal cover with a "multivalued coordinate" ζ.

[21] A more general definition is given in [OSS03], with sectorial neighbourhoods of the form $\mathcal{V} = \{r\,\mathrm{e}^{\mathrm{i}\theta} \mid \theta_0 - \alpha - 2\pi < \theta < \theta_0 + \alpha,\ 0 < r < h\}$ instead of spiralling neighbourhoods, giving rise to more general germs $\overset{\vee}{\varphi} \in \mathrm{ANA}_{\theta_0,\alpha}$ and singularities $\overset{\triangledown}{\varphi} \in \mathrm{SING}_{\theta_0,\alpha} = \mathrm{ANA}_{\theta_0,\alpha}/\mathbb{C}\{\zeta\}$. In practice, this is useful as an intermediary step to prove that the solution of a nonlinear equation is resurgent, but here we simplify the exposition.

Observe that the kernel of var : SING $\to$ ANA is isomorphic to the space of entire functions of $\frac{1}{\zeta}$ without constant term. It turns out that the map var is surjective (we omit the proof).

The simplest examples of singularities are poles

$$\delta = \operatorname{sing}_0\left(\frac{1}{2\pi i\zeta}\right), \qquad \delta^{(n)} = \operatorname{sing}_0\left(\frac{(-1)^n n!}{2\pi i \zeta^{n+1}}\right), \qquad n \geq 0$$

(observe that $\operatorname{var}\delta^{(n)} = 0$), and logarithmic singularities with regular variation, for which we use the notation

$$^{\flat}\hat{\varphi} = \operatorname{sing}_0\left(\frac{1}{2\pi i}\hat{\varphi}(\zeta)\log\zeta\right), \qquad \hat{\varphi}(\zeta) \in \mathbb{C}\{\zeta\} \tag{56}$$

($\log\zeta$ is well-defined since we work in $\underset{\bullet}{\mathbb{C}}$; anyway, another branch of the logarithm would define the same singularity since the difference of majors would be a multiple of $\hat{\varphi}(\zeta)$ which is regular). The last example is a particular case of "integrable singularity".

Definition 10 *An "integrable minor" is a germ $\hat{\varphi} \in$ ANA which is uniformly integrable at the origin in any sector $\theta_1 \leq \arg\zeta \leq \theta_2$, in the sense that for any $\theta_1 < \theta_2$ there exists a Lebesgue integrable function $f :]0, r^*] \to]0, +\infty[$ such that*

$$|\hat{\varphi}(\zeta)| \leq f(|\zeta|), \qquad \zeta \in S,$$

where $S = \{\zeta \in \underset{\bullet}{\mathbb{C}} \mid \theta_1 \leq \arg\zeta \leq \theta_2,\ |\zeta| \leq r^*\}$ and $r^* > 0$ is small enough so as to ensure that S be contained in the domain of analyticity of $\hat{\varphi}$. The corresponding subspace of ANA *is denoted by* $\mathrm{ANA}^{\mathrm{int}}$.

An "integrable singularity" is a singularity $\overset{\nabla}{\varphi} \in$ SING which admits a major $\check{\varphi}$ such that $\zeta\check{\varphi}(\zeta) \to 0$ as $\zeta \to 0$ uniformly in any sector $\theta_1 \leq \arg\zeta \leq \theta_2$ and for which $\operatorname{var}\overset{\nabla}{\varphi} \in \mathrm{ANA}^{\mathrm{int}}$. *The corresponding subspace of* SING *is denoted by* $\mathrm{SING}^{\mathrm{int}}$.

For example, the formulas

$$\overset{\nabla}{I}_\sigma = \operatorname{sing}_0(\check{I}_\sigma), \qquad \check{I}_\sigma(\zeta) = \frac{\zeta^{\sigma-1}}{(1 - e^{-2\pi i\sigma})\Gamma(\sigma)}, \qquad \sigma \in \mathbb{C} \setminus \mathbb{N}^* \tag{57}$$

define a family of singularities[22] among which the integrable ones correspond to $\Re e\,\sigma > 0$. Another example is provided by polynomials of $\log\zeta$, which can be viewed as integrable minors, and also as majors of integrable singularities.

---

[22] In view of the poles of the Gamma function, $\overset{\nabla}{I}_\sigma$ is well-defined for $\sigma = -n \in -\mathbb{N}$, and $\overset{\nabla}{I}_{-n} = \delta^{(n)}$. Besides, the reflection formula yields

$$\check{I}_\sigma(\zeta) = \tfrac{1}{2\pi i} e^{\pi i\sigma}\Gamma(1-\sigma)\zeta^{\sigma-1}.$$

This family of singularities admits a non-trivial analytic continuation with respect to σ at positive integers ([Eca81, Vol. 1, pp. 47–51]): for $n \in \mathbb{N}^*$, one may consider another major of $\overset{\nabla}{I}_\sigma$, which is analytic at $\sigma = n$, and define

$$\check{I}_n(\zeta) = \lim_{\sigma\to n} \frac{\zeta^{\sigma-1} - \zeta^{n-1}}{(1 - e^{-2\pi i\sigma})\Gamma(\sigma)} = \frac{\zeta^{n-1}\log\zeta}{2\pi i\,\Gamma(n)},$$

which yields $\overset{\nabla}{I}_n = {}^{\flat}\!\left(\frac{\zeta^{n-1}}{(n-1)!}\right)$. The formula for the minors

$$\hat{I}_\sigma(\zeta) = \frac{\zeta^{\sigma-1}}{\Gamma(\sigma)}$$

is thus valid for any $\sigma \in \mathbb{C} \setminus (-\mathbb{N})$ (while $\hat{I}_{-n} = 0$ if $n \in \mathbb{N}$).

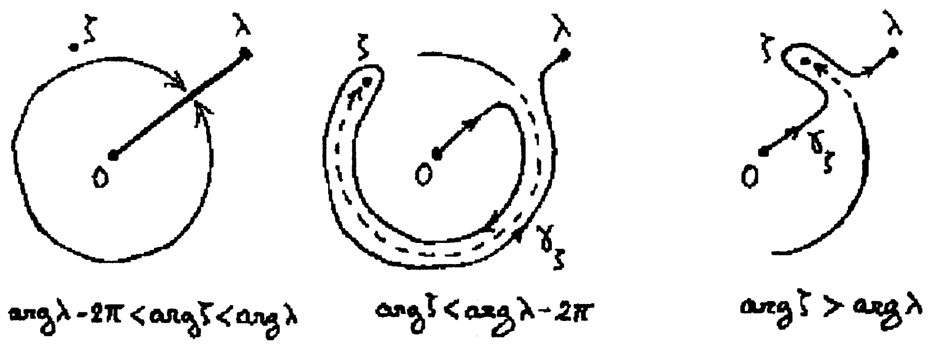


Figure 12: Cauchy integral for a major associated with an integrable minor.

Observe that, when a singularity $\overset{\triangledown}{\varphi}$ is integrable, any of its majors satisfies the condition which is stated in Definition 10 (since the difference between two majors, being regular, is $o(1/|\zeta|)$), and that its minor is by assumption an integrable minor.

Lemma 6 *By restriction, the variation map* var : SING $\to$ ANA *induces a linear isomorphism* $\mathrm{SING}^{\mathrm{int}} \to \mathrm{ANA}^{\mathrm{int}}$.

Proof. In view of the definition of $\mathrm{SING}^{\mathrm{int}}$, the variation map induces a linear map $\mathrm{var}^{\mathrm{int}}$: $\mathrm{SING}^{\mathrm{int}} \to \mathrm{ANA}^{\mathrm{int}}$. The injectivity of $\mathrm{var}^{\mathrm{int}}$ is obvious: $\mathrm{sing}_0(\check{\varphi})$ belongs to the kernel of var if and only if $\check{\varphi}(\zeta)$ is the sum of an entire function of $1/\zeta$ and of a regular germ, and the condition $\check{\varphi}(\zeta) = o(1/|\zeta|)$ leaves room for the regular germ only.

For the surjectivity, we suppose $\hat{\varphi} \in \mathrm{ANA}^{\mathrm{int}}$ and we only need to exhibit a germ $\check{\varphi} \in$ ANA with variation $\hat{\varphi}$ and with the property of being $o(1/|\zeta|)$ uniformly near the origin. If $\hat{\varphi}$ is regular, we can content ourselves with setting $\check{\varphi}(\zeta) = \frac{1}{2\pi \mathrm{i}}\hat{\varphi}(\zeta)\log\zeta$, but for the general case we resort to a Cauchy integral.

Let us fix an auxiliary point λ in the domain of analyticity $S \subset \underset{\bullet}{\mathbb{C}}$ of $\hat{\varphi}$. The integrability of $\hat{\varphi}$ allows us to define a holomorphic function by the formula

$$\check{\varphi}(\zeta) = -\frac{1}{2\pi \mathrm{i}} \int_0^\lambda \frac{\hat{\varphi}(\zeta_1)}{\zeta_1 - \zeta}\, \mathrm{d}\zeta_1, \qquad \arg\lambda - 2\pi < \arg\zeta < \arg\lambda. \tag{58}$$

This function admits an analytic continuation to $S \setminus \lambda\,[1, +\infty[$, as is easily seen by deforming the path of integration (see Figure 12):

$$\check{\varphi}(\zeta) = -\frac{1}{2\pi \mathrm{i}} \int_{\gamma_\zeta} \frac{\hat{\varphi}(\zeta_1)}{\zeta_1 - \zeta}\, \mathrm{d}\zeta_1$$

with a path γ_ζ connecting 0 and λ inside S and circumventing ζ to the right if $\arg\zeta < \arg\lambda - 2\pi$ or to the left if $\arg\zeta > \arg\lambda$ (turning around the origin as many times as necessary to reach the sheet of $\underset{\bullet}{\mathbb{C}}$ where ζ lies before turning back to sheet of λ). For $\arg\zeta$ slightly larger than $\arg\lambda$ and $|\zeta|$ small enough, the residuum formula yields

$$\check{\varphi}(\zeta) - \check{\varphi}(\zeta\, \mathrm{e}^{-2\pi \mathrm{i}}) = -\frac{1}{2\pi \mathrm{i}} \left(\int_{\gamma_\zeta} - \int_0^\lambda \right) \frac{\hat{\varphi}(\zeta_1)}{\zeta_1 - \zeta}\, \mathrm{d}\zeta_1 = \hat{\varphi}(\zeta)$$

(because this difference of paths is just a circle around ζ with clockwise orientation).

For the uniform $o(1/|\zeta|)$ property, it is sufficient to check that $\zeta\check{\varphi}(\zeta) \to 0$ as $\zeta \to 0$ uniformly in $\Sigma_{\lambda,\alpha} = \{\, \zeta \in \underset{\bullet}{\mathbb{C}} \mid \arg\lambda - 2\pi + \alpha \le \arg\zeta \le \arg\lambda - \alpha \,\}$, with arbitrarily small $\alpha > 0$, since changing λ only amounts to adding a regular germ to $\check{\varphi}$. We write $\zeta\check{\varphi}(\zeta) = -\frac{\lambda}{2\pi i}\int_0^1 \hat{\varphi}(t\lambda)\frac{\zeta}{t\lambda-\zeta}dt$ and observe that, for $\zeta \in \Sigma_{\lambda,\alpha}$, $\cos\big(\arg\frac{\lambda}{\zeta}\big) \le \cos\alpha$ hence

$$\left|t\tfrac{\lambda}{\zeta} - 1\right|^2 = t^2\left|\tfrac{\lambda}{\zeta}\right|^2 + 1 - 2t\left|\tfrac{\lambda}{\zeta}\right|\cos\big(\arg\tfrac{\lambda}{\zeta}\big) \ge F(t,|\zeta|),$$

$$F(t,r) = t^2\tfrac{|\lambda|^2}{r^2} + 1 - 2t\tfrac{|\lambda|}{r}\cos\alpha \ge \sin^2\alpha.$$

We can conclude by Lebesgue's dominated convergence theorem:

$$|\zeta\check{\varphi}(\zeta)| \le \frac{|\lambda|}{2\pi}\int_0^1 g(t,|\zeta|)\,dt, \qquad g(t,r) = \frac{1}{\sqrt{F(t,r)}}|\hat{\varphi}(t\lambda)| = \frac{r}{\big|t|\lambda|e^{i\alpha} - r\big|}|\hat{\varphi}(t\lambda)|,$$

with $g(t,r) \le \frac{1}{\sin\alpha}|\hat{\varphi}(t\lambda)|$ integrable and $g(t,r) \to 0$ as $r \to 0$ for each $t > 0$, hence $|\zeta\check{\varphi}(\zeta)| \le \varepsilon_{\lambda,\alpha}(|\zeta|)$ with $\varepsilon_{\lambda,\alpha}(r) \xrightarrow[r\to 0]{} 0$. □

Notation: The inverse map will be denoted

$$\hat{\varphi} \in \mathrm{ANA}^{\mathrm{int}} \mapsto {}^\flat\hat{\varphi} \in \mathrm{SING}^{\mathrm{int}},$$

a notation which is consistent with (56). In the spirit of Definition 6, we can define the "simple singularities" (at the origin) as those singularities of the form $c\,\delta + {}^\flat\hat{\varphi}$, with $c \in \mathbb{C}$ and $\hat{\varphi}(\zeta) \in \mathbb{C}\{\zeta\}$; we denote by $\mathrm{SING}^{\mathrm{simp}} = \mathbb{C}\,\delta \oplus {}^\flat(\mathbb{C}\{\zeta\})$ the subspace of SING that they form.

3.2 The convolution algebra SING

Starting with Section 1.3, we have dealt with convolution of regular germs. But the space $\mathbb{C}\{\zeta\}$ of regular germs is contained in the space $\mathrm{ANA}^{\mathrm{int}}$ of integrable minors, and we can extend the convolution law:

$$\hat{\varphi}_1, \hat{\varphi}_2 \in \mathrm{ANA}^{\mathrm{int}} \mapsto \hat{\varphi}_3 = \hat{\varphi}_1 * \hat{\varphi}_2 \in \mathrm{ANA}^{\mathrm{int}}, \qquad \hat{\varphi}_3(\zeta) = \int_0^\zeta \hat{\varphi}_1(\zeta_1)\hat{\varphi}_2(\zeta - \zeta_1).$$

Indeed, in any sector $\theta_1 \le \arg\zeta \le \theta_2$, using integrable functions $f_1, f_2 : \,]0, r^*] \to \,]0, +\infty[$ such that $|\hat{\varphi}_i(\zeta)| \le f_i(|\zeta|)$, we see that the formula makes sense for $|\zeta| \le r^*$ and defines a holomorphic function such that $|\hat{\varphi}_3(\zeta)| \le f_3(|\zeta|)$ with $f_3(r) = \int_0^1 f_1(tr)f_2((1-t)r)r\,dt$; the positive function $f_3 = f_1 * f_2$ itself is integrable, by virtue of the Fubini theorem: $\int_0^{r^*} f_3(r)\,dr = \iint_0^{r^*} f_1(r_1)f_2(r_2)\,dr_1\,dr_2 < \infty$.

We have for instance $\hat{I}_{\sigma_1} * \hat{I}_{\sigma_2} = \hat{I}_{\sigma_1+\sigma_2}$ for all complex σ_1, σ_2 with positive real part, whether integer or not (by the classical formula for the Beta function). The extended convolution is called "convolution of integrable minors"; it is still commutative and associative. We thus get an algebra $\mathrm{ANA}^{\mathrm{int}}$ (without unit), with $\mathbb{C}\{\zeta\}$ as a subalgebra.

Transporting this structure of algebra by $\mathrm{var}^{\mathrm{int}}$, we can view $\mathrm{SING}^{\mathrm{int}}$ as an algebra, with convolution law

$$\overset{\triangledown}{\varphi}_1 = {}^\flat\hat{\varphi}_1,\ \overset{\triangledown}{\varphi}_2 = {}^\flat\hat{\varphi}_2 \in \mathrm{SING}^{\mathrm{int}} \mapsto \overset{\triangledown}{\varphi}_1 * \overset{\triangledown}{\varphi}_2 := {}^\flat(\hat{\varphi}_1 * \hat{\varphi}_2) \in \mathrm{SING}^{\mathrm{int}}. \tag{59}$$

It turns out that the convolution law for integrable singularities can be extended to the whole space of singularities, so as to make SING an algebra, of which $\mathrm{SING}^{\mathrm{int}} = {}^\flat(\mathrm{ANA}^{\mathrm{int}})$ will appear as a subalgebra (and there will be a unit, namely δ).

Convolution with integrable singularities

As an introduction to the definition of the convolution of general singularities, let us begin with a more careful study of $\overset{\triangledown}{\varphi}_3 = \overset{\triangledown}{\varphi}_1 * \overset{\triangledown}{\varphi}_2$ in the integrable case (59): we shall indicate formulas for the minor and a major of $\overset{\triangledown}{\varphi}_3$, which do not make reference to the minor $\hat{\varphi}_1$ but only to a major of ${}^\flat\hat{\varphi}_1$.

Lemma 7 *Let $\hat{\varphi}_1, \hat{\varphi}_2 \in \mathrm{ANA}^{\mathrm{int}}$, $\hat{\varphi}_3 = \hat{\varphi}_1 * \hat{\varphi}_2$, and let $\check{\varphi}_1$ be any major of ${}^\flat\hat{\varphi}_1$. Then, for ζ with small enough modulus, one has*

$$\hat{\varphi}_3(\zeta) = \int_{\gamma_\zeta} \check{\varphi}_1(\zeta_1)\hat{\varphi}_2(\zeta - \overset{\bullet}{\zeta}_1)\,\mathrm{d}\zeta_1, \tag{60}$$

where γ_ζ is any path in $\underset{\bullet}{\mathbb{C}}$ which starts at $\zeta\,\mathrm{e}^{-2\pi\mathrm{i}}$, turns around the origin anticlockwise and ends at ζ, e.g. the circular path $t \in [0,1] \mapsto \zeta\,\mathrm{e}^{-2\pi\mathrm{i}(1-t)}$. In the above formula, $\zeta_2 = \zeta - \overset{\bullet}{\zeta}_1$ denotes the lift of $\overset{\bullet}{\zeta} - \overset{\bullet}{\zeta}_1$ which lies in the same sheet of $\underset{\bullet}{\mathbb{C}}$ as ζ; this point thus starts from and comes back to the origin after turning anticlockwise around ζ (rather than $\zeta\,\mathrm{e}^{-2\pi\mathrm{i}}$, or any other lift of $\overset{\bullet}{\zeta}$ in $\underset{\bullet}{\mathbb{C}}$).

Let $\lambda \in \underset{\bullet}{\mathbb{C}}$ belong to the domain of analyticity of $\check{\varphi}_1$, with small enough modulus so that $\lambda\,\mathrm{e}^{\mathrm{i}\pi}$ belongs to the domain of analyticity of $\hat{\varphi}_2$. Then the formula

$$\check{\varphi}_{3,\lambda}(\zeta) = \int_\lambda^\zeta \check{\varphi}_1(\zeta_1)\hat{\varphi}_2(\zeta_1\,\mathrm{e}^{\mathrm{i}\pi} + \overset{\bullet}{\zeta})\,\mathrm{d}\zeta_1, \qquad \arg\lambda - \pi < \arg\zeta < \arg\lambda + \pi, \quad |\zeta| \textit{ small enough} \tag{61}$$

defines by analytic continuation an element of ANA *which is a major of ${}^\flat\hat{\varphi}_3$. In formula* (61), *it is understood that $\zeta_2 = \zeta_1\,\mathrm{e}^{\mathrm{i}\pi} + \overset{\bullet}{\zeta}$ moves along the segment $[\lambda\,\mathrm{e}^{\mathrm{i}\pi} + \overset{\bullet}{\zeta}, 0]$, where $\lambda\,\mathrm{e}^{\mathrm{i}\pi} + \overset{\bullet}{\zeta}$ denotes the lift in $\underset{\bullet}{\mathbb{C}}$ of $-\overset{\bullet}{\lambda} + \overset{\bullet}{\zeta}$ which has its argument closest to $\arg\lambda + \pi$ (it is well-defined for $|\zeta| < |\lambda|$)—see the top of Figure 13.*

Proof. a) Let $\zeta \in \underset{\bullet}{\mathbb{C}}$ with small enough modulus. Observe that, in formula (60), $\arg\zeta_1$ takes all possible values beteween $\arg\zeta - 2\pi$ and $\arg\zeta$ at least, for whatever choice of γ_ζ, while $\arg(\zeta - \overset{\bullet}{\zeta}_1)$ (with the convention indicated) can be maintained arbitrarily close to $\arg\zeta$ by choosing γ_ζ close enough to the segments $[\zeta\,\mathrm{e}^{-2\pi\mathrm{i}}, 0]$ and $[0, \zeta]$. Let ε denote a positive function on $]0, |\zeta|]$ such that $\varepsilon(r) \xrightarrow[r\to 0]{} 0$ and

$$|\zeta_1\check{\varphi}_1(\zeta_1)| \le \varepsilon(|\zeta_1|), \qquad \arg\zeta - 2\pi \le \arg\zeta_1 \le \arg\zeta.$$

Deforming the contour, we rewrite the right-hand side of (60) as

$$\left(\int_{\zeta\,\mathrm{e}^{-2\pi\mathrm{i}}}^{a\,\mathrm{e}^{-2\pi\mathrm{i}}} + \int_{\gamma_a} + \int_a^\zeta\right) \check{\varphi}_1(\zeta_1)\hat{\varphi}_2(\zeta - \overset{\bullet}{\zeta}_1)\,\mathrm{d}\zeta_1,$$

with any auxiliary point $a \in \left]0, \frac{1}{2}\zeta\right]$. The two integrals over rectilinear segments contribute $\int_a^\zeta \hat{\varphi}_1(\zeta_1)\hat{\varphi}_2(\zeta - \overset{\bullet}{\zeta}_1)\,\mathrm{d}\zeta_1$, which tends to $\hat{\varphi}_3(\zeta)$ as $a \to 0$ by integrability of $\hat{\varphi}_1$ and $\hat{\varphi}_2$. The

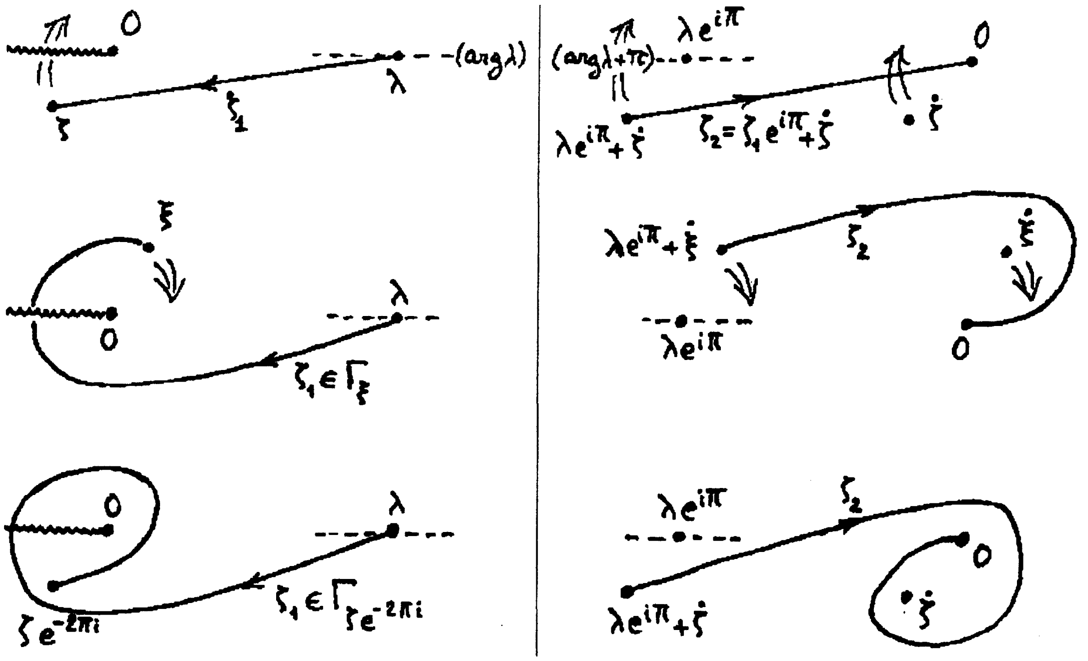


Figure 13: Analytic continuation of $\overset{\vee}{\varphi}_{3,\lambda}$. (Left: three examples of path Γ. Right: the corresponding paths followed by $\zeta_2 = \zeta_1\,\mathrm{e}^{\pi\mathrm{i}} + \dot\zeta$).

integral over γ_a tends to 0, since its modulus is not larger than $2\pi\,\varepsilon(|a|)\max_D|\hat\varphi_2|$, where D denotes the closed disc of radius $\frac{1}{2}|\zeta|$ centred at ζ (inside the same sheet of $\underset{\bullet}{\mathbb{C}}$ as ζ).

b) Formula (61) defines a function holomorphic near the origin of $\underset{\bullet}{\mathbb{C}}$ in the sheet (branch cut) which corresponds to the condition $0 \notin [\dot\lambda, \dot\zeta]$ and which contains λ, *i.e.* in $S_\lambda = \{\arg\lambda - \pi < \arg\zeta < \arg\lambda + \pi\}$ for small enough $|\zeta|$; for such ζ, the whole path of integration $[\lambda, \zeta]$ lies inside S_λ, while the corresponding $\zeta_2 = \zeta_1\,\mathrm{e}^{\mathrm{i}\pi} + \dot\zeta$ move along the segment $[\lambda\,\mathrm{e}^{\mathrm{i}\pi} + \dot\zeta, 0] \subset S_{\lambda\,\mathrm{e}^{\mathrm{i}\pi}}$ (here we use again the integrability of $\hat\varphi_2$). The analytic continuation is obtained by deforming continuously the contour of integration as ζ moves to different sheets of $\underset{\bullet}{\mathbb{C}}$:

$$\overset{\vee}{\varphi}_{3,\lambda}(\zeta) = \int_{\Gamma_\zeta} \overset{\vee}{\varphi}_1(\zeta_1)\hat\varphi_2(\zeta_1\,\mathrm{e}^{\mathrm{i}\pi} + \dot\zeta)\,\mathrm{d}\zeta_1,$$

with a path Γ_ζ which connects λ and ζ without touching the origin and without intersecting itself, it being understood that the symmetric path followed by $\zeta_2 = \zeta_1\,\mathrm{e}^{\mathrm{i}\pi} + \dot\zeta$ always starts on the same sheet of $\underset{\bullet}{\mathbb{C}}$—see Figure 13.

In particular, following the analytic continuation from ζ to $\zeta\,\mathrm{e}^{-2\pi\mathrm{i}}$, we obtain a path $\Gamma_{\zeta\,\mathrm{e}^{-2\pi\mathrm{i}}}$ which can be decomposed as the original path $[\lambda, \zeta]$ followed by $\tilde\gamma_\zeta$, where $\tilde\gamma_\zeta$ is the same as the above γ_ζ but with reverse orientation. Moreover, our convention for $\arg\zeta_2$ agrees[23] with that

[23]See Figure 13 or draw your own picture in the simpler case when $\arg\zeta$ is slightly smaller than $\arg\lambda + \pi$. Or start directly with the limit case $\arg\zeta = \arg\lambda + \pi$, with Γ_ζ and $\Gamma_{\zeta\,\mathrm{e}^{-2\pi\mathrm{i}}}$ starting at λ and following $[\dot\lambda, \dot\zeta]$ but circumventing 0 to the right or to the left, while the symmetric path starts at $\lambda\,\mathrm{e}^{\mathrm{i}\pi} + \dot\zeta$ and circumvents $\dot\zeta$ to the right or to the left; the point is that the symmetric path thus lies in the same sheet as ζ (rather than $\zeta\,\mathrm{e}^{-2\pi\mathrm{i}}$, or any other lift of $\dot\zeta$ in $\underset{\bullet}{\mathbb{C}}$), this is the origin of our convention on $\arg\zeta_2$.

of formula (60), hence $\overset{\vee}{\varphi}_{3,\lambda}(\zeta) - \overset{\vee}{\varphi}_{3,\lambda}(\zeta\, e^{-2\pi i}) = \hat{\varphi}_3(\zeta)$.

c) Let

$$\Sigma'_{\lambda,\alpha} = \{\, |\arg\zeta - \arg\lambda| \le \pi - \alpha \,\}, \qquad S_{\lambda,\alpha} = \Sigma'_{\lambda,\alpha} \cap \{\, |\zeta| \le \tfrac{1}{2}|\lambda| \,\}, \qquad \alpha \in \left]0, \tfrac{\pi}{2}\right[.$$

We observe that, if $0 < \arg\lambda' - \arg\lambda < 2\pi$, $(\overset{\vee}{\varphi}_{3,\lambda} - \overset{\vee}{\varphi}_{3,\lambda'})(\zeta)$ (which, for $\arg\lambda' - \pi \le \arg\zeta < \arg\lambda + \pi$, is given by an integral involving the values of $\hat{\varphi}_1$ close to $[\lambda, \lambda']$ and those of $\hat{\varphi}_2$ close to $[\lambda\, e^{i\pi} + \dot{\zeta}, \lambda' e^{i\pi} + \dot{\zeta}]$) extends to a holomorphic function which is regular near $\zeta = 0$. Thus, to prove that ${}^{\flat}\hat{\varphi}_3 = \operatorname{sing}_0(\overset{\vee}{\varphi}_{3,\lambda})$, it only remains to check that $\zeta\overset{\vee}{\varphi}_{3,\lambda}(\zeta) \to 0$ as $\zeta \to 0$, uniformly in any sector $\Sigma'_{\lambda,\alpha}$.

Let $r \mapsto \varepsilon(r)$ denote a positive function defined on $]0, |\lambda|]$ such that $\varepsilon(r) \xrightarrow[r\to 0]{} 0$ and $|\zeta_1\hat{\varphi}_1(\zeta_1)| \le \varepsilon(|\zeta_1|)$ whenever $\zeta_1 \in \Sigma'_{\lambda,\alpha}$ and $|\zeta_1| \le |\lambda|$. In fact, we shall only use the fact that one can take a *bounded* function ε: we only suppose $\varepsilon(r) \le \varepsilon^*$ (this is related to footnote 24). Let $r \mapsto f(r)$ denote a positive integrable function defined on $\left]0, \frac{3|\lambda|}{2}\right]$ such that $|\hat{\varphi}_2(\zeta_2)| \le f(|\zeta_2|)$ whenever $|\zeta_2| \le \frac{3}{2}|\lambda|$ and $\arg\lambda + \frac{\pi}{2} \le \arg\zeta_2 \le \arg\lambda + \frac{3\pi}{2}$. We can write

$$\overset{\vee}{\varphi}_{3,\lambda}(\zeta) = \int_0^{\lambda e^{i\pi} + \dot{\zeta}} \overset{\vee}{\varphi}_1(\zeta - \dot{\zeta}_2)\hat{\varphi}_2(\zeta_2)\, d\zeta_2, \qquad \zeta \in S_{\lambda,\alpha}.$$

For any $\zeta \in S_{\lambda,\alpha}$, letting $u_\zeta = \frac{\lambda e^{i\pi} + \dot{\zeta}}{|\lambda - \zeta|}$, we thus have

$$|\zeta\overset{\vee}{\varphi}_{3,\lambda}(\zeta)| \le \int_0^{|\lambda-\zeta|} \left|\tfrac{\zeta}{\zeta - \xi u_\zeta}\right| \varepsilon^* f(\xi)\, d\xi.$$

Elementary geometry shows that

$$\zeta \in S_{\lambda,\alpha} \;\Rightarrow\; \alpha' \le \arg\left(\tfrac{u_\zeta}{\zeta}\right) = \arg\left(\tfrac{\lambda e^{i\pi}}{\zeta} + 1\right) \le 2\pi - \alpha', \qquad \alpha' = \arg\left(2\, e^{i\alpha} + 1\right) \in \left]0, \alpha\right[.$$

Hence

$$\left|\tfrac{\zeta - \xi u_\zeta}{\zeta}\right|^2 = 1 + \left|\tfrac{\xi}{\zeta}\right|^2 - 2\left|\tfrac{\xi}{\zeta}\right| \cos\left(\arg\tfrac{u_\zeta}{\zeta}\right) \ge F(\xi, |\zeta|), \qquad F(\xi, r) = 1 + \tfrac{\xi^2}{r^2} - 2\tfrac{|\xi|}{r}\cos(\alpha').$$

But $g = 1/\sqrt{F}$ is continuous and $\le 1/\sin(\alpha')$ on $[0, \frac{3|\lambda|}{2}] \times]0, |\lambda|]$, with $g(\xi, r) \xrightarrow[r\to 0]{} 0$ for each $\xi > 0$, thus

$$|\zeta\overset{\vee}{\varphi}_{3,\lambda}(\zeta)| \le \varepsilon^* \int_0^{\frac{3|\lambda|}{2}} g(\xi, |\zeta|) f(\xi)\, d\xi = \varepsilon'_{\lambda,\alpha}(|\zeta|),$$

with $\varepsilon'_{\lambda,\alpha}(r) \xrightarrow[r\to 0]{} 0$ by Lebesgue's dominated convergence theorem. □

We now see how we can define the convolution product of a general singularity $\overset{\nabla}{\varphi}_1$ with an integrable one ${}^{\flat}\hat{\varphi}_2$: for any major $\overset{\vee}{\varphi}_1 \in$ ANA and any λ, formula (61) still defines an element $\overset{\vee}{\varphi}_{3,\lambda}$ of ANA by analytic continuation (the integrability of $\overset{\nabla}{\varphi}_1$ is not required for this), and we can set

$$\overset{\nabla}{\varphi}_1 * {}^{\flat}\hat{\varphi}_2 = \operatorname{sing}_0(\overset{\vee}{\varphi}_{3,\lambda}). \tag{62}$$

The choice of the major $\overset{\vee}{\varphi}_1$ does not matter (by linearity, adding to $\overset{\vee}{\varphi}_1$ a regular germ in (61) will add to $\overset{\vee}{\varphi}_{3,\lambda}$ a function which is regular, being a major of the null singularity, in view of the

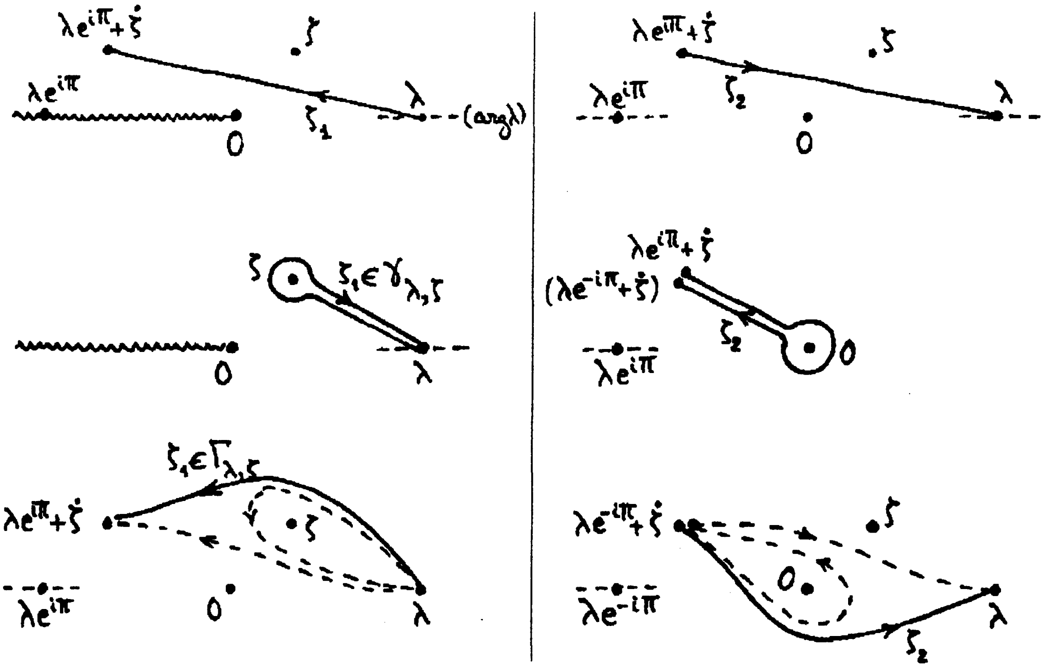


Figure 14: Top: integration path for $\overset{\vee}{\varphi}{}^{*}_{3,\lambda}(\zeta)$. Middle: path for $\overset{\vee}{\varphi}{}^{**}_{3,\lambda}(\zeta)$. Bottom: difference. (Left: paths followed by ζ_1. Right: corresponding paths followed by ζ_2.)

above), nor does the choice of λ because $\overset{\vee}{\varphi}_{3,\lambda} - \overset{\vee}{\varphi}_{3,\lambda'}$ is regular (as was mentioned in the above proof). The minor of $\overset{\triangledown}{\varphi}_1 * {}^\flat\widehat{\varphi}_2$ is still given by formula (60), but one must realize that $\overset{\triangledown}{\varphi}_1 * {}^\flat\widehat{\varphi}_2$ has no reason to be an integrable singularity when $\overset{\triangledown}{\varphi}_1$ is not.[24]

Convolution of general singularities. The convolution algebra SING

We just saw how the convolution of integrable minors gave rise to a convolution of integrable singularities $\mathrm{SING}^{\mathrm{int}} \times \mathrm{SING}^{\mathrm{int}} \to \mathrm{SING}^{\mathrm{int}}$ which could be extended to a convolution $\mathrm{SING} \times \mathrm{SING}^{\mathrm{int}} \to \mathrm{SING}$. We proceed with a further extension so as to view the space SING as an algebra, of which $\mathrm{SING}^{\mathrm{int}}$ will appear as a subalgebra.

To this end, it is sufficient to imitate the arguments leading to Lemma 7 and to express a major $\overset{\vee}{\varphi}{}^{*}_{3,\lambda}$ of the convolution product of two integrable singularities by a formula similar to (61), but referring to a major $\overset{\vee}{\varphi}_2$ rather than to the minor $\widehat{\varphi}_2$ of the second singularity. The new formula will then be taken as a definition of the convolution product when the singularities are no longer assumed to be integrable.

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[24] Notice however that, with this definition, $\delta * {}^\flat\widehat{\varphi}_2 = {}^\flat\widehat{\varphi}_2$ (compare $\overset{\vee}{\varphi}_{3,\lambda}(\zeta)$ when $\overset{\vee}{\varphi}_1(\zeta_1) = 1/2\pi i\zeta_1$ and the major $\overset{\vee}{\varphi}_2(\zeta) = \int_0^{\lambda e^{i\pi}} \frac{\widehat{\varphi}_2(\zeta_2)}{2\pi i(\zeta-\zeta_2)}\,d\zeta_2$ given by (58): the difference is regular at the origin). Hence, if $\overset{\triangledown}{\varphi}_1 = c\,\delta + {}^\flat\widehat{\varphi}_1$, $\overset{\triangledown}{\varphi}_1 * {}^\flat\widehat{\varphi}_2$ is still an integrable singularity (namely ${}^\flat(c\widehat{\varphi}_2 + \widehat{\varphi}_1 * \widehat{\varphi}_2)$). On the other hand, the minor of $\delta^{(n+1)} * {}^\flat\widehat{\varphi}_2$ is $\left(\frac{d}{d\zeta}\right)^{n+1}\widehat{\varphi}_2$, but this singularity is usually not integrable; for instance, if $\widehat{\varphi}_2 \in \mathbb{C}\{\zeta\}$, one can check that $\delta^{(n+1)} * {}^\flat\widehat{\varphi}_2 = \widehat{\varphi}_2(0)\,\delta^{(n)} + \frac{d\widehat{\varphi}_2}{d\zeta}(0)\,\delta^{(n-1)} + \cdots + \frac{d^n\widehat{\varphi}_2}{d\zeta^n}(0)\,\delta + {}^\flat\big(\big(\frac{d}{d\zeta}\big)^{n+1}\widehat{\varphi}_2\big)$.

Lemma 8 *Let* $\overset{\triangledown}{\varphi}_1, \overset{\triangledown}{\varphi}_2 \in \mathrm{SING}$ *have majors* $\check{\varphi}_1$ *and* $\check{\varphi}_2$. *Let* $\lambda \in \underset{\bullet}{\mathbb{C}}$ *belong to the intersection of the domains of analyticity of* $\check{\varphi}_1$ *and* $\check{\varphi}_2$, *with small enough modulus so that* $\lambda\,\mathrm{e}^{\mathrm{i}\pi}$ *also belongs to this intersection. Then an element of* ANA *can be defined by analytic continuation from the formula*

$$\check{\varphi}^*_{3,\lambda}(\zeta) = \int_{\lambda}^{\lambda\,\mathrm{e}^{\mathrm{i}\pi}+\dot{\zeta}} \check{\varphi}_1(\zeta_1)\check{\varphi}_2(\zeta_2)\,\mathrm{d}\zeta_1, \qquad \zeta \in H_\lambda, \quad |\zeta| \text{ small enough,} \tag{63}$$

where $H_\lambda = \{\, \zeta \in \underset{\bullet}{\mathbb{C}} \mid \arg\lambda < \arg\zeta < \arg\lambda + \pi \,\}$, *and where it is understood that* $\lambda\,\mathrm{e}^{\mathrm{i}\pi} + \dot{\zeta}$ *is the lift in* $\underset{\bullet}{\mathbb{C}}$ *of* $-\dot{\lambda} + \dot{\zeta}$ *which lies in* H_λ *and* ζ_2 *is the lift of* $\dot{\zeta} - \dot{\zeta}_1$ *which is also in* H_λ *(and thus moves backwards along the same segment* $[\lambda, \lambda\,\mathrm{e}^{\mathrm{i}\pi} + \dot{\zeta}]$*)—see the top of Figure 14. This germ gives rise to a singularity*

$$\overset{\triangledown}{\varphi}_3 := \mathrm{sing}_0(\check{\varphi}^*_{3,\lambda}) \tag{64}$$

which does not depend on λ, *nor on the choice of the majors* $\check{\varphi}_1$ *and* $\check{\varphi}_2$, *but only on the singularities* $\overset{\triangledown}{\varphi}_1$ *and* $\overset{\triangledown}{\varphi}_2$.

Moreover, when $\overset{\triangledown}{\varphi}_2 \in \mathrm{SING}^{\mathrm{int}}$, $\overset{\triangledown}{\varphi}_3$ *coincides with the singularity* $\overset{\triangledown}{\varphi}_1 * \overset{\triangledown}{\varphi}_2$ *defined by formula* (62) *in the previous section. (In particular, when both* $\overset{\triangledown}{\varphi}_1$ *and* $\overset{\triangledown}{\varphi}_2$ *are integrable singularities, we recover* ${}^b\hat{\varphi}_1 * {}^b\hat{\varphi}_2 = \overset{\triangledown}{\varphi}_3$.*)*

Proof. Formula (63) defines an analytic function, since the segment $[\lambda, \lambda\,\mathrm{e}^{\mathrm{i}\pi} + \dot{\zeta}]$ is contained in the domains of analyticity of $\check{\varphi}_1$ and $\check{\varphi}_2$. This is not the case when $\arg\zeta$ crosses $\arg\lambda$ or $\arg\lambda + \pi$, but the analytic continuation is then obtained by deforming the path so that the origin be avoided by ζ_1 and ζ_2. Another way of obtaining the analytic continuation is to observe that, whenever $|\arg\lambda' - \arg\lambda| < \pi$, the difference $(\check{\varphi}^*_{3,\lambda'} - \check{\varphi}^*_{3,\lambda})(\zeta)$ (which, for $\zeta \in H_\lambda \cap H_{\lambda'}$, is given by an integral involving the values of $\check{\varphi}_1$ and $\check{\varphi}_2$ close to $[\lambda, \lambda']$ and $[\lambda\,\mathrm{e}^{\mathrm{i}\pi}, \lambda'\,\mathrm{e}^{\mathrm{i}\pi}]$) extends analytically to a full neighbourhood of the origin. Thus $\check{\varphi}^*_{3,\lambda} \in \mathrm{ANA}$ and $\overset{\triangledown}{\varphi}_3$ does not depend on λ.

The fact that $\overset{\triangledown}{\varphi}_3$ does not depend on the choice of the major $\check{\varphi}_2$ follows by linearity from the last statement (a regular $\check{\varphi}_2$ can be viewed as the major of an integrable singularity, namely the null singularity, the convolution product of which with any $\overset{\triangledown}{\varphi}_1$ is the null singularity). The fact that it does not depend on the choice of the major $\check{\varphi}_1$ then follows from the commutativity of formula (63).

Consider

$$\check{\varphi}^{**}_{3,\lambda}(\zeta) = \int_{\gamma_{\lambda,\zeta}} \check{\varphi}_1(\zeta_1)\check{\varphi}_2(\zeta_2)\,\mathrm{d}\zeta_1, \qquad \arg\lambda - \pi < \arg\zeta < \arg\lambda + \pi, \quad |\zeta| \text{ small enough,} \tag{65}$$

where the path $\gamma_{\lambda,\zeta}$ starts at λ, goes towards ζ, follows a circle of small radius around ζ with clockwise orientation, and comes back to λ, and where ζ_2 is the lift of $\dot{\zeta} - \dot{\zeta}_1$ which starts at $\lambda\,\mathrm{e}^{\mathrm{i}\pi} + \dot{\zeta}$ and ends at $\lambda\,\mathrm{e}^{-\mathrm{i}\pi} + \dot{\zeta}$ after having turned clockwise around the origin (see the middle of Figure 14). We observe that, for $\zeta \in H_\lambda$ with small enough modulus,

$$-\check{\varphi}^*_{3,\lambda}(\zeta) + \check{\varphi}^{**}_{3,\lambda}(\zeta) = \int_{\Gamma_{\lambda,\zeta}} \check{\varphi}_1(\zeta_1)\check{\varphi}_2(\zeta_2)\,\mathrm{d}\zeta_1,$$

where the path $\Gamma_{\lambda,\zeta}$ starts at λ, circumvents both 0 and ζ to the right and ends at $\lambda\,\mathrm{e}^{\mathrm{i}\pi} + \dot{\zeta}$, while the lift ζ_2 of $\dot{\zeta} - \dot{\zeta}_1$ starts at $\lambda\,\mathrm{e}^{-\mathrm{i}\pi} + \dot{\zeta}$ and ends at λ after having circumvented both 0 and ζ to the right (see the bottom of Figure 14), and the function defined by the last integral is regular at the origin (because $\Gamma_{\lambda,\zeta}$ can keep off the origin even if ζ varies in a full neighbourhood of the origin). Hence

$$\operatorname{sing}_0(\overset{\vee}{\varphi}{}^{**}_{3,\lambda}) = \operatorname{sing}_0(\overset{\vee}{\varphi}{}^{*}_{3,\lambda}) = \overset{\triangledown}{\varphi}_3.$$

Suppose now that $\overset{\triangledown}{\varphi}_2 \in \mathrm{SING}^{\mathrm{int}}$ and define $\overset{\triangledown}{\varphi}_1 * \overset{\triangledown}{\varphi}_2$ by formula (62). When letting an auxiliary point a tend to ζ along $[\lambda,\zeta]$ and using a path $\gamma_{\lambda,\zeta} = [\lambda,a] \cup \gamma_{a,\zeta} \cup [a,\lambda]$, the alternative major $\overset{\vee}{\varphi}{}^{**}_{3,\lambda}$ of $\overset{\triangledown}{\varphi}_3$ appears to be nothing but the major $\overset{\vee}{\varphi}_{3,\lambda}$ of $\overset{\triangledown}{\varphi}_1 * \overset{\triangledown}{\varphi}_2$ delivered by (61). This ends the proof. □

Definition 11 *We define the convolution product*

$$\overset{\triangledown}{\varphi}_3 = \overset{\triangledown}{\varphi}_1 * \overset{\triangledown}{\varphi}_2$$

of any two singularities $\overset{\triangledown}{\varphi}_1, \overset{\triangledown}{\varphi}_2 \in \mathrm{SING}$ *by formulas* (63) *and* (64).

Proposition 8 *The convolution law just defined on the space* SING *is commutative and associative; it turns it into a commutative algebra, with unit* $\delta = \operatorname{sing}_0\left(\frac{1}{2\pi i \zeta}\right)$.

Proof. The commutativity is obvious. The relation $\overset{\triangledown}{\varphi}_1 * \delta = \overset{\triangledown}{\varphi}_1$ is immediate when using the alternative major $\overset{\vee}{\varphi}{}^{**}_{3,\lambda}$ of formula (65) with $\overset{\vee}{\varphi}_2(\zeta_2) = 1/2\pi i\zeta_2$, since the residuum formula gives $\overset{\vee}{\varphi}{}^{**}_{3,\lambda} = \overset{\vee}{\varphi}_1$.

For the associativity, the quickest proof consists in extending it from $\mathrm{SING}^{\mathrm{int}}$ (in restriction to which it is a mere consequence of the associativity of the convolution of integrable minors) to SING by *continuity* and *density*. Indeed, we may call a sequence $(\overset{\triangledown}{\varphi}_n)_{n\geq 0}$ of SING convergent if these singularities admit majors $\overset{\vee}{\varphi}_n$ analytic in the same spiralling neighbourhood of the origin $\mathcal{V}$ and if there exists $\overset{\vee}{\varphi}$ analytic in $\mathcal{V}$ such that $(\overset{\vee}{\varphi}_n)_{n\geq 0}$ converges uniformly towards $\overset{\vee}{\varphi}$ in every compact subset of $\mathcal{V}$; the singularity $\operatorname{sing}_0(\overset{\vee}{\varphi})$ is then unique and is called the limit of the sequence. It is easy to check that

$$\overset{\triangledown}{\varphi}_n \to \overset{\triangledown}{\varphi} \text{ and } \overset{\triangledown}{\psi}_n \to \overset{\triangledown}{\psi} \quad \Rightarrow \quad \overset{\triangledown}{\varphi}_n * \overset{\triangledown}{\psi}_n \to \overset{\triangledown}{\varphi} * \overset{\triangledown}{\psi}.$$

On the other hand, any singularity is the limit of a sequence of integrable singularities, majors of which can be chosen to be polynomials in $\log\zeta$ (this essentially amounts to the Weierstrass theorem in the variable $\log\zeta$). We thus obtain $\overset{\triangledown}{\varphi} * (\overset{\triangledown}{\psi} * \overset{\triangledown}{\chi}) = (\overset{\triangledown}{\varphi} * \overset{\triangledown}{\psi}) * \overset{\triangledown}{\chi}$ by passing to the limit in the corresponding identity for integrable singularities. □

Observe that we have two subalgebras without unit ${}^\flat(\mathbb{C}\{\zeta\}) \subset \mathrm{SING}^{\mathrm{int}} \subset \mathrm{SING}$, and that simple singularities form a subalgebra $\mathrm{SING}^{\mathrm{simp}} = \mathbb{C}\,\delta \oplus {}^\flat(\mathbb{C}\{\zeta\}) \subset \mathrm{SING}$. Here are a few properties of the algebra SING:

i) The family of singularities $(\overset{\triangledown}{I}_\sigma)_{\sigma\in\mathbb{C}}$ defined by (57) and

$$\overset{\triangledown}{I}_{-n} = \delta^{(n)}, \qquad \overset{\triangledown}{I}_{n+1} = {}^\flat\!\left(\tfrac{\zeta^n}{n!}\right), \qquad n \in \mathbb{N}$$

satisfies $\overset{\triangledown}{I}_{\sigma_1} * \overset{\triangledown}{I}_{\sigma_2} = \overset{\triangledown}{I}_{\sigma_1+\sigma_2}$ for all $\sigma_1, \sigma_2 \in \mathbb{C}$, as can be checked from the integrable case by analytic continuation[25] in (σ_1, σ_2).

ii) If $\alpha(\zeta)$ is a regular germ, the multiplication of majors by α obviously passes to the quotient:

$$\overset{\triangledown}{\varphi} = \operatorname{sing}_0(\overset{\vee}{\varphi}) \in \mathrm{SING} \;\mapsto\; \alpha\overset{\triangledown}{\varphi} := \operatorname{sing}_0(\alpha\overset{\vee}{\varphi}) \in \mathrm{SING}.$$

This turns SING into a $\mathbb{C}\{\zeta\}$-module.

iii) In particular, we have a linear operator of SING

$$\partial : \overset{\triangledown}{\varphi} \mapsto -\zeta\overset{\triangledown}{\varphi},$$

which turns out to be a derivation (multiply by $\overset{\bullet}{\zeta} = \overset{\bullet}{\zeta}_1 + \overset{\bullet}{\zeta}_2$ in formula (63)), the kernel of which is $\mathbb{C}\,\delta$.

iv) Differentiation of majors passes to the quotient and defines a linear operator $\frac{\mathrm{d}}{\mathrm{d}\zeta}$ which coincides with convolution by $\delta^{(1)} = \frac{\mathrm{d}}{\mathrm{d}\zeta}\delta$ (differentiate the relation $\overset{\vee}{\varphi}_1(\zeta) = \int_{\gamma_{\lambda,\zeta}} \overset{\vee}{\varphi}_1(\zeta_1)\overset{\vee}{\varphi}_2(\overset{\bullet}{\zeta} - \overset{\bullet}{\zeta}_1)\,\mathrm{d}\zeta_1$, where $\overset{\vee}{\varphi}_2(\zeta_2) = 1/2\pi\mathrm{i}\zeta_2$) and is invertible, with inverse $\left(\frac{\mathrm{d}}{\mathrm{d}\zeta}\right)^{-1}\overset{\triangledown}{\varphi} = \delta^{(-1)} * \overset{\triangledown}{\varphi}$. More generally

$$\left(\tfrac{\mathrm{d}}{\mathrm{d}\zeta}\right)^n\overset{\triangledown}{\varphi} = \delta^{(n)} * \overset{\triangledown}{\varphi}, \qquad \overset{\triangledown}{\varphi} \in \mathrm{SING}, \quad n \in \mathbb{Z}.$$

Notice that $\frac{\mathrm{d}}{\mathrm{d}\zeta}$ is not a derivation; its action on a convolution product is given by

$$\left(\tfrac{\mathrm{d}}{\mathrm{d}\zeta}\right)^n(\overset{\triangledown}{\varphi}_1 * \overset{\triangledown}{\varphi}_2) = \left(\left(\tfrac{\mathrm{d}}{\mathrm{d}\zeta}\right)^n\overset{\triangledown}{\varphi}_1\right) * \overset{\triangledown}{\varphi}_2 = \overset{\triangledown}{\varphi}_1 * \left(\left(\tfrac{\mathrm{d}}{\mathrm{d}\zeta}\right)^n\overset{\triangledown}{\varphi}_2\right).$$

We have for instance $\left(\frac{\mathrm{d}}{\mathrm{d}\zeta}\right)^n\overset{\triangledown}{I}_\sigma = \overset{\triangledown}{I}_{\sigma-n}$ for all $\sigma \in \mathbb{C}$ and $n \in \mathbb{N}$.

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[25] There is a notion of singularity $\overset{\triangledown}{\varphi}_s$ depending analytically on a parameter $s \in S$, where S is an open subset of $\mathbb{C}$: following [Eca81, Vol. 1, p. 48], we assume that for each $s_0 \in S$, there exist $r > 0$, an open subset V of S containing s_0 and a holomorphic function $\overset{\vee}{\varphi}(s,\zeta) = \overset{\vee}{\varphi}_s(\zeta)$ on $V \times D_r$, where $D_r = \{\,\zeta \in \overset{\bullet}{\mathbb{C}};\ |\zeta| < r\,\}$, such that $\overset{\triangledown}{\varphi}_s = \operatorname{sing}_0(\overset{\vee}{\varphi}_s)$ for each $s \in V$. According to footnote 22, the family $\left(\overset{\triangledown}{I}_\sigma\right)_{\sigma\in\mathbb{C}}$ satisfies this with $s = \sigma \in S = \mathbb{C}$, and this example shows that there may be no major $\overset{\vee}{\varphi}_s(\zeta)$ which satisfies the above with $V = S$.

With this definition, the uniqueness of the continuation of analytic identities is guaranteed by the following fact: if S is connected and $Z = \{\,s \in S \mid \overset{\triangledown}{\varphi}_s = 0\,\}$ has an accumulation point, then $Z = S$. (Let Z' denote the interior of Z. We first observe that if a non-stationary sequence of points $s_n \in Z$ converges to $s_\infty \in S$, then $s_\infty \in Z'$. Indeed, let V be an open connected neighbourhood of s_∞ in S, $r > 0$ and $\overset{\vee}{\varphi}_s(\zeta)$ be a major of $\overset{\triangledown}{\varphi}_s$ which is analytic in $(s,\zeta) \in V \times D_r$. For $n \geq N$ large enough, the functions $\overset{\vee}{\varphi}_{s_n}$ are regular at the origin; for any fixed $\zeta \in D_r$, the analytic identity $\overset{\vee}{\varphi}_s(\zeta) = \overset{\vee}{\varphi}_s(\zeta\,\mathrm{e}^{-2\pi\mathrm{i}})$ holds for $s = s_n$, $n \geq N$, thus for all $s \in V$; now each analytic function $s \in V \mapsto c_k(s) = \int \zeta^k \overset{\vee}{\varphi}_s(\zeta)\,\mathrm{d}\zeta$, where $k \in \mathbb{N}$ and the integral is taken over a circle centred at the origin, vanishes for $s = s_n$, $n \geq N$, thus for all $s \in V$; hence every $\overset{\vee}{\varphi}_s$, $s \in V$, is regular at the origin, *i.e.* $\overset{\triangledown}{\varphi}_s = 0$, and $s_\infty \in Z'$. The open subset Z' of S is thus non-empty and closed, and the conclusion follows from the connectedness of S.)

Moreover, if we are given two families of singularities $\overset{\triangledown}{\varphi}_s$ and $\overset{\triangledown}{\psi}_s$ which depend analytically on $s \in S$, formula (63) with $\lambda \in D_{r/2}$ and its analytic continuation for $\zeta \in D_{|\lambda|}$ show that $\overset{\triangledown}{\varphi}_s * \overset{\triangledown}{\psi}_s$ also depends analytically on $s \in S$. One can use these facts to continue analytically the identity $\overset{\triangledown}{I}_{\sigma_1} * \overset{\triangledown}{I}_{\sigma_2} - \overset{\triangledown}{I}_{\sigma_1+\sigma_2} = 0$ from $\Re e\,\sigma_1 > 0$ to arbitrary $\sigma_1 \in \mathbb{C}$ with a fixed σ_2 of positive real part, and then from $\Re e\,\sigma_2 > 0$ to arbitrary $\sigma_2 \in \mathbb{C}$ with any fixed σ_1.

Extensions of the formal Borel transform

The formal Borel transform that we have used so far was defined on the space of Gevrey-1 formal series $\mathbb{C}[[z^{-1}]]_1$. In the language of singularities, this means that we have an algebra isomorphism

$$\mathcal{B} : \tilde\varphi = \sum_{n\geq 0} c_n z^{-n} \in \mathbb{C}[[z^{-1}]]_1 \mapsto \overset{\triangledown}{\varphi} = c_0\,\delta + {}^\flat\hat\varphi \in \mathrm{SING}^{\mathrm{simp}}, \qquad \hat\varphi(\zeta) = \sum_{n\geq 1} c_n \tfrac{\zeta^{n-1}}{(n-1)!}.$$

The field of fractions of $\mathbb{C}[[z^{-1}]]_1$ is $\mathbb{C}((z^{-1}))_1 = \mathbb{C}[[z^{-1}]]_1[z]$, the space of sums of a polynomial in z and a Gevrey-1 series in z^{-1} (because, when $c_0 \neq 0$, the above $\tilde\varphi$ admits a multiplicative inverse in $\mathbb{C}[[z^{-1}]]_1$), and it is natural to extend the formal Borel transform by setting $\mathcal{B}(z^n) = \delta^{(n)}$: we get an algebra isomorphism

$$\mathcal{B} : \tilde\varphi = \sum_{n\geq -N} c_n z^{-n} \in \mathbb{C}((z^{-1}))_1 \mapsto \overset{\triangledown}{\varphi} = \sum_{k=0}^{N} c_{-k}\,\delta^{(k)} + {}^\flat\hat\varphi \in \mathrm{SING}^{\mathrm{s.ram.}},$$

with N depending on $\tilde\varphi$ and the same $\hat\varphi(\zeta)$ as above, and where $\mathrm{SING}^{\mathrm{s.ram.}} \subset \mathrm{SING}$ is the subalgebra of "simply ramified singularities", consisting of those singularities which admit a major of the form $P(1/\zeta) + \frac{1}{2\pi \mathrm{i}}\hat\varphi(\zeta)\log\zeta$ with P polynomial and $\hat\varphi(\zeta) \in \mathbb{C}\{\zeta\}$ (notice that this subalgebra is smaller than $\mathrm{var}^{-1}(\mathbb{C}\{\zeta\})$, which consists of those singularities which admit a major of the same form but with P entire function).

In $\mathbb{C}((z^{-1}))_1$, we have well-defined difference operators $\tilde\varphi(z) \mapsto \tilde\varphi(z+1) - \tilde\varphi(z)$ and $\tilde\varphi(z) \mapsto \tilde\varphi(z+1) - 2\tilde\varphi(z) + \tilde\varphi(z-1)$, the counterpart of which are $\overset{\triangledown}{\varphi} \mapsto (\mathrm{e}^{-\zeta}-1)\overset{\triangledown}{\varphi}$ and $\overset{\triangledown}{\varphi} \mapsto 4\sinh^2(\frac{\zeta}{2})\overset{\triangledown}{\varphi}$.

The inverse formal Borel transform is not defined in the space of all singularities, but further extensions are possible beyond $\mathrm{SING}^{\mathrm{s.ram.}}$. For instance, setting

$$\mathcal{B}(z^{-\sigma}) = \overset{\triangledown}{I}_\sigma, \qquad \mathcal{B}\big((-1)^m z^{-\sigma}(\log z)^m\big) = \overset{\triangledown}{J}_{\sigma,m} := \big(\tfrac{\partial}{\partial\sigma}\big)^m \overset{\triangledown}{I}_\sigma, \qquad \sigma \in \mathbb{C},\ m \in \mathbb{N},$$

allows one to deal with formal expansions involving non-integer powers of z and integer powers of $\log z$ (*cf.* footnote 25 for the differentiation with respect to a parameter in an analytic family of singularities). In practice, when studying the formal solutions of a problem, one chooses a suitable subset of SING according to one's needs. This choice can be dictated by the shape of the formal solutions one finds and of their formal Borel transforms, and also by the nature of the singularities of the analytic continuation of the minors of these Borel transforms.

Laplace transform of majors

Let us denote by R_θ, for any $\theta \in \mathbb{R}$, the ray $]0, \mathrm{e}^{\mathrm{i}\theta}\infty[$ in $\underset{\bullet}{\mathbb{C}}$. If the minor $\hat\varphi$ of a singularity $\overset{\triangledown}{\varphi} = \mathrm{sing}_0(\check\varphi)$ extends analytically along R_θ and has at most exponential growth at infinity in this direction, one can define the Laplace transform by

$$(\mathcal{L}^\theta\,\overset{\triangledown}{\varphi})(z) = \int_{a\,\mathrm{e}^{\mathrm{i}(\theta-2\pi)}}^{a\,\mathrm{e}^{\mathrm{i}\theta}} \mathrm{e}^{-z\zeta}\,\check\varphi(\zeta)\,\mathrm{d}\zeta + \int_{a\,\mathrm{e}^{\mathrm{i}\theta}}^{\mathrm{e}^{\mathrm{i}\theta}\infty} \mathrm{e}^{-z\zeta}\,\hat\varphi(\zeta)\,\mathrm{d}\zeta,$$

where $a > 0$ is chosen small enough and the first integral is taken over a circle centred at the origin; the result then does not depend on the choice of a nor on the chosen major, it is a function holomorphic in a half-plane of the form $\Re e(z\,e^{i\theta}) > \tau$. Observe that if $\overset{\triangledown}{\varphi}$ is an integrable singularity, one can let a tend to 0, which yields the usual formula:

$$\mathcal{L}^\theta({}^\flat\hat{\varphi})(z) = \int_0^{e^{i\theta}\infty} e^{-z\zeta}\,\hat{\varphi}(\zeta)\,d\zeta.$$

The idea is in fact very similar to the one which was used to define the convolution of general singularities: integration of the minor up to the origin, being possible only in the integrable case, must be replaced by an integration of a major around the origin. Thus extended, the Laplace transform is compatible with the convolution of general singularities: $\mathcal{L}^\theta(\overset{\triangledown}{\varphi}*\overset{\triangledown}{\psi}) = (\mathcal{L}^\theta\overset{\triangledown}{\varphi})(\mathcal{L}^\theta\overset{\triangledown}{\psi})$. If the singularity admits a major which has at most exponential growth along R_θ and $R_{\theta-2\pi}$, one can then resort to the Laplace transform of majors:

$$(\mathcal{L}^\theta\overset{\triangledown}{\varphi})(z) = \left(-\int_{a\,e^{i(\theta-2\pi)}}^{e^{i(\theta-2\pi)}\infty} + \int_{a\,e^{i(\theta-2\pi)}}^{a\,e^{i\theta}} + \int_{a\,e^{i\theta}}^{e^{i\theta}\infty}\right) e^{-z\zeta}\,\check{\varphi}(\zeta)\,d\zeta$$

(with the second integral taken over the same circle as above, *i.e.* the sum of the three parts amounts to a Hankel's contour integral).

Laplace transforms in nearby directions $\theta_1 < \theta_2$ can be glued together and yield a function $\mathcal{L}^{]\theta_1,\theta_2[}\,\overset{\triangledown}{\varphi}$ analytic in a sectorial neighbourhood of infinity when the minor has no singularity in the sector $\theta_1 \le \arg\zeta \le \theta_2$; it is then more convenient to consider z as element of $\underset{\bullet}{\mathbb{C}}$, with $-\theta_2 - \frac{\pi}{2} < \arg z < -\theta_1 + \frac{\pi}{2}$ and $|z|$ large enough. The difference with what we saw at the beginning (Section 1.1) is in the possible asymptotic expansions of $\mathcal{L}^\theta\,\overset{\triangledown}{\varphi}$.

In the simply ramified case, when a Laplace transform $\mathcal{L}^{]\theta_1,\theta_2[}\,\overset{\triangledown}{\varphi}$ can be defined for $\overset{\triangledown}{\varphi} \in \mathrm{SING}^{\mathrm{s.ram.}}$, one finds $\mathcal{B}^{-1}\overset{\triangledown}{\varphi} \in \mathbb{C}((z^{-1}))_1$ as asymptotic expansion. Similarly, if the case of one of the extensions of the formal Borel transform mentioned in the previous section, $\mathcal{L}^{]\theta_1,\theta_2[}\,\overset{\triangledown}{\varphi}(z) \sim \mathcal{B}^{-1}\overset{\triangledown}{\varphi}(z)$, as a consequence of the formula

$$(-1)^m z^{-\sigma}(\log z)^m = \mathcal{L}^\theta\,\overset{\triangledown}{J}_{\sigma,m}(z), \qquad z\in\underset{\bullet}{\mathbb{C}}, \quad -\theta-\tfrac{\pi}{2} < \arg z < -\theta+\tfrac{\pi}{2}.$$

3.3 General resurgent functions and alien derivations

We are now ready to give definitions which are more general than in Sections 1.4 and 2.3. We shall not provide many details; the reader is referred to [Eca81, Vol. 3], [Eca92], [Eca93] or [CNP93].

We first define the space of "resurgent minors", $\widehat{\mathrm{RES}}_{2\pi i\mathbb{Z}}$, as the set of all the germs of ANA which extend to the universal cover $(\widetilde{\mathbb{C}\setminus 2\pi i\,\mathbb{Z}},1)$ (using the notation of footnote 3, meaning that the holomorphic function $\hat{\varphi}$ determined by the germ in a spiralling neighbourhood of the origin $\mathcal{V}$ extends analytically along any path of $\underset{\bullet}{\mathbb{C}}$ which starts in $\mathcal{V}$ and avoids the lift of $2\pi i\,\mathbb{Z}^*$ in $\underset{\bullet}{\mathbb{C}}$). Some resurgent minors are integrable minors, among these some are even regular at the origin; this gives rise to subspaces $\widehat{\mathrm{RES}}_{2\pi i\mathbb{Z}} \cap \mathrm{ANA}^{\mathrm{int}}$ and $\widehat{\mathcal{H}}(\mathcal{R}) = \widehat{\mathrm{RES}}_{2\pi i\mathbb{Z}} \cap \mathbb{C}\{\zeta\}$, which are both stable by convolution (the former for reasons similar to what was explained in Section 1.3 for the latter).

Next, we define the "convolutive model of resurgent functions" as the space of all the singularities of SING, the minors of which belong to $\widehat{\mathrm{RES}}_{2\pi i\mathbb{Z}}$:

$$\overset{\triangledown}{\mathrm{RES}}_{2\pi i\mathbb{Z}} := \mathrm{var}^{-1}(\widehat{\mathrm{RES}}_{2\pi i\mathbb{Z}}) \subset \mathrm{SING}.$$

This space is stable by convolution (we omit the proof): $\overset{\triangledown}{\mathrm{RES}}_{2\pi i\mathbb{Z}}$ *is a subalgebra of* SING, which obviously contains the unit δ. We may call

$$\overset{\triangledown}{\mathrm{RES}}{}^{\mathrm{int}}_{2\pi i\mathbb{Z}} = {}^{\flat}\big(\widehat{\mathrm{RES}}_{2\pi i\mathbb{Z}} \cap \mathrm{ANA}^{\mathrm{int}}\big) \subset \overset{\triangledown}{\mathrm{RES}}_{2\pi i\mathbb{Z}}$$

the "minor model"; it is a subalgebra of $\mathrm{SING}^{\mathrm{int}}$ (without unit), the elements of which are determined by their minor, so that there is no loss in information when reasoning on the minors only. The convolution algebra $\widehat{\mathcal{H}}(\mathcal{R})$ of Section 1.4, being isomorphic to ${}^{\flat}\big(\widehat{\mathrm{RES}}_{2\pi i\mathbb{Z}} \cap \mathbb{C}\{\zeta\}\big)$, can now be considered as a subalgebra of $\overset{\triangledown}{\mathrm{RES}}{}^{\mathrm{int}}_{2\pi i\mathbb{Z}}$.

The algebra $\mathrm{RES}^{\mathrm{simp}}$ of Section 2.3

$$\mathrm{RES}^{\mathrm{simp}} \simeq \overset{\triangledown}{\mathrm{RES}}{}^{\mathrm{simp}}_{2\pi i\mathbb{Z}} := \{\, \overset{\triangledown}{\varphi} \in \overset{\triangledown}{\mathrm{RES}}_{2\pi i\mathbb{Z}} \cap \mathrm{SING}^{\mathrm{simp}} \mid \mathrm{var}(\overset{\triangledown}{\varphi}) \text{ has only simple singularities} \,\} \quad (66)$$

corresponds to singularities which are determined by a regular minor up to the addition of a multiple of δ, such that the minor extends to $\mathcal{R}$, and *with the further restriction that the analytic continuation of the minor possesses only simple singularities.* This restriction made it possible to define the alien derivations Δ_ω in Section 2.3 as internal operators of $\mathrm{RES}^{\mathrm{simp}}$. Relaxing the conditions of regularity at the origin and on the shape of the singularities, we are now in a position to define the alien derivations Δ_ω in a somewhat enlarged framework, as internal operators of $\overset{\triangledown}{\mathrm{RES}}_{2\pi i\mathbb{Z}}$ and not only of $\overset{\triangledown}{\mathrm{RES}}{}^{\mathrm{simp}}_{2\pi i\mathbb{Z}}$.

Because of the possible ramification of the minor at the origin, the alien derivations will now be indexed by all $\omega \in \underset{\bullet}{\mathbb{C}}$ such that $\overset{\bullet}{\omega} \in 2\pi i\,\mathbb{Z}^*$. Here is the generalisation of Definition 8, with notations similar to those of Section 2.3:

For $\overset{\triangledown}{\varphi} \in \overset{\triangledown}{\mathrm{RES}}_{2\pi i\mathbb{Z}}$ *and* $\omega = 2\pi m\, e^{i\theta}$ *with* $m \in \mathbb{N}^*$ *and* $\theta \in \frac{\pi}{2} + \pi\mathbb{Z}$,

$$\Delta_\omega \overset{\triangledown}{\varphi} = \sum_{\varepsilon_1,\dots,\varepsilon_{m-1}\in\{+,-\}} \frac{p(\varepsilon)!\,q(\varepsilon)!}{m!}\, \mathrm{sing}_0\big(\overset{\vee}{\Phi}_{\gamma(\varepsilon)}\big), \tag{67}$$

where the path $\gamma(\varepsilon)$ *connects* $\left]0, \frac{1}{m}\omega\right[$ *and* $\left]\frac{m-1}{m}\omega, \omega\right[$ *and circumvents* $2\pi r\, e^{i\theta} = \frac{r}{m}\omega$ *to the right if* $\varepsilon_r = +$ *and to the left if* $\varepsilon = -$, *and where the analytic continuation of the minor* $\hat{\varphi}$ *of* $\overset{\triangledown}{\varphi}$ *determines the major*

$$\overset{\vee}{\Phi}_{\gamma(\varepsilon)}(\zeta) = \big(\mathrm{cont}_{\gamma(\varepsilon)}\, \hat{\varphi}\big)(\omega + \overset{\bullet}{\zeta}), \qquad \arg\omega - 2\pi < \arg\zeta < \arg\omega, \quad |\zeta| < 2\pi.$$

One can check that the operators Δ_ω are *derivations* of $\overset{\triangledown}{\mathrm{RES}}_{2\pi i\mathbb{Z}}$, which satisfy the rules of alien calculus that we have indicated in the case of simple resurgent functions. Notice that if $\overset{\bullet}{\omega}_1 = \overset{\bullet}{\omega}_2$, the restrictions of Δ_{ω_1} and Δ_{ω_2} to $\mathrm{var}^{-1}\big(\widehat{\mathcal{H}}(\mathcal{R})\big)$, and a fortiori to $\overset{\triangledown}{\mathrm{RES}}{}^{\mathrm{simp}}_{2\pi i\mathbb{Z}}$, coincide.

Up to now we have restricted ourselves to minors which extend analytically provided one avoids always the same fixed set of potentially singular points, namely $2\pi i\,\mathbb{Z}$ (or its lift in $\underset{\bullet}{\mathbb{C}}$). But one can consider other lattices $\Omega \subset \mathbb{C}$ of singular points and define accordingly the space $\widehat{\mathrm{RES}}_\Omega$ of germs which extend to $(\widetilde{\mathbb{C}\setminus\Omega}, 1)$, the algebra $\overset{\triangledown}{\mathrm{RES}}_\Omega = \mathrm{var}^{-1}\big(\widehat{\mathrm{RES}}_\Omega\big)$, and the alien derivations Δ_ω with $\overset{\bullet}{\omega} \in \Omega$ (Ω must be an additive semi-group of $\mathbb{C}$ to ensure stability by convolution, and a group to ensure stability by alien derivations[26]). The subalgebra of simple resurgent functions with singular support in Ω will be defined by

$$\overset{\triangledown}{\mathrm{RES}}{}^{\mathrm{simp}}_{\Omega} = \{\, \overset{\triangledown}{\varphi} \in \overset{\triangledown}{\mathrm{RES}}_{\Omega} \cap \mathrm{SING}^{\mathrm{simp}} \mid \forall r \geq 1, \forall \omega_1, \ldots, \omega_r,\ \Delta_{\omega_r} \cdots \Delta_{\omega_1} \overset{\triangledown}{\varphi} \in \mathrm{SING}^{\mathrm{simp}} \,\}$$

(which is consistent with (66)—*cf.* footnote 14). Similarly, replacing $\mathrm{SING}^{\mathrm{simp}}$ by $\mathrm{SING}^{\mathrm{s.ram.}}$ in the above formula, one can define the larger subalgebra $\overset{\triangledown}{\mathrm{RES}}{}^{\mathrm{s.ram.}}_{\Omega}$ of simply ramified resurgent functions with singular support in Ω.

Finally, the most general algebra we can construct at this level is the space $\overset{\triangledown}{\mathrm{RES}}$ of all the singularities, the minors of which are *endlessly continuable along broken lines*: a singularity $\overset{\triangledown}{\varphi}$ is said to belong to $\overset{\triangledown}{\mathrm{RES}}$ if, on any broken line L of finite length drawn on $\underset{\bullet}{\mathbb{C}}$ and starting in the domain of analyticity of the minor $\hat{\varphi}$, there exists a finite set Ω_L (depending on $\hat{\varphi}$) such that $\hat{\varphi}$ admits an analytic continuation along the paths which follow L but circumvent the points of Ω_L to the right or to the left.

This means that we do not impose the location of possibly singular points in advance, nor any constraint on the shape of the possible singularities. The alien derivations Δ_ω acting in $\overset{\triangledown}{\mathrm{RES}}$ are thus indexed by all $\omega \in \underset{\bullet}{\mathbb{C}}$; they are defined by the same formula as (67), but with $m-1 = \operatorname{card}\Omega_L$, for a segment $L = [\tau\omega, (1-\tau)\omega]$ with $\tau > 0$ small enough so that the points of $\Omega_L = \{\omega_1, \ldots, \omega_{m-1}\}$ be the only singular points encountered in the analytic continuation of the minor along $]0,\omega[$ (instead of $\{\frac{1}{m}\omega, \ldots, \frac{m-1}{m}\omega\}$), and with $\overset{\vee}{\Phi}_{\gamma(\varepsilon)}$ denoting the 2^{m-1} corresponding branchs of the minor near ω (of course, if none of them is singular at ω, then $\Delta_\omega \overset{\triangledown}{\varphi} = 0$). One can check that the modified formula defines an operator Δ_ω which is a derivation of the convolution algebra $\overset{\triangledown}{\mathrm{RES}}$.

Bridge equation for non-degenerate parabolic germs in the case $\rho \neq 0$

As an illustration of this enlarged formalism with more general resurgent functions than simple ones, let us return to non-degenerate parabolic germs with arbitrary resiter, as defined in Section 2.1: the holomorphic germ F at the origin gives rise to a germ at infinity

$$f(z) = z + 1 + a(z), \qquad a(z) = -\rho z^{-1} + \mathcal{O}(z^{-2}) \in \mathbb{C}\{z^{-1}\},$$

with $\rho \in \mathbb{C}$. As was mentioned after Proposition 4, Abel's equation $v \circ f = v + 1$ admits a formal solution

$$\tilde{v}(z) = z + \tilde{\psi}(z), \qquad \tilde{\psi} = \rho \log z + \sum_{n \geq 0} c_n z^{-n},$$

which is unique if we impose $c_0 = 0$. This "iterator" is formally invertible, with inverse

$$\tilde{u}(z) = z + \tilde{\varphi}(z), \qquad \tilde{\varphi}(z) = -\rho \log z + \sum_{\substack{n,m \geq 0 \\ n+m \geq 1}} C_{n,m} z^{-n} (z^{-1} \log z)^m.$$

[26] If $\overset{\bullet}{\omega} \in \Omega$ and $\overset{\triangledown}{\varphi} \in \overset{\triangledown}{\mathrm{RES}}_{\Omega}$, some branchs of the minor of $\Delta_{2\omega}\overset{\triangledown}{\varphi}$ will usually be singular at $-\overset{\bullet}{\omega}$, which should thus be included in Ω (for instance, with the minor $\hat{\varphi}(\zeta) = \frac{\omega}{\zeta - \omega} \log(1 - \frac{\zeta}{2\omega})$, which is regular at the origin, $\Delta_{2\omega}\overset{\triangledown}{\varphi} = {}^\flat(1 + \frac{\zeta}{\omega})$).

The inverse iterator allows one to describe all the solutions of the difference equation $f \circ u(z) = u(z+1)$ in the set $\{z - \rho \log z + \chi(z),\ \chi(z) \in \mathbb{C}[[z^{-1}, z^{-1}\log z]]\}$: they are the series of the form $\tilde{u}(z+c)$ with arbitrary $c \in \mathbb{C}$.

The space $\mathbb{C}[[z^{-1}, z^{-1}\log z]]$ is one of those spaces to which the formal Borel transform can be extended (*cf.* Section 3.2). For the Borel transform of $J_{\sigma,m}(z) = (-1)^m z^{-\sigma}(\log z)^m$, we have the formula

$$\overset{\triangledown}{J}_{\sigma,m} = {}^{\flat}\hat{J}_{\sigma,m}, \qquad \hat{J}_{\sigma,m}(\zeta) = \sum_{k=0}^{m} \binom{m}{k} \left(\frac{1}{\Gamma}\right)^{(k)}(\sigma)\, \zeta^{\sigma-1}(\log\zeta)^{m-k}, \qquad \Re e\, \sigma > 0, \quad m \in \mathbb{N}.$$

The analogue of Theorem 2 is that the series

$$\hat{\psi}_1(\zeta) = \sum c_n \frac{\zeta^{n-1}}{(n-1)!}, \qquad \hat{\varphi}_1(\zeta) = \sum (-1)^m C_{n,m} \hat{J}_{n+m,m}(\zeta)$$

converge for $|\zeta|$ small enough and define germs of ANA, which belong to $\widehat{\mathrm{RES}}_{2\pi i\mathbb{Z}}$, with at most exponential growth at infinity in the non-vertical directions. Thus

$$\overset{\triangledown}{v} = \delta' - \rho \overset{\triangledown}{J}_{0,1} + {}^{\flat}\hat{\psi}_1 \in \overset{\triangledown}{\mathrm{RES}}_{2\pi i\mathbb{Z}}, \qquad \overset{\triangledown}{u} = \delta' + \rho \overset{\triangledown}{J}_{0,1} + {}^{\flat}\hat{\varphi}_1 \in \overset{\triangledown}{\mathrm{RES}}_{2\pi i\mathbb{Z}}.$$

The analogue of Theorem 3 is the existence of complex numbers A_ω, the "analytic invariants" of f, such that, when pulled back in the z-variable, the action of the alien derivations is given by

$$\Delta_\omega \tilde{u} = A_\omega \frac{\mathrm{d}\tilde{u}}{\mathrm{d}z}, \qquad \Delta_\omega \tilde{v} = -A_\omega\, \mathrm{e}^{-\omega(\tilde{v}(z)-z)}.$$

Equivalently, we can say that $\tilde{\varphi}, \tilde{\psi}$ have formal Borel tansforms $\overset{\triangledown}{\varphi}, \overset{\triangledown}{\psi} \in \overset{\triangledown}{\mathrm{RES}}_{2\pi i\mathbb{Z}}$ and satisfy

$$\Delta_\omega \tilde{\varphi} = A_\omega\Big(1 + \frac{\mathrm{d}\tilde{\varphi}}{\mathrm{d}z}\Big), \qquad \Delta_\omega \tilde{\psi} = -A_\omega\, z^{-\rho\omega}\, \mathrm{e}^{-\omega\tilde{\psi}_1}.$$

The successive alien derivatives can also be computed, giving rise to formulas analogous to (44) and (45). This means in particular that, near ω, any branch of $\hat{\varphi}(\zeta)$ is of the form $\frac{B}{2\pi i(\zeta-\omega)} + \check{\chi}(\zeta - \omega) + \mathrm{reg}(\zeta-\omega)$, with a complex number B and an integrable singularity $\mathrm{sing}_0\big(\check{\chi}(\zeta)\big)$ which can be computed from $\tilde{\varphi}$ and from the invariants for each chosen branch), and any branch of $\hat{\psi}(\zeta)$ is of the form $-B\big(\check{I}_{\rho\omega}(\zeta) + b_1\check{I}_{\rho\omega+1}(\zeta) + b_2\check{I}_{\rho\omega+2}(\zeta) + \cdots\big) + \mathrm{reg}(\zeta-\omega)$ with computable complex numbers $b_1, b_2, \ldots$ When $\rho \neq 0$, neither $\overset{\triangledown}{\varphi}, \overset{\triangledown}{\psi}$ nor their alien derivatives[27] are simple singularities (and $\overset{\triangledown}{\psi}_1 = {}^{\flat}\hat{\psi}_1$ is a simple singularity, but $\Delta_\omega \overset{\triangledown}{\psi}_1 = \Delta_\omega \overset{\triangledown}{\psi}$ is not).

In the above, ω is any point of $\underset{\bullet}{\mathbb{C}}$ with $\overset{\bullet}{\omega} \in 2\pi i\mathbb{Z}^*$, but it turns out that the numbers A_ω depend on $\overset{\bullet}{\omega}$ only, since $\hat{\psi} = -\rho\hat{J}_{0,1} + \hat{\psi}_1$ where $\hat{\psi}_1(\zeta) \in \mathbb{C}\{\zeta\}$ and $\hat{J}_{0,1}(\zeta) = 1/\zeta$ carries no singularity outside the origin.[28] This fact can also be deduced from the existence of a resurgent $\tilde{U}(z) \in \mathrm{Id} + z^{-1}\mathbb{C}[[z^{-1}]]$ such that $\tilde{u}(z) = \tilde{U}\big(z + \chi(z^{-1}, z^{-1}\log z)\big)$ with $\chi(u,v) \in \mathbb{C}\{u,v\}$ (the alien derivatives $\Delta_\omega \tilde{U}$ depend on the projections $\overset{\bullet}{\omega}$ only, since $\hat{U}(\zeta) \in \mathbb{C}\{\zeta\}$, and the "alien

[27] Except the ones corresponding to ω such that $A_\omega = 0$, in case certain invariants vanish.

[28] Using the formula for $\check{I}_\sigma$ indicated in footnote 22, one finds $\overset{\triangledown}{J}_{0,1} = \mathrm{sing}_0\big(\frac{\log\zeta + \gamma + i\pi}{2\pi i\zeta}\big)$, where γ is Euler's constant.

chain rule" yields $\Delta_\omega \tilde{u} = \Delta_\omega(\tilde{U} \circ (\mathrm{Id} + \chi)) = \mathrm{e}^{-\dot{\omega}\chi}(\Delta_\omega \tilde{U}) \circ (\mathrm{Id} + \chi))$. The series $\tilde{U}$ and χ can be found by using the operator of "formal monodromy", *i.e.* the substitution $z \mapsto z\,\mathrm{e}^{2\pi\mathrm{i}}$ in the solution $\tilde{u}(z)$ defined by $z^{-n}(\log z)^m \mapsto z^{-n}(\log z + 2\pi\mathrm{i})^m$, which leads to another solution $\tilde{u}(z\,\mathrm{e}^{2\pi\mathrm{i}}) = z - \rho\log z - 2\pi\mathrm{i}\rho + o(1)$, whence $\tilde{u}(z\,\mathrm{e}^{2\pi\mathrm{i}}) = \tilde{u}(z - 2\pi\mathrm{i}\rho)$ (because all the solutions are known to be of the form $\tilde{u}(z+c)$); one then observes that the (convergent) transformation $z_* \mapsto z = z_* + \rho\log z_*$, $\tilde{u}(z) = \tilde{U}(z_*)$, transforms this monodromy relation into the trivial one $\tilde{U}(z_*\,\mathrm{e}^{2\pi\mathrm{i}}) = \tilde{U}(z_*)$ and has a convergent inverse of the form $z_* = z + \chi(z^{-1}, z^{-1}\log z)$.

The reader is referred to [Eca81, Vol. 2] for the results of this section.

4 Splitting problems

4.1 Second-order difference equations and complex splitting problems

We now wish to present the results of the article [GS01], and hint at some of the more general results to be found in the work in progress [GS06], in the context of 2-dimensional holomorphic transformations with a parabolic fixed point.

The aim is to understand a part of the local dynamics for a germ

$$F : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x + y + f(x,y) \\ y + f(x,y) \end{pmatrix}, \tag{68}$$

where $f(x,y) \in \mathbb{C}\{x,y\}$ is of the form

$$f(x,y) = -x^2 - \gamma xy + \mathcal{O}_3(x,y), \qquad \gamma \in \mathbb{C}.$$

It is shown in [GS06] that any germ of holomorphic map of $(\mathbb{C}^2, 0)$ with double eigenvalue 1 but non-identity differential at the origin can be reduced, by a local analytic change of coordinates, to the form (68) with $f(x,y) = b_{20}x^2 + b_{11}xy + \mathcal{O}_3(x,y)$; in the non-degenerate case, *i.e.* when $b_{20} \neq 0$, it is easy to normalise further the map so as to have $b_{20} = -1$ (observe that one can always remove the y^2-term, but it is not so for the xy-term: the complex number γ is in fact a *formal invariant* of the non-degenerate parabolic germ under consideration).

The article [GS01] is devoted to the particular map one obtains when $f(x,y) = -x^2$, which is an instance of the Hénon map (for a special choice of parameters). It is an invertible quadratic map $\mathbb{C}^2 \to \mathbb{C}^2$, the only fixed point of which is the origin, and which is symplectic for the standard symplectic structure $\mathrm{d}x \wedge \mathrm{d}y$.

Formal separatrix

Our main subject of investigation will be the *formal separatrix* of the map (68), which is a pair of formal series $\tilde{p}(z) = (\tilde{x}_0(z), \tilde{y}_0(z))$ satisfying $\tilde{p}(z+1) = F(\tilde{p}(z))$ and formally asymptotic to the origin, and its Borel sums $p^+(z)$ and $p^-(z)$. A slightly more geometric way of introducing $\tilde{p}(z)$ is to consider first the *formal infinitesimal generator* of F.

Indeed, it turns out that there exists a unique formal vector field

$$\tilde{X} = (y + \tilde{A}(x,y))\frac{\partial}{\partial x} + \tilde{B}(x,y)\frac{\partial}{\partial y}, \qquad \tilde{A}(x,y), \tilde{B}(x,y) = \mathcal{O}_2(x,y) \in \mathbb{C}[[x,y]],$$

the formal flow of which $\tilde{\Phi}(t,\cdot,\cdot) = \exp_t \tilde{X}$ satisfies $\exp_1 \tilde{X} = F$ (the flow of a formal vector field like $\tilde{X}$ is determined as the unique $\tilde{\Phi} \in (\mathbb{C}[t][[x,y]])^2$ such that $\tilde{\Phi}_{|t=0} = \mathrm{Id}$ and $\partial_t\tilde{\Phi} = \tilde{X} \circ \tilde{\Phi}$).

This is the 2-dimensional analogue of the infinitesimal generator mentioned in footnote 16. Now, for any 2-dimensional vector field with a singularity at the origin, there exists at least one "separatrix", *i.e.* a solution of the vector field which is asymptotic to the origin. This is the celebrated Camacho-Sad theorem for analytic vector fields; it is thus not a surprise that for our formal vector field there exists a formal solution $\tilde{p}(z)$ which is formally asymptotic to the origin. In practice, one finds that both components of $\tilde{p}(z)$ belong to the space $z^{-2}\mathbb{C}[[z^{-1}]]$ when $\gamma = 0$, and to the space $z^{-2}\mathbb{C}[[z^{-1}, z^{-1}\log z]]$ in the general case; moreover this solution is unique (under our non-degeneracy hypothesis) up to a time-shift $z \mapsto z + a$.

In some sense the dominant part in $\tilde{X}$ is $y\frac{\partial}{\partial x} - x^2\frac{\partial}{\partial y}$, which is the Hamiltonian vector field generated by $h(x,y) = \frac{1}{2}y^2 + \frac{1}{3}x^3$ for the standard symplectic structure (however the whole vector field $\tilde{X}$ itself is Hamiltonian only when F is symplectic, *i.e.* when the function $f(x,y)$ depends on its first argument only). The separatrix for this dominant part is given by the cusp $\{h(x,y) = 0\}$ (zero energy level), the time-parametrisation of which is $(-6z^{-2}, 12z^{-3})$. This is the leading term of the formal separatrix $\tilde{p}(z)$.

It turns out that the formal separatrix $\tilde{p}(z) = \big(\tilde{x}_0(z), \tilde{y}_0(z)\big)$ can be found directly from F, without any reference to the formal vector field $\tilde{X}$, *i.e.* without solving the formal differential equation $\partial_z\tilde{p}(z) = \tilde{X}\big(\tilde{p}(z)\big)$. One just needs to consider the difference equation

$$\tilde{p}(z+1) = F\big(\tilde{p}(z)\big) \tag{69}$$

(in fact the equations $\tilde{p}(z+t) = \exp_t\tilde{X}\big(\tilde{p}(z)\big)$ with $t \neq 0$ are all equivalent). This vector equation, in turn, is equivalent to the scalar equations

$$P\tilde{x}_0 = f(\tilde{x}_0, D\tilde{x}_0), \qquad \tilde{y}_0 = D\tilde{x}_0,$$

with difference operators P and D defined by

$$P\varphi(z) = \varphi(z+1) - 2\varphi(z) + \varphi(z-1), \qquad D\varphi(z) = \varphi(z) - \varphi(z-1) \tag{70}$$

(thanks to the special form (68) that we gave to the map F). One can thus eliminate $\tilde{y}_0$ and work with the equation $P\tilde{x}_0 = f(\tilde{x}_0, D\tilde{x}_0)$ alone, which is nothing but the nonlinear second-order difference equation (9) of Section 1.2 (up to a slight change of notation). The equation which corresponds to the Hénon map and is studied in [GS01] is simply $P\tilde{x}_0 = -\tilde{x}_0^2$.

When one is given an analytic vector field, the corresponding separatrix is convergent: the series $\tilde{x}_0(z)$ and $\tilde{y}_0(z)$ converge for $|z|$ large enough. One should not expect convergence for a general map F. The case of an entire map, like the Hénon map, is of particular interest, as one can prove divergence in this case. We shall see that the components of the separatrix are always resurgent and generically divergent.

We speak of "separatrix splitting" because the separatrix, which was convergent in the case of an analytic vector field, breaks (becomes formal) when one passes to maps and gives rise to two distinct curves (two Borel sums of $\tilde{p}(z)$, none of which is the analytic continuation of the other).

We shall now proceed and describe the results concerning $\tilde{x}_0(z)$ and its resurgent structure which are given in [GS01] for the Hénon map. We shall see that one of the novelties of the 2-dimensional case with respect to Section 2 is the necessity of considering a formal solution more general than the formal series $\tilde{x}_0(z)$, namely the "formal integral" $\tilde{x}(z,b)$ which depends on a further formal variable and allows one to write a Bridge equation. The terminology comes from [Eca81, Vol. 3], as well as the ideas for the resurgent approach (although the case of parabolic maps like (68) is not covered by this reference).

First resurgence relations

From now on we thus set $f(x,y) = -x^2$ and we keep using the notation (70) for the difference operators D and P.

Theorem 5 *The nonzero solutions $x(z) \in \mathbb{C}[[z^{-1}]]$ of the equation*

$$Px = -x^2 \tag{71}$$

are the formal series $x(z) = \tilde{x}_0(z+a)$, where $a \in \mathbb{C}$ is arbitrary and $\tilde{x}_0(z)$ is the unique nonzero even formal solution:

$$\tilde{x}_0(z) = -6z^{-2} + \frac{15}{2}z^{-4} - \frac{663}{40}z^{-6} + \cdots$$

They are resurgent and Borel summable: the formal Borel transform $\hat{x}_0 = \mathcal{B}\tilde{x}_0$ has positive radius of convergence and extends to $\mathcal{R}$ ($\hat{x}_0(\zeta) \in \widehat{\mathcal{H}}(\mathcal{R}) = \mathbb{C}\{\zeta\} \cap \widehat{\mathrm{RES}}_{2\pi i\mathbb{Z}}$), with at most exponential growth at infinity along non-vertical directions.

Idea of the proof. The formal part of the statement can be obtained by substitution of $x(z) = \sum_{n \geq n_0} a_n z^{-n}$ with $a_{n_0} \neq 0$ into (71): one finds that necessarily $n_0 = 2$ and $a_2 = -6$, then a_3 is free whereas all the successive coefficients are uniquely determined. Choosing $a_3 = 0$ yields the even solution $\tilde{x}_0(z)$, while the general solution must coincide with $\tilde{x}_0(z + \frac{a_3}{12})$.

The Borel transforms of the solutions are studied through the equation they satisfy: the counterpart of P is multiplication by $\alpha(\zeta) = \mathrm{e}^{-\zeta} - 2 + \mathrm{e}^{\zeta}$, hence

$$\alpha\,\hat{x} = -\hat{x} * \hat{x}, \qquad \alpha(\zeta) = 4\sinh^2\frac{\zeta}{2}. \tag{72}$$

We know in advance that this equation has a unique formal solution of the form $\hat{x}_0(\zeta) = -6\zeta + \hat{v}(\zeta)$ with $\hat{v}(\zeta) \in \zeta^3\mathbb{C}[[\zeta]]$. The corresponding equation for $\hat{v}$ is

$$\alpha\,\hat{v} - 12\zeta * \hat{v} = 6\big(\zeta\alpha(\zeta) - \zeta^3\big) - \hat{v} * \hat{v}.$$

As in the proof of Theorem 2 in Section 2.1, one can devise a method of majorants to prove that $\hat{v}(\zeta)$ has positive radius of convergence and extends analytically to the sets $\mathcal{R}_c^{(0)}$ which were defined there, with at most exponential growth at infinity (and, in fact, exponential decay). The method can also be adapted to reach the union $\mathcal{R}^{(1)}$ of the half-sheets which are contiguous to the principal sheet; analyticity is then propagated to the rest of $\mathcal{R}$ through the resurgence relations to be shown below. See [GS01] for the details (or [OSS03] for an analogous proof). □

Observe that the only source of singularities in the Borel plane is the division by $\alpha(\zeta)$ when solving equation (72), this is why the only possible singular points are the points of $2\pi i\,\mathbb{Z}$.

We shall see that the first singularities of $\hat{x}_0(\zeta)$, *i.e.* the singularities at $\zeta = \pm 2\pi i$, are not apparent ones, thus $\tilde{x}_0(z)$ is divergent. The coefficients of $\tilde{x}_0(z)$ are real numbers and $\hat{x}_0(\zeta)$ is thus real-analytic; therefore, the singularity at $-2\pi i$ can be deduced by symmetry from the singularity at $2\pi i$. We now use alien calculus to analyse the singularity at $2\pi i$.

Let $\overset{\triangledown}{x}_0 = {}^\flat\hat{x}_0$; we thus have $\overset{\triangledown}{x}_0 \in {}^\flat\big(\mathbb{C}\{\zeta\} \cap \widehat{\mathrm{RES}}_{2\pi i\mathbb{Z}}\big)$, solution of

$$\alpha\,\overset{\triangledown}{x}_0 = -\overset{\triangledown}{x}_0 * \overset{\triangledown}{x}_0. \tag{73}$$

A major of the singularity $\overset{\triangledown}{\chi} = \Delta_{2\pi i}\overset{\triangledown}{x}_0$ can be defined by $\overset{\vee}{\chi}(\zeta) = \hat{x}_0(2\pi i + \zeta)$, $-\frac{3\pi}{2} < \arg\zeta < \frac{\pi}{2}$, $|\zeta| < 2\pi$. We shall show that $\overset{\triangledown}{\chi}$ can be expressed as the linear combinations of two elementary singularities deduced from $\overset{\triangledown}{x}_0$. The key point is the possibility of "alien-differentiating" equation (73): for any $\omega \in 2\pi i\mathbb{Z}$, the singularity $\overset{\triangledown}{\varphi} = \Delta_\omega\overset{\triangledown}{x}_0$ must satisfy $\alpha\overset{\triangledown}{\varphi} = -2\overset{\triangledown}{x}_0 * \overset{\triangledown}{\varphi}$ (because Δ_ω is a derivation which commutes with the multiplication by α).[29]

Proposition 9 *The linear difference equation*

$$P\varphi = -2\tilde{x}_0\varphi \tag{74}$$

admits a unique even solution $\tilde{\varphi}_2(z) \in \mathbb{C}[[z^{-1}]][z]$ of the form $\frac{1}{84}z^4\big(1+\mathcal{O}(z^{-1})\big)$. It belongs in fact to the space $\mathbb{C}((z^{-1}))_1 = \mathbb{C}[[z^{-1}]]_1[z]$ and has a formal Borel transform of the form

$$\overset{\triangledown}{\varphi}_2 = \frac{1}{84}\delta^{(4)} + \frac{17}{840}\delta^{(2)} - \frac{17}{2240}\delta + {}^\flat\hat{\varphi}_2$$

with $\hat{\varphi}_2(\zeta) \in \widehat{\mathcal{H}}(\mathcal{R})$. Moreover, the solutions of the linear equation

$$\alpha\overset{\triangledown}{\varphi} = -2\overset{\triangledown}{x}_0 * \overset{\triangledown}{\varphi}, \qquad \overset{\triangledown}{\varphi} \in \mathrm{SING} \tag{75}$$

are the linear combinations (with constant coefficients) of $\overset{\triangledown}{\varphi}_1 = \partial\overset{\triangledown}{x}_0$ and $\overset{\triangledown}{\varphi}_2$. In particular, there exist $\mu \in \mathbb{R}$ and $\Theta \in i\mathbb{R}$ such that

$$\Delta_{2\pi i}\overset{\triangledown}{x}_0 = \mu\overset{\triangledown}{\varphi}_1 + \Theta\overset{\triangledown}{\varphi}_2, \qquad \Delta_{-2\pi i}\overset{\triangledown}{x}_0 = -\mu\overset{\triangledown}{\varphi}_1 + \Theta\overset{\triangledown}{\varphi}_2. \tag{76}$$

Proof. Let us first consider equation (74) in the space of formal series $\mathbb{C}[[z^{-1}]][z]$. This equation being the linearization of equation (71), the ordinary derivative $\tilde{\varphi}_1 = \partial\tilde{x}_0 = 12z^{-3}\big(1+\mathcal{O}(z^{-2})\big)$ is obviously a particular solution (thus its formal Borel transform $\overset{\triangledown}{\varphi}_1 = \partial\overset{\triangledown}{x}_0$, which we know belongs to ${}^\flat\big(\mathbb{C}\{\zeta\}\big) \subset \mathrm{SING}$, satisfies (75)).

Standard tools of the theory of second-order linear difference equations (see [GL01]) allow us to find an independent solution $\tilde{\varphi}_2$: $\tilde{\varphi}_2 = \tilde{\psi}\tilde{\varphi}_1$ is solution as soon as $D\tilde{\psi} = \tilde{\chi}$, where $\tilde{\chi}(z) = \frac{1}{\tilde{\varphi}_1(z)\tilde{\varphi}_1(z-1)} = \frac{1}{144}z^6\big(1+\mathcal{O}(z^{-1})\big)$. The latter equation determines $\tilde{\psi}(z)$ up to an additive constant, since it can be rewritten $\partial\,\beta(\partial)\tilde{\psi} = \tilde{\chi}$, with an invertible power series $\beta(X) = \frac{1-e^{-X}}{X} = 1+\mathcal{O}(X)$. We just need to choose a primitive $\partial^{-1}\tilde{\chi}$ and we get the corresponding $\tilde{\psi} = \gamma(\partial)\,\partial^{-1}\tilde{\chi}$, with $\gamma(X) = \frac{1}{\beta(X)} = 1 + \gamma_1X + \gamma_2X^2 + \cdots$. It turns out that the primitives of $\tilde{\chi}(z)$ belong to $\mathbb{C}[[z^{-1}]][z]$: there is no $\log z$ term because the coefficient of z^{-1} in $\tilde{\chi}(z)$ is zero (this is due to the special form of $\tilde{\chi}(z) = \tilde{\chi}_1(z)\tilde{\chi}_1(z-1)$, with $\tilde{\chi}_1 = 1/\tilde{\varphi}_1$ odd: use the Taylor formula and observe that $\tilde{\chi}_1\tilde{\chi}_1^{(k)}$ is even when k is even, and that it is the derivative of an element of $\mathbb{C}[[z^{-1}]][z]$ when k is odd). These solutions $\tilde{\psi}\tilde{\varphi}_1$ are thus elements of $\mathbb{C}[[z^{-1}]][z]$, of the form $\frac{1}{84}z^4\big(1+\mathcal{O}(z^{-1})\big)$, differing one from the other by a multiple of $\tilde{\varphi}_1$, and it is easy to check that exactly one of them is our even solution $\tilde{\varphi}_2$. The coefficients can be determined inductively from those of $\tilde{x}_0$; one finds $\tilde{\varphi}_2(z) = \frac{1}{84}z^4 + \frac{17}{840}z^2 - \frac{17}{2240} + \mathcal{O}(z^{-2})$.

[29] We do not present the arguments in full rigour here. For instance, the method of majorants we alluded to for the proof of Theorem 5 yields the analyticity of $\hat{x}_0$ in $\mathcal{R}^{(1)}$, hence the above $\overset{\vee}{\chi}$, at this level, is only a major of "sectorial" singularity: $\overset{\vee}{\chi} \in \mathrm{ANA}_{\frac{\pi}{2},\delta}$ for $0 < \delta < \frac{\pi}{2}$ with the notation of footnote 21. The subsequent arguments should thus be rephrased in the corresponding space $\mathrm{SING}_{\frac{\pi}{2},\delta}$ rather than SING, in order to establish the relations (76). One would then use these relations to propagate the analyticity of $\hat{x}_0$ in $\mathcal{R}^{(2)}$, and argue similarly to reach farther and farther half-sheets of $\mathcal{R}$, using gradually all the resurgent relations expressed by the Bridge Equation of Theorem 6 below.

We have $\tilde{\varphi}_2 \in \mathbb{C}((z^{-1}))_1$ and the minor of its formal Borel transform is in $\widehat{\mathcal{H}}(\mathcal{R})$, because the same is true of the above series $\tilde{\chi}$, $\partial^{-1}\tilde{\chi}$ and $\tilde{\psi}$. Indeed, we can write $\tilde{\varphi}_1(z)\tilde{\varphi}_1(z-1) = 144z^{-6}\big(1-\tilde{w}(z)\big)$ with $\tilde{w}(z) \in z^{-1}\mathbb{C}[[z^{-1}]]$ and $\hat{w}(\zeta) = -\frac{1}{144}\big(\frac{\mathrm{d}}{\mathrm{d}\zeta}\big)^6\big[(\zeta\hat{x}_0) * (\zeta \mathrm{e}^{\zeta}\hat{x}_0)\big] \in \widehat{\mathcal{H}}(\mathcal{R})$; Proposition 3 then ensures that $(1-\tilde{w})^{-1}$ is in $\mathbb{C}[[z^{-1}]]_1$ (thus $\tilde{\chi}(z) = \frac{1}{144}z^6\big(1-\tilde{w}(z)\big)^{-1}$ and its primitives lie in $\mathbb{C}((z^{-1}))_1$) and that the minor of the formal Borel transform of $(1-\tilde{w})^{-1}$ is in $\widehat{\mathcal{H}}(\mathcal{R})$ (thus the minor $\hat{\chi}$ of $\overset{\triangledown}{\chi} = \mathcal{B}\tilde{\chi}$ lies also in $\widehat{\mathcal{H}}(\mathcal{R})$, and so does $\frac{1}{\zeta}\hat{\chi}(\zeta)$ which is the minor corresponding to the primitives of $\tilde{\chi}$); the conclusion for $\tilde{\psi}$ and $\tilde{\varphi}_2$ follows easily (the operator $\gamma(\partial)$ amounts to the multiplication by $-\frac{\zeta}{1-\mathrm{e}^{\zeta}}$ in the Borel plane).

We can now easily describe the solutions of equation (75), or even those of the inhomogeneous equation

$$\alpha\overset{\triangledown}{\varphi} + 2\overset{\triangledown}{x}_0 * \overset{\triangledown}{\varphi} = \overset{\triangledown}{\psi}, \tag{77}$$

with any given $\overset{\triangledown}{\psi} \in \mathrm{SING}$. It is sufficient to use the finite-difference Wronskian

$$\mathcal{W}(\tilde{\psi}_1,\tilde{\psi}_2)(z) := \det\begin{pmatrix}\tilde{\psi}_1(z-1) & \tilde{\psi}_2(z-1)\\ \tilde{\psi}_1(z) & \tilde{\psi}_2(z)\end{pmatrix}, \qquad \tilde{\psi}_1,\tilde{\psi}_2 \in \mathbb{C}[[z^{-1}]][z]$$

or rather its Borel counterpart

$$\mathcal{W}(\overset{\triangledown}{\psi}_1,\overset{\triangledown}{\psi}_2) = (\mathrm{e}^{\zeta}\overset{\triangledown}{\psi}_1) * \overset{\triangledown}{\psi}_2 - \overset{\triangledown}{\psi}_1 * (\mathrm{e}^{\zeta}\overset{\triangledown}{\psi}_2), \qquad \overset{\triangledown}{\psi}_1,\overset{\triangledown}{\psi}_2 \in \mathrm{SING}.$$

One can indeed check that $\mathcal{W}(\tilde{\varphi}_1,\tilde{\varphi}_2) = 1$ and that equation (77) is thus reduced to

$$(\mathrm{e}^{-\zeta}-1)\overset{\triangledown}{c}_1 = -\overset{\triangledown}{\psi} * \overset{\triangledown}{\varphi}_2, \qquad (\mathrm{e}^{-\zeta}-1)\overset{\triangledown}{c}_2 = \overset{\triangledown}{\psi} * \overset{\triangledown}{\varphi}_1 \tag{78}$$

by the change of unknown

$$\begin{cases} \overset{\triangledown}{\varphi} = \overset{\triangledown}{c}_1 * \overset{\triangledown}{\varphi}_1 + \overset{\triangledown}{c}_2 * \overset{\triangledown}{\varphi}_2 \\ \mathrm{e}^{\zeta}\overset{\triangledown}{\varphi} = \overset{\triangledown}{c}_1 * (\mathrm{e}^{\zeta}\overset{\triangledown}{\varphi}_1) + \overset{\triangledown}{c}_2 * (\mathrm{e}^{\zeta}\overset{\triangledown}{\varphi}_2) \end{cases} \Leftrightarrow \begin{cases} \overset{\triangledown}{c}_1 = \mathcal{W}(\overset{\triangledown}{\varphi},\overset{\triangledown}{\varphi}_2) \\ \overset{\triangledown}{c}_2 = \mathcal{W}(\overset{\triangledown}{\varphi}_1,\overset{\triangledown}{\varphi}). \end{cases}$$

For the homogeneous equation (75) we have $\overset{\triangledown}{\psi} = 0$, thus the only solutions of (78) are $(\overset{\triangledown}{c}_1,\overset{\triangledown}{c}_2) = (C_1\,\delta, C_2\,\delta)$ with arbitrary $C_1, C_2 \in \mathbb{C}$. Indeed, the equation for $\overset{\triangledown}{c}_1$ for instance amounts to $\overset{\triangledown}{c}_1 = \mathrm{sing}_0\big(\frac{\mathrm{reg}(\zeta)}{\mathrm{e}^{-\zeta}-1}\big)$ with an arbitrary $\mathrm{reg}(\zeta) \in \mathbb{C}\{\zeta\}$, of which the value at 0 will be $-\frac{C_1}{2\pi\mathrm{i}}$.

We already saw that $\Delta_{2\pi\mathrm{i}}\overset{\triangledown}{x}_0$ was solution of (75), this yields complex numbers $C_1 = \mu$ and $C_2 = \Theta$. Since $\hat{x}_0$ is real-analytic and odd, it is purely imaginary on the imaginary axis; since the coefficients of $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are real, the parity properties imply that $\mu \in \mathbb{R}$ and $\Theta \in \mathrm{i}\mathbb{R}$. The statement for $\Delta_{-2\pi\mathrm{i}}\overset{\triangledown}{x}_0$ is obtained by symmetry. □

The fact that $\Delta_{\pm 2\pi\mathrm{i}}\overset{\triangledown}{x}_0 \in \mathrm{SING}^{\mathrm{s.ram.}}$ means that the first singularities of $\hat{x}_0$ are of the form {polar part} + {logarithmic singularity with regular variation}. More precisely, with the notation

$$\tilde{\varphi}_1(z) = \partial\tilde{x}_0(z) = \sum_{k\geq 1} b_k z^{-2k-1}, \qquad \tilde{\varphi}_2(z) = \sum_{k\geq -2} d_k z^{-2k},$$

the principal branch of $\hat{x}_0$ satisfies

$$\hat{x}_0(2\pi\mathrm{i}+\zeta) = \frac{\Theta}{2\pi\mathrm{i}}\left(d_{-2}\frac{4!}{\zeta^5} + d_{-1}\frac{2!}{\zeta^3} + d_0\frac{1}{\zeta}\right) + \frac{1}{2\pi\mathrm{i}}\big(\Theta\hat{\varphi}_2(\zeta) + \mu\hat{\varphi}_1(\zeta)\big)\log\zeta + \mathrm{reg}(\zeta), \tag{79}$$

$$\hat{x}_0(-2\pi\mathrm{i}+\zeta) = \frac{\Theta}{2\pi\mathrm{i}}\left(d_{-2}\frac{4!}{\zeta^5} + d_{-1}\frac{2!}{\zeta^3} + d_0\frac{1}{\zeta}\right) + \frac{1}{2\pi\mathrm{i}}\big(\Theta\hat{\varphi}_2(\zeta) - \mu\hat{\varphi}_1(\zeta)\big)\log\zeta + \mathrm{reg}(\zeta). \tag{80}$$

The coefficients b_k and d_k can be computed inductively, whereas the constants Θ and μ must be considered as transcendent: in the case of a more general map (68), there would be relations analogous to (76) in which the corresponding constants do not depend on finitely many coefficients of f only. In the case of the Hénon map, a positivity argument leads to

Proposition 10 *The constant $\Theta \in \mathrm{i}\mathbb{R}$ determined in equation* (76) *of Proposition 9 satisfies*

$$\Im m\, \Theta < 0.$$

Proof. We first show that $\tilde{x}_0(z) = \sum_{k\geq 1} a_k z^{-2k}$ with $(-1)^k a_k > 0$. Consider the auxiliary series $\tilde{U}(t) = \tilde{x}_0(\mathrm{i}t) = \sum_{k\geq 1}(-1)^k a_k t^{-2k}$: it is the unique nonzero even solution of

$$-\tilde{U}(t+\mathrm{i}) + 2\tilde{U}(t) - \tilde{U}(t-\mathrm{i}) = \tilde{U}(t)^2.$$

This equation can be rewritten $\partial_t^2 \tilde{U} = \Gamma(\partial_t)(\tilde{U}^2)$, with a convergent power series $\Gamma(X) = \frac{X^2}{4\sin^2\frac{X}{2}} = 1 + \Gamma_1 X^2 + \Gamma_2 X^4 + \cdots$ which has only non-negative coefficients (as can be seen from the decomposition of the meromorphic function $1/\sin^2\frac{X}{2}$ as a series of second-order poles). One can thus write induction formulas for the coefficients of $\tilde{U}(t)$ which show that they are positive.

As a consequence, the Borel transform $\hat{U}(\tau) = \mathrm{i}\,\hat{x}_0(\mathrm{i}\tau) = \sum_{k\geq 1}(-1)^k a_k \frac{\tau^{2k-1}}{(2k-1)!}$ (which is convergent at least in the disc of radius 2π) is positive and increasing on the segment $]0, 2\pi[$. But this function satisfies

$$\left(4\sin^2\frac{\tau}{2}\right)\hat{U}(\tau) = \hat{U} * \hat{U}(\tau),$$

hence it cannot be bounded on $]0, 2\pi[$ (if it were, the left-hand side would tend to 0 as $\tau \xrightarrow{<} 2\pi$, whereas the right-hand side is positive increasing).

Now, in view of (79), the fact that $\hat{x}_0(\zeta)$ is not bounded on $]0, 2\pi\mathrm{i}[$ shows that $\Theta \neq 0$ (because $\hat{\varphi}_1(\zeta) = \mathcal{O}(\zeta^2)$). Moreover, $0 < \hat{U}(\tau) = \mathrm{i}\,\hat{x}_0(\mathrm{i}\tau) \sim \frac{1}{2\pi}\frac{\Theta}{84}\frac{4!}{(\mathrm{i}\tau - 2\pi\mathrm{i})^5}$ for $\tau \xrightarrow{<} 2\pi$ implies $\mathrm{i}\Theta > 0$. □

Numerically, one finds $|\Theta| \simeq 2.474 \cdot 10^6$, $\mu \simeq 4.909 \cdot 10^3$ (much better accuracy can be achieved thanks to the precise information we have on the form of the singularity—see [GS01]).

The parabolic curves $p^+(z)$ and $p^-(z)$ and their splitting

We pause here in the description of the resurgent structure of $\tilde{x}_0$ to give a look at the analytic consequences we can already deduce from the above.

Borel-Laplace summation yields two analytic solutions $x^+(z)$ and $x^-(z)$ of equation (71):

$$x^{\pm}(z) = \mathcal{L}^{\pm}\hat{x}_0(z) \sim \tilde{x}_0(z), \qquad z \in \mathcal{D}^{\pm}$$

(with notations analogous to those of Section 2). The analysis of the first singularities of $\hat{x}_0(\zeta)$ is sufficient to describe the asymptotic behaviour of the difference $(x^+ - x^-)(z)$ when $z \in \mathcal{D}^+ \cap \mathcal{D}^-$ with $\Im m\, z > 0$ or $\Im m\, z < 0$: when $\Im m\, z < 0$ for instance, one can argue as at the end of Section 2.4 and write

$$(x^+ - x^-)(z) = \int_{\gamma_1} \mathrm{e}^{-z\zeta}\,\hat{x}_0(\zeta)\,\mathrm{d}\zeta + \int_{\Gamma} \mathrm{e}^{-z\zeta}\,\hat{x}_0(\zeta)\,\mathrm{d}\zeta,$$

with the same path γ_1 as on Figure 11 (with $\omega = 0$), and with a path Γ coming from $\mathrm{e}^{\mathrm{i}\theta'}\infty$, passing slightly below $4\pi\mathrm{i}$ and going to $\mathrm{e}^{\mathrm{i}\theta}\infty$. The first integral is exactly $\mathrm{e}^{-2\pi\mathrm{i}z}\mathcal{L}^-(\Delta_{2\pi\mathrm{i}}\check{x}_0)(z)$ (*cf.*

the section on the Laplace transform of majors), while the second integral is $\mathcal{O}(\mathrm{e}^{-(4\pi-\delta)|\Im m\, z|})$ for any $\delta > 0$. We thus obtain

$$\mathrm{e}^{2\pi \mathrm{i} z}(x^+ - x^-)(z) \sim \Delta_{2\pi\mathrm{i}}\tilde{x}_0(z) = \Theta\tilde{\varphi}_2(z) + \mu\tilde{\varphi}_1(z), \qquad z \in \mathcal{D}^+ \cap \mathcal{D}^-, \quad \Im m\, z < 0$$

(and similarly with $\Delta_{-2\pi\mathrm{i}}\tilde{x}_0(z)$ for $\Im m\, z > 0$). Since $\Theta \neq 0$, the leading term for this exponentially small discrepancy is

$$x^+(z) - x^-(z) \sim \tfrac{\Theta}{84} z^4\, \mathrm{e}^{\pm 2\pi\mathrm{i} z}, \qquad z \in \mathcal{D}^+ \cap \mathcal{D}^-, \quad \pm\Im m\, z > 0.$$

Observe that both x^+ and x^- extend from $\mathcal{D}^\pm$ to entire functions, as would any solution of the difference equation (71) analytic in a half-plane bounded by a non-horizontal line (for instance, the analytic continuation of x^+ is defined by iterating $x^+(z-1) = 2x^+(z) - x^+(z+1) - \big(x^+(z)\big)^2$). But the simple asymptotic behaviour of $x^\pm$ described by $\tilde{x}_0$ does not extend beyond $\mathcal{D}^\pm$: this is the Stokes phenomenon we have just described.

We finally return to the dynamics to the Hénon map F and supplement the solutions $\tilde{x}_0(z)$ and $x^\pm(z)$ of the scalar equation (71) with the appropriate y-components, $\tilde{y}_0 = D\tilde{x}_0$ and $y^\pm = Dx^\pm$, in order to define solutions of equation (69):

Proposition 11 *There exist two holomorphic maps $p^+ : \mathbb{C} \to \mathbb{C}^2$ and $p^- : \mathbb{C} \to \mathbb{C}^2$ satisfying equation* (69)*:*

$$p^\pm(z+1) = F\big(p^\pm(z)\big), \qquad z \in \mathbb{C},$$

and admitting the same asymptotic expansion in different domains:

$$p^\pm(z) \sim \tilde{p}(z), \qquad z \in \mathcal{D}^\pm,$$

where $\tilde{p}(z) = \big(-6z^{-2} + \mathcal{O}(z^{-4}), 12z^{-3} + \mathcal{O}(z^{-4})\big)$ is a formal solution of (69). *Moreover,*

$$\mathrm{e}^{\pm 2\pi\mathrm{i} z}\big(p^+(z) - p^-(z)\big) \sim \Theta\tilde{N}(z) \pm \mu\frac{\mathrm{d}\tilde{p}}{\mathrm{d}z}(z), \qquad z \in \mathcal{D}^+ \cap \mathcal{D}^-, \quad \pm\Im m\, z < 0,$$

with the notations of Proposition 9 and $\tilde{N} = (\tilde{\varphi}_2, D\tilde{\varphi}_2) = \big(\frac{1}{84}z^4 + \mathcal{O}(z^2), \frac{1}{21}z^3 + \mathcal{O}(z^2)\big)$.

Observe that the symplectic 2-form $\mathrm{d}x \wedge \mathrm{d}y$ yields 1 when evaluated on $\big(\frac{\mathrm{d}\tilde{p}}{\mathrm{d}z}(z), \tilde{N}(z)\big)$. The constant Θ thus describes the normal component of the splitting, while μ describes the tangential component.

Because of equation (69), the curves $\mathcal{W}^+ = \{p^+(z),\ z \in \mathbb{C}\}$ and $\mathcal{W}^- = \{p^-(z),\ z \in \mathbb{C}\}$ are invariant by F with

$$F^n\big(p^\pm(z)\big) = p^\pm(z+n) \xrightarrow[n\to\pm\infty]{} 0$$

(in view of their common asymptotic series). They may be called "stable and unstable separatrices" (by analogy with the stable and unstable manifolds of a hyperbolic fixed point), or "parabolic curves" (as is more common in the litterature on 2-dimensional holomorphic maps).

Formal integral and Bridge equation

We end with the complete description of the resurgent structure of the formal solution $\tilde{x}_0(z)$ of equation (71). Taking for granted the possibility of following the analytic continuation of $\hat{x}_0(\zeta)$ in $\mathcal{R}$, as ascertained by Theorem 5, and consequently the possibility of defining the singularities $\Delta_{\omega_1}\cdots\Delta_{\omega_r}\overset{\triangledown}{x}_0$ for all $r\geq 1$ and $\omega_1,\dots,\omega_r\in 2\pi\mathrm{i}\,\mathbb{Z}^*$, we see that the there must exist complex numbers $A_{\omega,0}$, $B_{\omega,0}$ such that $A_{\pm 2\pi\mathrm{i},0}=\pm\mu$, $B_{\pm 2\pi\mathrm{i},0}=\Theta$ and

$$\Delta_\omega\overset{\triangledown}{x}_0 = A_{\omega,0}\,\overset{\triangledown}{\varphi}_1 + B_{\omega,0}\,\overset{\triangledown}{\varphi}_2, \qquad \omega\in 2\pi\mathrm{i}\,\mathbb{Z}^*. \tag{81}$$

Indeed, the same arguments as those which led to Proposition 9 apply and each $\Delta_\omega\overset{\triangledown}{x}_0$ is a solution of the linearized equation (75). But this is not a closed system of resurgence relations: in order to compute $\Delta_{\omega_1}\Delta_{\omega_2}\overset{\triangledown}{x}_0$ for instance, we need to know the alien derivatives of $\overset{\triangledown}{\varphi}_2$ (on the other hand the alien derivatives of $\overset{\triangledown}{\varphi}_1$ can be deduced from (81) and from the commutation relation (32) of Proposition 6).

This problem is solved by the notion of "formal integral". Recalling that $\tilde{\varphi}_1=\partial\tilde{x}_0\in z^{-3}\mathbb{C}[[z^{-1}]]$ and setting $\tilde{x}_1=\tilde{\varphi}_2\in z^4\mathbb{C}[[z^{-1}]]$, we can consider $\tilde{x}_0(z)+b\tilde{x}_1(z)$, where b is a small deformation parameter, as a solution of equation (71) up to $\mathcal{O}(b^2)$. We can also consider this expression as the beginning of an exact solution belonging to $\big(\mathbb{C}[[z^{-1}]][z]\big)[[b]]$:

Proposition 12 *There exist formal series $\tilde{x}_2(z),\tilde{x}_3(z),\dots$ in $\mathbb{C}[[z^{-1}]][z]$ such that*

$$\tilde{x}(z,b)=\sum_{n\geq 0} b^n\tilde{x}_n(z) \tag{82}$$

solves equation (71)*. The $\tilde{x}_n$'s are uniquely determined by the further requirement that they be even and do not contain any z^4-term. Moreover,*

$$\tilde{x}_n(z)\in z^{6n-2}\mathbb{C}[[z^{-1}]]_1, \qquad \overset{\triangledown}{x}_n=\mathcal{B}\tilde{x}_n\in \overset{\triangledown}{\mathrm{RES}}_{2\pi\mathrm{i}\mathbb{Z}}.$$

Proof. Plugging (82) into (71) and expanding in powers of b, we recover the known equations $P\tilde{x}_0=-\tilde{x}_0^2$ and $P\tilde{x}_1+2\tilde{x}_0\tilde{x}_1=0$ at orders 0 and 1, and then a system of inhomogeneous linear equations to be solved inductively:

$$P\tilde{x}_n+2\tilde{x}_0\tilde{x}_n=\tilde{\psi}_n, \qquad \tilde{\psi}_n=-\sum_{n_1=1}^{n-1}\tilde{x}_{n_1}\tilde{x}_{n-n_1}, \qquad n\geq 2. \tag{83}$$

In the course of the proof of Proposition 9, we saw how to solve such equations (admittedly their Borel counterparts, but this makes no difference to the algebraic structure of their solution): $\tilde{x}_n=\tilde{c}_1\tilde{\varphi}_1+\tilde{c}_2\tilde{\varphi}_2$ is solution of (83) as soon as

$$\tilde{c}_j(z+1)-\tilde{c}_j(z)=\tilde{\chi}_j(z), \qquad \tilde{\chi}_1=-\tilde{\varphi}_2\tilde{\psi}_n, \qquad \tilde{\chi}_2=\tilde{\varphi}_1\tilde{\psi}_n.$$

Cancellations occur so that the coefficient of z^{-1} in $\tilde{\chi}_1$ and $\tilde{\chi}_2$ vanishes,[30] hence the primitives of $\tilde{\chi}_j$ have no $\log z$ term. The end of the proof is just a matter of selecting appropriately the

[30]See the proof of Proposition 6 in [GS01] (beware of the misprint in the last formula on p. 551, which corresponds to an incorrect expansion of the (correct) identity indicated in the footnote on that page). Besides, these cancellations are special to the case of a symplectic map (68), *i.e.* with f depending on its first argument only; the formalism of Section 3 is perfectly capable to handle the general case with $\log z$ terms—see [GS06].

primitives $\partial^{-1}\tilde{\chi}_j$, setting $\tilde{c}_j = \gamma(-\partial)\,\partial^{-1}\tilde{\chi}_j$ (with the notations of the proof of Proposition 9) and counting the valuations—observe that the Borel tranforms $\overset{\triangledown}{c}_j = -\frac{\zeta}{e^{-\zeta}-1}\mathcal{B}(\partial^{-1}\tilde{\chi}_j)$, whence the analyticity in $\mathcal{R}$ of the minors follows by induction. The reader is referred to [GS01, §5.1] for the details. □

The series $\tilde{x}(z,b) = \sum b^n\tilde{x}_n(z)$ is called a *formal integral* of equation (71). We thus have a kind of two-parameter family of solutions of (71), namely $\tilde{x}(z+a,b)$, which is consistent with the fact that we are dealing with a second-order equation.[31] At the level of the map F, this corresponds to a parametrisation of the "formal invariant foliation" of the formal infinitesimal generator $\tilde{X}$. Here is the implication for the resurgent structure of $\tilde{x}_0$:

Theorem 6 *For each $\omega \in 2\pi i\,\mathbb{Z}^*$, there exist formal series*

$$A_\omega(b) = \sum_{n\geq 0} A_{\omega,n}b^n,\quad B_\omega(b) = \sum_{n\geq 0} B_{\omega,n}b^n \in \mathbb{C}[[b]]$$

such that

$$\Delta_\omega\tilde{x}(z,b) = \big(A_\omega(b)\partial_z + B_\omega(b)\partial_b\big)\tilde{x}(z,b), \tag{84}$$

this "Bridge equation" being understood as a compact writing of infinitely many "resurgence relations"

$$\Delta_\omega\overset{\triangledown}{x}_n = \sum_{n_1+n_2=n}\big(A_{\omega,n_1}\partial\overset{\triangledown}{x}_{n_2} + (n_2+1)B_{\omega,n_1}\overset{\triangledown}{x}_{n_2+1}\big),\qquad n\geq 0. \tag{85}$$

Proof. The point is that $\partial_z\tilde{x}(z,b)$ and $\partial_b\tilde{x}(z,b)$ are independent solutions of the linearized equation $P\tilde{\varphi} + 2\tilde{x}(z,b)\tilde{\varphi} = 0$; what we shall check amounts to the fact that their formal Borel transforms $\partial\overset{\triangledown}{x}$ and $\partial_b\overset{\triangledown}{x}$, which lie in $\mathrm{SING}^{\text{s.ram.}}[[b]]$, span the space of solutions of the equation

$$\alpha\overset{\triangledown}{\varphi} + 2\overset{\triangledown}{x} * \overset{\triangledown}{\varphi} = 0,\qquad \overset{\triangledown}{\varphi} \in \mathrm{SING}[[b]], \tag{86}$$

a particular solution of which is $\Delta_\omega\overset{\triangledown}{x} = \sum b^n\Delta_\omega\overset{\triangledown}{x}_n$.

We prove (85) by induction on n and suppose that the coefficients of

$$A^*(b) = \sum_{n=0}^{N-1} A_{\omega,n}b^n,\quad B^*(b) = \sum_{n=0}^{N-1} B_{\omega,n}b^n$$

were already determined so as to satisfy (85) for $n = 0,\dots,N-1$. The right-hand side of (85) with $n = N$ can be written as $\overset{\triangledown}{\chi}_N + A_{\omega,N}\,\partial\overset{\triangledown}{x}_0 + B_{\omega,N}\,\overset{\triangledown}{x}_1$, where $\overset{\triangledown}{\chi}_N \in \mathrm{SING}^{\text{s.ram.}}$ is known in terms of the coefficients of $\overset{\triangledown}{x}(z,b)$, $A^*(b)$ and $B^*(b)$. Thus, we only need to check that $\Delta_\omega\overset{\triangledown}{x}_N - \overset{\triangledown}{\chi}_N$ is a linear combination of $\partial\overset{\triangledown}{x}_0 = \overset{\triangledown}{\varphi}_1$ and $\overset{\triangledown}{x}_1 = \overset{\triangledown}{\varphi}_2$.

The singularity $\Delta_\omega\overset{\triangledown}{x}_N - \overset{\triangledown}{\chi}_N$ is the coefficient of b^N in $\overset{\triangledown}{\varphi} = \Delta_\omega\overset{\triangledown}{x} - \big(A^*(b)\partial + B^*(b)\partial_b\big)\overset{\triangledown}{x}$. On the one hand, $\overset{\triangledown}{\varphi} = \mathcal{O}(b^N)$ by our induction hypothesis; on the other hand, $\overset{\triangledown}{\varphi}$ is solution of (86) (because Δ_ω, ∂ and ∂_b are derivations of $\mathrm{SING}[[b]]$ that commute with the multiplication by α). Therefore $\Delta_\omega\overset{\triangledown}{x}_N - \overset{\triangledown}{\chi}_N$ is solution of (75), hence of the form $A_{\omega,N}\,\overset{\triangledown}{\varphi}_1 + B_{\omega,N}\,\overset{\triangledown}{\varphi}_2$ according to Proposition 9. □

[31]For the first-order difference equation (8), we had the one-parameter family of solutions $\tilde{\varphi}(z+c)$.

Now we can compute all the successive alien derivatives $\Delta_{\omega_1}\cdots\Delta_{\omega_r}\overset{\triangledown}{x}_n$ of all the components of the formal integral. Since the Δ_ω's commute with ∂_b and satisfy the commutation rule (32) with ∂, we get

$$\Delta_{\omega_1}\cdots\Delta_{\omega_r}\overset{\triangledown}{x} = \big(A_{\omega_r}(\partial-\check{\omega}_{r-1})+B_{\omega_r}\partial_b\big)\big(A_{\omega_{r-1}}(\partial-\check{\omega}_{r-2})+B_{\omega_{r-1}}\partial_b\big)\cdots$$
$$\cdots\big(A_{\omega_1}(\partial-\check{\omega}_0)+B_{\omega_1}\partial_b\big)\overset{\triangledown}{x}$$

with $\check{\omega}_j := \omega_1+\cdots+\omega_j$, $\check{\omega}_0 := 0$. By expanding in powers of b, we have access to the resurgent structure of each $\overset{\triangledown}{x}_n$. In particular, we see that all of them are simply ramified resurgent functions:

$$\overset{\triangledown}{x} \in \overset{\triangledown}{\mathrm{RES}}{}^{\mathrm{s.ram.}}_{2\pi i\mathbb{Z}}\,[[b]].$$

4.2 Real splitting problems

We now give a brief account of the work by V. Lazutkin and V. Gelfreich on the exponentially small splitting for area-preserving planar maps, indicating the connection with Section 4.1.

Two examples of exponentially small splitting

Let us consider a one-parameter family of real 2-dimensional symplectic maps

$$\mathcal{G}_\varepsilon : \begin{pmatrix}X\\Y\end{pmatrix} \longmapsto \begin{pmatrix}X+Y+\varepsilon^2 g(X)\\ Y+\varepsilon^2 g(X)\end{pmatrix},$$

where $\varepsilon \geq 0$, and $g:\mathbb{R}\to\mathbb{R}$ is analytic with $g(X) = X+\mathcal{O}(X^2)$. More specifically, we have in mind two examples in which g is a (possibly trigonometric) polynomial:

- $g(X) = g^{\mathrm{qu}}(X) = X(1-X)$ gives rise to $\mathcal{G}_\varepsilon = \mathcal{G}_\varepsilon^{\mathrm{qu}}$, which is a normal formal for non-trivial quadratic diffeomorphisms of the plane which are symplectic and possess two fixed points (see [GS01, § 1.2] and the references therein);
- $g(X) = g^{\mathrm{sm}}(X) = \sin X$ gives rise to $\mathcal{G}_\varepsilon = \mathcal{G}_\varepsilon^{\mathrm{sm}}$, the so-called *standard map* (see [GL01] for a survey and references to the litterature); in this case we consider X as an angular variable, *i.e.* the phase space of $\mathcal{G}_\varepsilon^{\mathrm{sm}}$ is $(\mathbb{R}/2\pi\mathbb{Z})\times\mathbb{R}$.

For $\varepsilon = 0$, we have an invariant foliation by horizontal lines, with fixed points for $Y=0$. In both examples, we wish to study the behaviour of $\mathcal{G}_\varepsilon$ with $\varepsilon>0$ small, in a region $|Y| = \mathcal{O}(\varepsilon)$ of the phase space.

Rescaling the variables by $x = X$, $y = \varepsilon Y$, we get

$$\mathcal{F}_\varepsilon : \begin{pmatrix}x\\y\end{pmatrix} \longmapsto \begin{pmatrix}x+\varepsilon\big(y+\varepsilon g(x)\big)\\ y+\varepsilon g(x)\end{pmatrix},$$

which has a hyperbolic fixed point at the origin. But the hyperbolicity is weak when ε is small (the eigenvalues are $\mathrm{e}^{\pm h}$ with $h = \varepsilon+\mathcal{O}(\varepsilon^2)$) and this is the source of an exponentially small phenomenon: parts of the stable and unstable manifolds $\mathcal{W}_\varepsilon^+$ and $\mathcal{W}_\varepsilon^-$ of the origin are very close one to the other (see [FS90]). However, it turns out that they do not coincide and there is a homoclinic point h_ε at which they intersect transversely,[32] with an exponentially small angle α:

$$\alpha^{\mathrm{qu}} = \frac{\Omega^{\mathrm{qu}}}{\varepsilon^7}\mathrm{e}^{-\frac{2\pi^2}{\varepsilon}}\big(1+\mathcal{O}(\varepsilon)\big), \qquad \alpha^{\mathrm{sm}} = \frac{\Omega^{\mathrm{sm}}}{\varepsilon^3}\mathrm{e}^{-\frac{\pi^2}{\varepsilon}}\big(1+\mathcal{O}(\varepsilon)\big), \qquad (87)$$

[32]Moreover, in both examples $\mathcal{W}_\varepsilon^-$ can be deduced from $\mathcal{W}_\varepsilon^+$ by a linear symmetry and h_ε lies on the symmetry line.

where $\Omega^{\mathrm{qu}} = \frac{64\pi}{9}|\Theta|$ and $\Omega^{\mathrm{sm}} = \pi|\Theta'|$, with the same constant Θ as the one discussed in Propositions 9 and 10 in the study of the Hénon map of Section 4.1, and with an analogous constant Θ' stemming from the study of another map without parameter.

The proof of the result by Vladimir Lazutkin and his coworkers in the case of the standard map has a long story, which starts with his VINITI preprint in 1984 and ends with an article by Vassili Gelfreich in 1999 filling all the remaining gaps. In the quadratic case (and also for other cases, corresponding to other algebraic or trigonometric polynomials $g(X)$ in the map $\mathcal{G}_\varepsilon$), the result was indicated by V. Gelfreich without a complete proof (which should be, in principle, a mere adaptation of the proof for the standard map).

The map F as "inner system"

The strategy for proving (87) is to represent the invariant manifolds $\mathcal{W}_\varepsilon^\pm$ as parametrised curves $P_\varepsilon^\pm(t) = \big(x_\varepsilon^\pm(t), y_\varepsilon^\pm(t)\big)$, which are real-analytic, extend holomorphically to a half-plane $\pm\Re e\, t \gg 0$, with $x_\varepsilon^\pm(t) \sim \mathrm{const}\, \mathrm{e}^t$ as $\pm\Re e\, t \to \infty$, and satisfy $P_\varepsilon^\pm(t+\varepsilon) = \mathcal{F}_\varepsilon\big(P_\varepsilon^\pm(t)\big)$. The second component can be eliminated: $y_\varepsilon^\pm(t) = x_\varepsilon^\pm(t) - x_\varepsilon^\pm(t-\varepsilon)$, and the first is then solution of a nonlinear second-order difference equation with small step size:

$$x_\varepsilon^\pm(t+\varepsilon) - 2x_\varepsilon^\pm(t) + x_\varepsilon^\pm(t-\varepsilon) = \varepsilon^2 g\big(x_\varepsilon^\pm(t)\big). \tag{88}$$

Pictures in [GL01] or [GS01] show that both curves are close to the separatrix solution $P_0(t)$ of the Hamiltonian vector field generated by $H(x,y) = \frac{1}{2}y^2 + G(x)$, where $G' = -g$. Indeed, the differential equation $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = g(x)$, which is the singular limit of (88) when $\varepsilon \to 0$, has a particular solution $x_0(t)$ wich tends to 0 both for $t \to +\infty$ and $t \to -\infty$ (with the identification $0 \equiv 2\pi$ in the case of the standard map), corresponding to the upper part of the separatrix of the pendulum in the second example and to a homoclinic loop in the first one:

$$x_0^{\mathrm{qu}}(t) = \frac{3}{2\cosh^2 \frac{t}{2}} \quad \text{or} \quad x_0^{\mathrm{sm}}(t) = 4\arctan \mathrm{e}^t. \tag{89}$$

The exponentially small phenomenon that we wish to understand can be described as the splitting of this separatrix: when passing from the Hamiltonian flow corresponding to $\varepsilon = 0$ to the map $\mathcal{F}_\varepsilon$ with $\varepsilon > 0$, the separatrix solution $x_0(t)$ is replaced by two solutions $x_\varepsilon^+(t)$ and $x_\varepsilon^-(t)$ of equation (88), the difference of which is exponentially small but not zero. Proving (87) amounts essentially to estimating $x_\varepsilon^+(t) - x_\varepsilon^-(t)$.

It is easy to see that the separatrix must split for an entire map like $\mathcal{F}_\varepsilon$: since g is entire, equation (88) affords analytic continuation to the whole t-plane of the functions x_ε^+ and x_ε^- and Liouville's theorem prevents them from coinciding (see [Ush80] for a more general result of the same kind).

The method which was successfully developed by Vladimir Lazutkin and his coworkers can then be described in the language of complex matching asymptotics. The separatrix solution $x_0(t)$ is a good approximation of $x_\varepsilon^\pm(t)$ for real values of t, but it is no so in the complex domain, if only because x_ε^+ and x_ε^- are entire functions, whereas x_0 has singularities in the complex plane. It turns out that the asymptotics that we want to capture is governed by the singularities of x_0 which are closest to the real axis,

$$t^* = \mathrm{i}\pi \quad \text{or} \quad \frac{\mathrm{i}\pi}{2}, \qquad \text{and} \quad \bar{t}^* = -t^*$$

(the exponential in the final result (87) is nothing but $e^{\frac{2\pi i t^*}{\varepsilon}}$). One is thus led to look for a better approximation of $x_\varepsilon^\pm(t)$ when t lies in a domain called "inner domain", close to t^*.

In the case of the quadratic map, one can perform the change of variables $u = \varepsilon^2 x - \frac{\varepsilon^2}{2}$, $v = \varepsilon^3 y$, which transforms $\mathcal{F}_\varepsilon$ into

$$F_\varepsilon : \begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} u + v - u^2 + \frac{\varepsilon^4}{4} \\ v - u^2 + \frac{\varepsilon^4}{4} \end{pmatrix}.$$

We recognize here an ε^4-perturbation of the Hénon map of Section 4.1 (*i.e.* the map (68) with $f(x,y) = -x^2$), and the second-order difference equation with small step size (88) can be treated as a perturbation of equation (71) with $t = t^* + \varepsilon z$. The Hénon map, being obtained by forgetting the ε^4-terms in F_ε and providing good complex approximations of $x_\varepsilon^+(t)$ and $x_\varepsilon^-(t)$ through its parabolic curves, is called the *inner system* of the family $\mathcal{F}_\varepsilon$.

The parabolic fixed point of the Hénon map appears as the organising centre of our perturbation problem; the detailed study of the parabolic curves, including Proposition 11, is an important step in the proof of the estimate (87) for the splitting in the quadratic case; it is here that the non-zero constant Θ appears. In the case of the standard map, the inner system is the so-called "semi-standard map", which one could also study by resurgent methods, although this is not exactly what Lazutkin and his coworkers did.

Once the separatrix splitting for the inner map has been analysed, an extra work is required to reach the result for $\mathcal{G}_\varepsilon$, which was not yet written in full details for the quadratic family $\mathcal{G}_\varepsilon^{\mathrm{qu}}$, contrarily to the case of the standard map $\mathcal{G}_\varepsilon^{\mathrm{sm}}$.

Towards parametric resurgence

One can suggest another approach to the proof of (87). Equation (88) admits a formal solution

$$\tilde{x}_\varepsilon(t) = \sum_{n \geq 0} \varepsilon^{2n} x_n(t), \tag{90}$$

where the first term is the separatrix solution x_0 of the limit flow, and where the subsequent functions can be computed inductively. The formal solution is unique if one imposes the condition $x_n(0) = 0$ for each $n \geq 1$. One finds that each function $x_n(t)$ tends to 0 when t tends to $+\infty$ or to $-\infty$. The formal solution is thus a candidate to represent the stable solution $x_\varepsilon^+(t)$ and the unstable one $x_\varepsilon^-(t)$ as well. What happens is that the formal series (90) is divergent for $t \neq 0$ and only provides an asymptotic expansion both for $x_\varepsilon^+(t)$ and $x_\varepsilon^-(t)$.

The formal series (90) truncated far enough is used in the work by Lazutkin et al. as an approximation of $x_\varepsilon^\pm(t)$, but one can envisage a more radical use of this formal solution. Unpublished computations performed in collaboration with Stefano Marmi tend to indicate that it is resurgent in $z = \frac{1}{\varepsilon}$, that $x_\varepsilon^+(t)$ and $x_\varepsilon^-(t)$ can be recovered from it by Borel-Laplace summation for $\varepsilon > 0$ as

$$x_\varepsilon^+(t) = x_0(t) + \int_0^{+\infty} e^{-\zeta/\varepsilon}\, \hat{x}(\zeta,t)\, d\zeta \quad \text{if } t > 0, \quad x_\varepsilon^-(t) = x_0(t) + \int_0^{+\infty} e^{-\zeta/\varepsilon}\, \hat{x}(\zeta,t)\, d\zeta \quad \text{if } t < 0,$$

where $\hat{x}(\zeta,t) = \sum_{n \geq 1} \frac{\zeta^{2n-1}}{(2n-1)!} x_n(t)$, and that

$$x_\varepsilon^+(t) - x_\varepsilon^-(t) \sim e^{-\omega_*(t)/\varepsilon}\, \Delta_{\omega_*(t)} \tilde{x} - e^{-\overline{\omega}_*(t)/\varepsilon}\, \Delta_{\overline{\omega}_*(t)} \tilde{x}, \qquad t > 0,$$

where $\omega_*(t) = 2\pi i(t - t^*)$ and $\overline{\omega}_*(t) = -2\pi i(t + t^*)$ are the singularities of $\hat{x}(\,.\,, t)$ with the smallest positive real part.

We shall try to explain in the next section why singularities of the principal branch of $\hat{x}(\zeta, t)$ should appear at the points $\omega_{a,b}(t) = 2\pi i a\big(t - (2b + 1)t^*\big)$, $a \in \mathbb{Z}^*$, $b \in \mathbb{Z}$. This kind of resurgence we expect is called *parametric resurgence*, because the resurgent variable $1/\varepsilon$ appears as a parameter in equation (88), whereas the "dynamical" variable is t. This makes the analysis more complicated, since, for instance, the singular points of the formal Borel transform with respect to $1/\varepsilon$ are "moving singular points" (they move along with t).

In the above conjectural statements, the Borel summability is the most accessible. It amounts to the fact that, for fixed t, the formal Borel transform $\hat{x}(\zeta, t)$ has positive radius of convergence and extends holomorphically, with at most exponential growth at infinity, to the branch cut obtained by removing from the complex plane the moving singular half-lines $\pm\omega_{1,b}(t)\,[1, +\infty[$. See [MS03, § 5.3] for such a statement concerning a slightly simpler second-order difference equation with small step size (related to the semi-standard map).

4.3 Parametric resurgence for a cohomological equation

Since we just alluded to a possible phenomenon of parametric resurgence, we end with two linear examples which are comparable to equations (5) and (6) of Section 1.2 and can help to understand the origin of this phenomenon in difference equations with small step size.

Proposition 13 *Let U be an open connected and simply connected subset of $\mathbb{C}$ and let $\mathcal{H}(U)$ denote the space of all the functions holomorphic in U. Let $\alpha, \beta \in \mathcal{H}(U)$ and consider the two difference equations*

$$\varphi(t+\varepsilon) - \varphi(t) = \varepsilon\alpha(t), \qquad \psi(t+\varepsilon) - 2\psi(t) + \psi(t-\varepsilon) = \varepsilon^2\beta(t). \tag{91}$$

Then there exist sequences $(\varphi_n)_{n\ge1}$ and $(\psi_n)_{n\ge1}$ of elements of $\mathcal{H}(U)$ such that the solutions of the first equation in $\mathcal{H}(U)[[\varepsilon]]$ are the formal series

$$\tilde{\varphi}_\varepsilon(t) = c_0(\varepsilon) + \sum_{n\ge0} \varepsilon^n \varphi_n(t),$$

where $c_0(\varepsilon) \in \mathbb{C}[[\varepsilon]]$ is arbitrary and φ_0 is any primitive of α in U, and the solutions of the second equation in $\mathcal{H}(U)[[\varepsilon]]$ are the formal series

$$\tilde{\psi}_\varepsilon(t) = c_1(\varepsilon) + c_2(\varepsilon)t + \sum_{n\ge0} \varepsilon^{2n} \psi_n(t),$$

where $c_1(\varepsilon), c_2(\varepsilon) \in \mathbb{C}[[\varepsilon]]$ are arbitrary and ψ_0 is any second primitive of β in U (i.e. $\psi_0'' = \beta$). Moreover the formal Borel transforms

$$\hat{\varphi}(\zeta, t) = \sum_{n\ge1} \frac{\zeta^{n-1}}{(n-1)!}\varphi_n(t), \qquad \hat{\psi}(\zeta, t) = \sum_{n\ge1} \frac{\zeta^{2n-1}}{(2n-1)!}\psi_n(t)$$

have positive radius of convergence for any fixed $t \in U$ and define holomorphic functions of (ζ, t), which can be expressed as

$$\hat{\varphi}(\zeta, t) = -\frac{1}{2}\alpha(t) - \sum_{\nu\in2\pi i\mathbb{Z}^*} \frac{1}{\nu}\Big(\alpha(t + \tfrac{\zeta}{\nu}) - \alpha(t)\Big), \qquad \hat{\psi}(\zeta, t) = \sum_{\nu\in2\pi i\mathbb{Z}^*} \frac{\zeta}{\nu^2}\beta(t + \tfrac{\zeta}{\nu})$$

and extend holomorphically to

$$\mathcal{E} = \{ (\zeta, t) \in \mathbb{C} \times U \mid [t - \tfrac{\zeta}{2\pi i}, t + \tfrac{\zeta}{2\pi i}] \subset U \}.$$

When $\alpha(t) = g(e^t)$ with $g(z) \in z\mathbb{C}\{z\}$, equation (91a) is a particular case of the *cohomological equation* studied in [MS03]. This is the equation

$$f(qz) - f(z) = g(z),$$

with an unknown function $f(z) \in z\mathbb{C}\{z\}$ depending on the complex parameter q. The article [MS03] investigates the nature of the dependence of f upon q when q crosses the unit circle. Roots of unity appear as "resonances", at which the solution $f(q,z)$ has sorts of simple poles: $f(q,z) = \sum \frac{\mathcal{L}_\Lambda(z)}{q-\Lambda}$ with a sum extending to all roots of unity Λ and explicit "residua" $\mathcal{L}_\Lambda(z)$, except that these singular points are not isolated and the unit circle is to be considered as a natural boundary of analyticity. Still, $(q-\Lambda)f(q,z)$ tends to $\mathcal{L}_\Lambda(z)$ when q tends to Λ non-tangentially with respect to the unit circle (uniformly in z), and there is even a kind of Laurent series at Λ: an asymptotic expansion $\tilde{f}_\Lambda$ which is valid near Λ, inside or outside the unit circle, and which must be divergent due to the presence of arbitrarily close singularities.

The asymptotic series $\tilde{f}_\Lambda$ can be found by setting $q = \Lambda e^\varepsilon$ and $\varphi(t) = \varepsilon f(z)$ (notice that $\varepsilon \sim \frac{q-\Lambda}{\Lambda}$); this yields the equation

$$\varphi(t + 2\pi i \tfrac{N}{M} + \varepsilon) - \varphi(t) = \varepsilon\alpha(t), \tag{92}$$

where $\Lambda = \exp\left(2\pi i \frac{N}{M}\right)$. The formal solution corresponding to $\tilde{f}_\Lambda$ and its Borel transform are described in [MS03, Chap. 4], whith a statement which generalizes Proposition 13.

If $\alpha(t)$ is a meromorphic function,[33] then so is $\hat{\varphi}(\zeta, t)$ as a function of ζ for any fixed t, and similarly with $\hat{\psi}(\zeta, t)$ when $\beta(t)$ is meromorphic. This provides us with elementary examples of parametric resurgence.

Proof of Proposition 13. As previously mentioned, equation (91a) and the more general equation (92) are dealt with in [MS03], so we content ourselves with the case of equation (91b) (which is not very different anyway).

The equation can be written $\left(4\sinh^2 \frac{\varepsilon\partial_t}{2}\right)\psi = \varepsilon^2\beta$. We set $4\sinh^2\frac{X}{2} = \frac{X^2}{1+\Gamma(X)}$ and introduce the Taylor series

$$\Gamma(X) = -1 + \frac{X^2}{4\sinh^2\frac{X}{2}} = \sum_{n\geq 0} \Gamma_n X^{2n+2}$$

and its Borel transform $\hat{\Gamma}(\xi) = \sum_{n\geq 0} \Gamma_n \frac{\xi^{2n+1}}{(2n+1)!}$. With these notations,

$$\text{(91b)} \;\Leftrightarrow\; \partial_t^2\psi = \beta + \Gamma(\varepsilon\partial_t)\beta \;\Leftrightarrow\; \psi = \psi_0 + \sum_{n\geq 0}\Gamma_n\,\varepsilon^{2n+2}\partial_t^{2n}\beta \text{ with } \partial_t^2\psi_0 = \beta.$$

This gives all the formal solutions, with $\psi_n = \Gamma_{n-1}\,\partial_t^{2(n-1)}\beta$ for $n \geq 1$. Now,

$$\hat{\psi}(\zeta, t) = \sum_{n\geq 0}\Gamma_n \frac{\zeta^{2n+1}}{(2n+1)!}\partial_t^{2n}\beta = \hat{\Gamma}(\zeta\partial_t)\,\partial_t^{-1}\beta, \tag{93}$$

[33]This is the case when $\alpha(t) = g(e^t)$ with a meromorphic function g; this is worth of interest since $g_*(z) = \frac{z}{1-z}$ is the unit of the Hadamard product, from which the case of any $g(z) \in z\mathbb{C}\{z\}$ can be deduced (see [MS03]).

where $\partial_t^{-1}\beta$ denotes any primitive of β. A classical identity yields

$$\Gamma(X) = \sum_{\nu\in 2\pi\mathrm{i}\mathbb{Z}^*} \frac{X^2}{(X-\nu)^2} = \sum_{\nu\in 2\pi\mathrm{i}\mathbb{Z}^*} \nu^{-2}X^2(1-\nu^{-1}X)^{-2} = \sum_{\nu\in 2\pi\mathrm{i}\mathbb{Z}^*}\sum_{n\geq 0}(n+1)\nu^{-n-2}X^{n+2},$$

whence $\hat{\Gamma}(\xi) = \sum \nu^{-2}\,\xi\,\mathrm{e}^{\nu^{-1}\xi}$. Inserting this into (93) and using the Taylor formula in the form $\mathrm{e}^{\nu^{-1}\zeta\,\partial_t}\,\beta = \beta(t+\nu^{-1}\zeta)$ yields the conclusion. □

We can now explain why we expect parametric resurgence for the formal solution $\tilde{x}_\varepsilon(t)$ of equation (88), with singularities of the Borel transform located at the points $\omega_{a,b}(t) = 2\pi\mathrm{i}a\big(t-(2b+1)t^*\big)$, $a\in\mathbb{Z}^*$, $b\in\mathbb{Z}$. The idea is simply that equation (91b) with $\beta(t) = g\big(x_0(t)\big)$ can be considered as a non-trivial approximation of (88); but $g\circ x_0 = \frac{\mathrm{d}^2x_0}{\mathrm{d}t^2}$ is meromorphic in the two cases we are interested in, in view of (89), with $t^* + 2t^*\mathbb{Z}$ as set of poles. This yields a resurgent solution $\tilde{\psi}_\varepsilon(t)$, with a meromorphic Borel transform $\hat{\psi}(\zeta,t)$ for which the set of poles is $2\pi\mathrm{i}\,\mathbb{Z}^*(-t+t^*+2t^*\mathbb{Z})$. We expect $\hat{x}(\zeta,t)$ to resemble somewhat $\hat{\psi}(\zeta,t)$; we do not expect it to be meromorphic, because of the nonlinear character of equation (88), but a majorant method can be devised to control at least its principal branch.

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