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Author(s)	Kato, Hisao
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Topological entropy and a theorem of Misiurewicz, Szlenk and Young

筑波大学・数理物質科学研究科 加藤久男 (Hisao Kato)
Institute of Mathematics, University of Tsukuba

1 Introduction

Recently, many geometric and dynamical properties of fractal sets have been studied. In this note, we study dynamical pr operties of maps on regular curves, which are contained in the class of fractal sets. It is well known that in the dynamics of a piecewise strictly monotone (= piecewise embedding) map f on an interval, the topological entropy can be expressed in terms of the growth of the number (= the lap number) of strictly monotone intervals for f^n (M. Misiurewicz, W. Szlenk and L. S. Young). We generalize the theorem of M. Misiurewicz, W. Szlenk and L. S. Young to the cases of regular curves and dendrites.

For a metric space X, $\mathrm{Comp}(X)$ denotes the set of all components of X. A map $f : X \to Y$ of compacta is *monotone* if for each $y \in f(X)$, $f^{-1}(y)$ is connected. A continuum X is a *regular continuum* (=*regular curve*) if for each $x \in X$ and each open neighborhood V of x in X, there is an open neighborhood U of x in X such that $U \subset V$ and the boundary set $Bd(U)$ of U is a finite set. Clearly, each regular curve is a *Peano curve* (= 1-dimensional locally connected continuum). For each $p \in X$, we define the cardinal number $ls_X(p)$ of p as follows: $ls_X(p) \leq \alpha$ (α is a cardinal number) if and only if for any neighborhood V of p there is a neighborhood $U \subset V$ of p in X such that $|\mathrm{Comp}(U - \{p\})| \leq \alpha$, and $ls_X(p) = \alpha$ if and only if $ls_X(p) \leq \alpha$ and the inequality $ls_X(p) \leq \beta$ for $\beta < \alpha$ does not hold. We define $ls(X) < \infty$ if $ls_X(p) < \infty$ for each $p \in X$.

For example, the Sierpinski triangle S is a well-known regular curve with $ls_S(p) \leq 2$ for each $p \in S$. The Menger curve and the Sierpinski carpet are not regular curves.

2 Topological Entropy

The notion of topological entropy provided a numerical measure for the complexity of map of a compactum. First, we introduce topological entropy by Adler, Konheim and McAndrew. Let $\mathcal{A}, \mathcal{B}$ be finite open coverings of a compactum X, and let $N(\mathcal{A})$ denote the minimum cardinality of subcovering of $\mathcal{A}$. For any map $f : X \to X$, put $f^{-k}(\mathcal{A}) = \{f^{-k}(U) | U \in \mathcal{A}\}$. Define $\mathcal{A} \vee \mathcal{B} = \{U \bigcap V | U \in \mathcal{A}, V \in \mathcal{B}\}$. Consider the following

$$h(f, \mathcal{A}) = \lim_{n\to\infty}(1/n) \cdot \log N(\mathcal{A} \vee f^{-1}(\mathcal{A}) \vee ... \vee f^{-(n-1)}(\mathcal{A})).$$

Then *topological entropy* $h(f)$ of f is then

$$h(f) = \sup\{h(f, \mathcal{A}) | \mathcal{A} \text{ is an open covering of} X\}.$$

Related to this representation of topological entropy, recently we obtained a theorem about topological dimension. Pontrjagin and Schniremann characterized dimension of a compact metric space X as follows: For a metric ρ on X and $\epsilon > 0$, let

$$N(\epsilon, \rho) = \min\{|\mathcal{U}| \mid \mathcal{U} \text{ is a finite open covering of } X \text{ with } \mathrm{mesh}(\mathcal{U}) \leq \epsilon\}$$

and

$$\kappa(X, \rho) = \sup\{\inf\{\frac{\log N(\epsilon, \rho)}{-\log \epsilon} | \, 0 < \epsilon < \epsilon_0\} \mid \epsilon_0 > 0\},$$

where $|A|$ denotes the cardinality of a set A. Then

$$\dim X = \inf\{\kappa(X, \rho) | \, \rho \text{ is a metric for } X\}.$$

Bruijning and Nagata introduced an index $\Delta_k(X)$ for a (topological) space X and a natural number k, and they determined the value of $\Delta_k(X)$: The function $\Delta_k(X)$ is defined as the least natural number m such that for evey (cozero-set) open covering $\mathcal{U}$ of X with $|\mathcal{U}| = k$ there is an (cozero-set) open covering $\mathcal{V}$ of X with $|\mathcal{V}| \leq m$ such that $\mathcal{V}$ is a delta-refinement of $\mathcal{U}$. They proved that for every infinite normal space X with $\dim X = n$ and a natural number k,

$$\Delta_k(X) = \begin{cases} 2^k - 1, & \text{if } k \leq n+1 \\ \Sigma_{j=1}^{n+1} \begin{pmatrix} k \\ j \end{pmatrix}, & \text{if } k \geq n+1 \end{cases}$$

By use of $\Delta_k(X)$ they gave an interesting characterization of the covering dimension $\dim X$:

$$\dim X = \lim_{k \to \infty} \frac{\log \Delta_k(X)}{\log k} - 1.$$

Hashimoto and Hattori determined the value of an index $\star_k(X)$, which was also introduced by Nagata:

$$\star_k(X) = \begin{cases} k \cdot 2^{k-1}, & \text{if } k \leq n+1 \\ \Sigma_{j=1}^{n+1} \begin{pmatrix} k \\ j \end{pmatrix} j, & \text{if } k \geq n+1 \end{cases}$$

Also, Nagata defined an index $\Delta_k^p(X)$ and gave a problem on the determination of the index $\Delta_k^p(X)$. For a finite open cover $\mathcal{U}$ of a normal space X, we define indices:

$$\Delta^p(X, \mathcal{U}) = \min\{|\mathcal{V}| \mid \mathcal{V} \text{ is a finite open covering of } X \text{ such that } \mathcal{V}^{\Delta^p} \leq \mathcal{U}\},$$

and

$$\star^p(X,\mathcal{U}) = \min\{|\mathcal{V}| \mid \mathcal{V} \text{ is a finite open covering of } X \text{ such that } \mathcal{V}^{\star^p} \le \mathcal{U}\}.$$

Also, the function $\Delta_k^p(X)$ is defined as the least natural number m such that for every open covering $\mathcal{U}$ of X with $|\mathcal{U}| = k$, there is an open covering $\mathcal{V}$ of X such that $|\mathcal{V}| \le m$ and $\mathcal{V}^{\Delta^p} \le \mathcal{U}$. Similarly, the function $\star_k^p(X)$ is defined as the least natural number m such that for every open covering $\mathcal{U}$ of X with $|\mathcal{U}| = k$, there is an open covering $\mathcal{V}$ of X such that $|\mathcal{V}| \le m$ and $\mathcal{V}^{\star^p} \le \mathcal{U}$.

For natural numbers $k, m, p \ge 1$ with $k \ge m$, we define the natural numbers

$$\tilde{\Delta}(k;m;p) = \Sigma_{m \ge j_1 \ge j_2 \ge \dots \ge j_p \ge 1} \binom{k}{j_1}\binom{j_1}{j_2}\dots\binom{j_{p-1}}{j_p}$$

and

$$\tilde{\star}(k;m;p) = \Sigma_{m \ge j_1 \ge j_2 \ge \dots \ge j_p \ge 1} \binom{k}{j_1}\binom{j_1}{j_2}\dots\binom{j_{p-1}}{j_p} j_p.$$

Theorem 2.1. *(H. Kato and M. Matsumoto) Let X be a normal space and* $\dim X = n$ *and let k and p be natural numbers. Then*

$$\Delta_k^p(X) = \begin{cases} \tilde{\Delta}(k;k;2^{p-1}) = (2^{p-1}+1)^k - (2^{p-1})^k, & \text{if } k \le n+1 \\ \tilde{\Delta}(k;n+1;2^{p-1}), & \text{if } k \ge n+1 \end{cases}$$

and

$$\star_k^p(X) = \begin{cases} \tilde{\star}(k;k;(1/2)(3^p-1)) = k[(1/2)(3^p-1)+1)]^{k-1}, & \text{if } k \le n+1 \\ \tilde{\star}(k;n+1;(1/2)(3^p-1)), & \text{if } k \ge n+1. \end{cases}$$

In particular,

$$\dim X = \sup\{\limsup_{p\to\infty} \frac{\log_2 \Delta^p(X,\mathcal{U})}{p} \mid \mathcal{U} \textit{ is a finite open covering of } X\},$$

and

$$\dim X = \sup\{\limsup_{p\to\infty} \frac{\log_3 \star^p(X,\mathcal{U})}{p} \mid \mathcal{U} \textit{ is a finite open covering of } X\}.$$

Next, we shall introduce the definition of topological entropy by Bowen. Let $f : X \to X$ be a map of a compactum X and let $K \subset X$ be a closed subset of X. We define the topological entropy $h(f, K)$ of f with respect to K as follows. Let n be a natural number and $\epsilon > 0$. A subset F of K is an *(n,ϵ)-spanning set* for f with respect to K if for each $x \in K$, there is $y \in F$ such that

$$\max\{d(f^i(x), f^i(y)) \mid 0 \le i \le n-1\} < \epsilon.$$

A subset E of K is an *(n,ϵ)-separated set* for f with respect to K if for each $x, y \in E$ with $x \neq y$, there is $0 \leq j \leq n-1$ such that

$$d(f^j(x), f^j(y)) > \epsilon.$$

Let $r_n(\epsilon, K)$ be the smallest cardinality of all (n,ϵ)-spanning sets for f with respect to K. Also, let $s_n(\epsilon, K)$ be the maximal cardinality of all (n,ϵ)-separated sets for f with respect to K. Put

$$r(\epsilon, K) = \limsup_{n\to\infty}\ (1/n)\log r_n(\epsilon, K)$$

and

$$s(\epsilon, K) = \limsup_{n\to\infty}\ (1/n)\log s_n(\epsilon, K).$$

Also, put

$$h(f, K) = \lim_{\epsilon\to 0} r(\epsilon, K).$$

Then it is well known that $h(f, K) = \lim_{\epsilon\to 0} s(\epsilon, K)$. Finally, put

$$h(f) = h(f, X).$$

It is well known that $h(f)$ is equal to the topological entropy which was defined by Adler, Konheim and McAndrew.

Let X be a regular continuum. A finite closed covering $\mathcal{A}$ of a regular curve X is a *regular partition* of X provided that if $A, A' \in \mathcal{A}$ and $A \neq A'$, then $\mathrm{Int}(A) \neq \phi$, $A \cap A' = Bd(A) \cap Bd(A')$, and $Bd(A)$ is a finit set. We can easily see that if X is a regular curve and $\epsilon > 0$, then there is a regular partition $\mathcal{A}$ of X such that mesh $\mathcal{A} < \epsilon$, that is, diam $A < \epsilon$ for each $A \in \mathcal{A}$.

For a regular partition $\mathcal{A}$ of X, moreover, $\mathcal{A}$ is called a *strongly regular partition* if $ls_X(a) < \infty$ for each $a \in \bigcup\{Bd(A) |\ A \in \mathcal{A}\}$.

A map $f : X \to X$ is a *piecewise embedding map* with respect to a regular partition $\mathcal{A}$ if the restriction $f|A : A \to X$ is an embedding (= injective) map for each $A \in \mathcal{A}$. A map $f : X \to X$ is a *piecewise monotone map* with respect to $\mathcal{A}$ if the restriction $f|A : A \to f(A)$ is a monotone map for each $A \in \mathcal{A}$.

The following theorem of M. Misiurewicz, W. Szlenk and L. S. Young is well known.

Theorem 2.2. (Misiurewicz-Szlenk and Young) *If $f : I = [0,1] \to I$ is a piecewise embedding map (i.e., there is a finite sequence $c_1, c_2, ..., c_k$ of I such that $c_0 = 0 < c_1 < c_2 < ... < c_k = 1$, each restriction $f|[c_i, c_{i+1}] : [c_i, c_{i+1}] \to I$ is an embedding (=strictly monotone) map and each c_i $(i = 1, 2, .., k-1)$ is a turning point of f, then*

$$h(f) = \lim_{n\to\infty}(1/n)\log l(f^n),$$

where $l(f^n)$ denotes the lap number of f^n.

Let $f : X \to X$ be a map of a regular curve X and let $\mathcal{A} = \{A_1, A_2, ..., A_m\}$ be a regular partition of X. For each $n \geq 0$, consider the itinerary set $It(f, n; \mathcal{A})$ for f and n defined by

$$It(f,n;\mathcal{A}) = \{(x_0, x_1, ..., x_{n-1}) \mid x_i \in \{1,2,...,m\} \text{ and } \bigcap_{i=0}^{n-1} f^{-i}(\text{Int}(A_{x_i})) \neq \phi\}.$$

Put $I(f,n;\mathcal{A}) = |It(f,n;\mathcal{A})|$. Note that $I(f,n+m;\mathcal{A}) \leq I(f,n;\mathcal{A}) \cdot I(f,m;\mathcal{A})$. Hence we see that the limit $\lim_{n\to\infty}(1/n)\log I(f,n;\mathcal{A})$ exists. Note that if $f : I \to I$ is a piecewise embedding map of the unit interval I, then $l(f^{n-1}) = I(f,n;\mathcal{A})$, where $\mathcal{A} = \{[c_i, c_{i+1}] \mid i = 0, 1, ..., k-1\}$.

We can generalize the theorem of Misiurewicz-Szlenk and Young to the case of piecewise embedding maps with respect to strongly regular partitions of regular curves.

Theorem 2.3. *Let X be a regular curve. If a map $f : X \to X$ is a piecewise embedding map with respect to a strongly regular partition $\mathcal{A}$ of X, then*

$$h(f) = \lim_{n\to\infty}(1/n)\log I(f,n;\mathcal{A}).$$

For the proof of the above theorem, we need the following Bowen's result.

Proposition 2.4. (Bowen) *Let X and Y be compacta, and let $f : X \to X$, g: $Y \to Y$ be maps. If $\pi : X \to Y$ is an onto map such that $\pi \cdot f = g \cdot \pi$, then*

$$h(g) \leq h(f) \leq h(g) + \sup_{y\in Y} h(f, \pi^{-1}(y)).$$

Theorem 2.5. *Let X be a regular curve. If a map $f : X \to X$ is a piecewise embedding map with respect to a regular partition $\mathcal{A}$ of X, then*

$$h(f) \leq \lim_{n\to\infty}(1/n)\log I(f,n;\mathcal{A}).$$

Let $f : X \to X$ be a piecewise embedding map of a regular curve X with respect to a regular partition $\mathcal{A} = \{A_1, A_2, ..., A_m\}$ of X. Note that $m = |\mathcal{A}|$. Define an $m \times m$ matrix $M_f = (a_{ij})$ by the following; $a_{ij} = 1$ if $f(\text{Int}(A_i)) \supset \text{Int}(A_j)$, and $a_{ij} = 0$ otherwise. Also, define an $m \times m$ matrix $N_f = (b_{ij})$ by the following; $b_{ij} = 1$ if $f(\text{Int}(A_i)) \cap \text{Int}(A_j) \neq \phi$, and $b_{ij} = 0$ otherwise. Let $\lambda(M_f)$ be the real eigenvalue of M_f such that $\lambda(M_f) \geq |\lambda|$ for all the other eigenvalue λ of M_f. Then we have the following corollary.

Corollary 2.6. *Let X be a regular curve. If a map $f : X \to X$ is a piecewise embedding map with respect to a strongly regular partition $\mathcal{A}$ of X, then*

$$\lambda(M_f) \leq h(f) \leq \lambda(N_f).$$

Remark. (1) The assertion of Theorem 2.3 is not true for piecewise embedding maps on Peano curves. Let $X = \mu^1$ be the Menger curve. We can choose a homeomorphism $f : X \to X$ such that $h(f) \neq 0$. Then f is also a piecewise embedding map with respect to $\mathcal{A} = \{X\}$ and

$$h(f) > 0 = \lim_{n\to\infty} (1/n) \log I(f, n; \mathcal{A}).$$

(2) There is a piecewise embedding map $f : X \to X$ of a dendrite X with respect to a regular partition $\mathcal{A}$ of X such that

$$h(f) < \lim_{n\to\infty} (1/n) \log I(f, n; \mathcal{A}).$$

The assertion of Theorem 2.3 is not true for piecewise embedding maps with respect to regular partitions of regular curves.
(3) Moreover, there is a homeomorphism $f : X \to X$ of a dendrite X such that

$$h(f) < \lim_{n\to\infty} (1/n) \log I(f, n; \mathcal{A})$$

for some regular partition $\mathcal{A}$ of X.

For a map $f : X \to X$ of a regular curve X and a regular partition $\mathcal{A} = \{A_i \mid i = 1, 2, ..., m\}$ of X, we put

$$\sum(f, \mathcal{A}) = \{(x_i)_{i=0}^{\infty} \mid A_{x_i} \in \mathcal{A} \text{ and } \bigcap_{i=0}^{n} f^{-i}(\text{Int}(A_{x_i})) \neq \phi \text{ for all } n = 0, 1, 2, ...\}.$$

Also, let $\sigma_{(f,\mathcal{A})} : \sum(f, \mathcal{A}) \to \sum(f, \mathcal{A})$ be the shift map defined by

$$\sigma_{(f,\mathcal{A})}((x_i)_{i=0}^{\infty}) = (x_{i+1})_{i=0}^{\infty}.$$

Then we have

Theorem 2.7. *Let X be a dendrite. If a map $f : X \to X$ is a piecewise monotone map with respect to a strongly regular partition $\mathcal{A}$ of X, then*

$$h(f) = h(\sigma_{(f,\mathcal{A})}).$$

For each map $f : X \to X$ of a compactum X and a natural number n, put

$$\varphi(f, n) = \sup\{|Comp(f^{-n}(y))| \mid y \in X\}.$$

Then we have the following theorem.

Theorem 2.8. *If $f : X \to X$ is a map of a regular curve X, then*

$$h(f) \leq \limsup_{n\to\infty} (1/n) \log \varphi(f, n).$$

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