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New Results on Uniform Covers

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ABSTRACT. We present new results of the author on the localization of uniform covers of product spaces. Localization is considered more generally in the context of filters of covers (pre-uniformities) or *monoids of covers.*

Introduction. Let us start from the following well-known question that H. Tamano asked in the first number of the Mathematical Journal of the Kyoto University.

Question [Tamano,1962]: *Let X be a paracompact space. For what spaces Y is the product $X \times Y$ paracompact?*

Also consider the well-known theorem by [Tamano,1960]: X is paracompact if, and only if, $X \times \beta X$ is normal. In our early studies of this theorem, we considered open covers of products $X \times K$, where K is a compact space. For any Tychonoff space S, we denote by $\mathcal{F}(S)$ the collection of all normal covers of S (also called the fine uniformity of S). Then we have

$$\boxed{\mathcal{F}(X \times K) = [\mathcal{F}(X) \times \mathcal{F}(K)]^{(1)}}$$

where the symbol (1) denotes that we have formed the uniformly locally uniform covers from the collection $\mu = \mathcal{F}(X) \times \mathcal{F}(K)$. (We say that a cover $\mathcal{U}$ is *uniformly local uniform* if there is a uniform cover $\mathcal{V}$ such that each restriction $\mathcal{U} \restriction \mathcal{V}$, is a uniform cover of $V \in \mathcal{V}$.)

• The description of normal covers is very simple in terms of the factors. What if K is replaced by a space which is just locally compact and paracompact? Then

$$\boxed{\mathcal{F}(X \times K) = [\mathcal{F}(X) \times \mathcal{F}(K)]^{(2)}}$$

Here naturally $\mu^{(2)} = (\mu^{(1)})^{(1)}$. What would be the next step? Let Y be a paracompact space such that there is a locally compact closed $L \subset Y$ such that $Y \setminus L$ is locally compact. Then

$$\boxed{\mathcal{F}(X \times Y) = [\mathcal{F}(X) \times \mathcal{F}(Y)]^{(3)}}$$

Proceeding systematically, with both the localization on the right-hand side and the structure of the space Y, we obtain two concepts at the limit:

- We get a class of spaces called C-scattered spaces (every closed subspace has a point with a compact neighbourhood).
- We also get, by transfinitely repeating the localization of the filter of coverings, a *closure* denoted by $\lambda\mu$. This operation was already considered in 1959 by Ginsburg and Isbell.

• We get: If X is a paracompact space, and Y is a C-scattered paracompact space, then

(L)
$$\boxed{\mathcal{F}(X \times Y) = \lambda([\mathcal{F}(X) \times \mathcal{F}(Y)])}$$

In fact, we get more because $X \times Y$ is paracompact:

(L′)
$$\boxed{\mathcal{O}(X \times Y) = \lambda([\mathcal{O}(X) \times \mathcal{O}(Y)])}$$

Here $\mathcal{O}(Z)$ denotes the *fine* monoid of all open-refinable covers of Z. (In general, $\mathcal{O}(Z)$ is not a uniformity anymore (precisely when Z is not paracompact).

Consequences. This result (similar to the Künneth theorem from algebraic topology) tells us that the open covers of a product are "combinatorially" obtained from the factors alone. They are *almost rectangular* in the following sense:

(1) $\mu = \mu^{(0)}$ has a basis consisting of *product covers* $\mathcal{U} \times \mathcal{V}$, where $\mathcal{U} \in \mathcal{F}(X), \mathcal{V} \in \mathcal{F}(Y)$.

(2) $\mu^{(1)}$ has a basis of covers obtained by applying a product cover over elements of a fixed product cover.

(3) Finally, the closure $\lambda\mu$ (the localization) has a basis of covers obtained as a *Noetherian tree* of applications in (2) (each branch is finite).

In addition to being rectangular in this combinatorial sense (a very strong sense of rectangularity), the products satisfying the condition **(L')** are rectangular in the classical sense of Pasynkov (1975); in fact, they are strongly rectangular in the sense of Yajima (1981) (finite cozero covers are refinable by locally finite covers consisting of cozero-rectangles).

• Please notice that

$$\mathcal{F}(X \times Y) = \text{all normal covers of } X \times Y$$

Each finite cozero cover is normal, so therefore the condition is even stronger than (Yajima's) strong rectangularity.

As for Tamano's question, we published several years ago the following result:

Theorem [Hohti,1990]: *For a paracompact space X, $\mathcal{F}(X \times Y) = \lambda(\mathcal{F}(X) \times \mathcal{F}(Y))$ for all paracompact spaces Y if, and only, the space X is C-scattered.*

• The question is, *why* to bother? Is it not true that **(L)** is just some *technical* condition? What is natural about **(L)**?

• It turns out that localization λ is a natural operation that appears in several important places in mathematics (and logic).

• Let us consider a pre-ordered set $(S, \leq)$. A covering relation **Cov** is a relation between the elements $a \in S$ and subsets $U \subset S$, i.e., $\mathbf{Cov} \subseteq S \times P(S)$.

We have the following axioms:

(1) if $a \in U$, then $\mathbf{Cov}(a, U)$.

(2) if $a \leq b$, then $\mathbf{Cov}(a, \{b\})$.

(3) if $\mathbf{Cov}(a, U)$ and $\mathbf{Cov}(b, V)$, then $\mathbf{Cov}(a \wedge b, U \wedge V)$, where $U \wedge V$ denotes the cover consisting of all elements $w \in S$ majorized by both U and V.

We need one more axiom:

(4) if $\mathbf{Cov}(a, U)$ and $\mathbf{Cov}(u, V)$ for all $u \in U$, then $\mathbf{Cov}(a, V)$ [Transitivity].

The first 3 axioms are very general, while (4) is the essential one. In fact, it directly corresponds to the condition that a filter μ of covers is closed under localization, i.e., $\lambda\mu = \mu$. Indeed, given a set X and a filter μ of covers of X, then we define a relation $R \subset P(X) \times P(P(X))$ by setting $(A, \mathcal{U}) \in R$ if there is a cover $\mathcal{V} \in \mu$ such that $\mathcal{V} \restriction A \prec \mathcal{U}$. Then R satisfies the conditions (1)-(3), and (4) is satisfied iff μ is closed under λ.

These axioms appear in several places:

(A) *Formal Spaces* of a propositional language (Fourman & Grayson,1982)

(B) *Modal Logics* ((4) corresponds to the classical system S4)

(C) *Grothendieck topologies* of pre-ordered sets

(D) *Locales* ((4) corresponds to the Heyting axiom: $x \wedge \bigvee y_\alpha = \bigvee x \wedge y_\alpha$ - right distributivity)

(E) *Kuratowski axioms* for the closure operation in topology.

Given only a *set of generators* $G \subset S \times P(S)$, the associated covering relation $\mathbf{Cov}_G$ is obtained by closing G under the conditions C1) – C4). This means forming all Noetherian trees T such that for each element x of T, the immediate successors are derived by using one of the four conditions. This corresponds to the idea of using Noetherian trees to construct 'recursively defined' refinements of open covers of uniform spaces, in particular in the products of paracompact spaces. Such constructions start from the basis of uniform covers, which is a *commutative monoid* under the operation of meet, and closes the collection under the condition of transitivity equivalent to the locally fine condition. (To underscore the independence of general filters of covers from uniformities, we often use "monoid of covers" instead of "pre-uniformity".)

• We are particularly interested in situations in which $\lambda\mu$ *reaches* the topology of the underlying space X, i.e., $\lambda\mu$ is the fine monoid $\mathcal{O}(X)$. (For example, if M is a complete metric space, then the localization of all uniform covers gives all open covers.)

Infinite Products. The first essential result was proved by the late Jan Pelant.

Theorem [Pelant,1987]: *If (M_α) is an arbitrary family of completely metrizable spaces, then*

$$\boxed{\lambda\left(\Pi_\alpha \mathcal{F}(M_\alpha)\right) = \mathcal{F}\left(\Pi_\alpha M_\alpha\right)} \tag{L}$$

In a recent paper by Hohti, Hušek and Pelant, this has been extended to paracompact, σ-partition complete spaces:

Theorem [Hohti-Hušek-Pelant,2003]: *Let the spaces X_α be paracompact and suppose that each X_α is a countable union of closed, partition complete subspaces. Then the condition* (**L**) *holds.*

(Partition-complete spaces were defined by Telgársky and Wicke in 1988. By a result of Michael, a space is partition-complete iff it has a complete sequence of exhaustive covers.)

Remark. *As partition-complete ⇒ Čech-scattered ⇒ C-scattered, the result is also valid for σ-C-scattered spaces.*

For countable powers, we can prove **(L)** *with respect to $\mathcal{O}(X)$ (the fine monoid of open-refinable covers):*

Theorem [HHP,2003]: *Let $(X_i : i \in I)$ be a countable family of partition-complete regular spaces. Then*

$$(L') \qquad \boxed{\lambda\left(\Pi_{i\in\omega}\mathcal{O}(X_i)\right) = \mathcal{O}(\Pi X_i)}$$

Notice that the spaces are regular, and there is no assumption of paracompactness!

Consequences. The filters of covers refinable by point-finite, locally finite, countable, point-countable etc. covers are locally fine (preserved by λ). Therefore, if the X_i are regular, partition-complete spaces which are paracompact, Lindelöf, metacompact, meta-Lindelöf, then the same is true of $\Pi_{i\in\omega}X_i$.
This corollary was already obtained by (Plewe,1996), because he proved that the product of countably many partition-complete *locales* is *spatial*. Essentially this means that the locale of ΠX_i, denoted by $T(\Pi X_i)$ (the lattice of open subsets) is the same as the localic product of the $T(X_i)$:

$$\otimes_{i\in\omega}T(X_i) \simeq T(\Pi_{i\in\omega}X_i)$$

Earlier, Dowker & Strauss (1977) had proved that products of regular paracompact locales are paracompact (they proved the same result for Lindelöf, metacompact etc. spaces). Combining these two results, we get Plewe's corollary.
This result has a major deficiency, because it does not cover important cases such as the rationals: The product $\mathbb{Q} \otimes \mathbb{Q}$ is not spatial (and therefore so

is not the infinite product $\otimes_{i\in\omega}\mathbb{Q}$). Using known techniques for uniform spaces (inherited from the work of Frolík et al. of the Prague school) we can prove a similar result for σ-partition complete spaces.

• We call a cover $\mathcal{V}$ σ-uniform if there is a countable closed cover $\{F_i : i \in \omega\}$ of the underlying space such that each restriction $\mathcal{V} \restriction F_i$ is a uniform cover of the subspace F_i. Given a uniformity μ on X, then $m\mu$ denotes the collection of all σ-uniform covers of X relative to μ. (This is the same as the *metric-fine* uniformity defined by Hager (1974).) It is easy to see that this metric-fine modification can be extended to monoids of covers. We obtain the following result:

Theorem [HHP, 2003]: *Let $(X_i : i \in \omega)$ be a countable family of σ-partition-complete, regular spaces. Then*

$$\boxed{\lambda m(\Pi_{i\in\omega}\mathcal{O}(X_i)) = \mathcal{O}(\Pi_{i\in\omega}X_i)}$$

If the μ_i are monoids of covers such that $\lambda\mu_i = \mathcal{O}(X_i)$, then similarly

$$\boxed{\lambda m(\Pi_{i\in\omega}\mu_i) = \mathcal{O}(\Pi_{i\in\omega}X_i)}$$

For a paracompact space X, $\mathcal{O}(\mathcal{X})$ is the fine uniformity $\mathcal{F}(X)$. For completely regular spaces, m and therefore λm preserve uniformities. Thus, it follows in particular that the countable product of σ-partition-complete paracompact spaces is again paracompact.

Conclusion. What is the "ultimate" connection between the result of Plewe and our results on condition **(L)**? This is completely determined by our latest result:

Theorem [Hohti 2005(?)]: *The localic product of a family (X_i) of regular topological spaces is spatial if, and only if, $\lambda(\Pi\mathcal{O}(X_i)) = \mathcal{O}(\Pi X_i)$.*

Thus, for products of regular topological spaces, spatiality and the condition (**L**) (for open covers) are equivalent.

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