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Author(s)	Maitani, Fumio; Yamaguchi, Hiroshi
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Variation of Bergman metrics in different views

京都工芸繊維大学　米谷　文男 (Fumio Maitani)
Kyoto Institute of Technology
Pohang 工科大学　山口　博史 (Hiroshi Yamaguchi)
Pohang University of Science and Technology

We shall show two views on the variation of Bergman metrics. The one view comes from a quasiconformal deformation of a Riemann surface holomorphically depended and the other view comes from a holomorphic family of Riemann surfaces shaping a pseudoconvex domain.

1 Quasiconformally holomorphic movement of a Riemann surface

Let R_t be a Riemann surface which moves quasiconformally depended on a complex parameter t about 0 in the complex plane. We say R_t has quasiconformally holomorphic movement if the Beltrami differential

$$\mu_t = \mu(z,t)\frac{d\bar{z}}{dz} = \frac{(h_t)_{\bar{z}}d\bar{z}}{(h_t)_z dz}$$

of quasiconfomal mapping

$$h_t : R_0 \longrightarrow R_t \quad (w = h_t(z))$$

satisfies the following conditions:

(i) $\mu(z,0) = 0$, $\mu(z,t)$ is measurable, $esssup_R|\mu(z,t)| < 1$,

(ii) For every t there exist constants ϵ_t and M_t such that

$$|\epsilon| < \epsilon_t \Longrightarrow esssup_R|\mu(z,t+\epsilon) - \mu(z,t)| < |\epsilon|M_t,$$

(iii) For almost all $z \in R_0$ $\mu(z,t)$ is holomorphic with respect to t.

2 holomorphic family

Let

$$\pi : S \longrightarrow B$$

be a holomorphic family of a Riemann surfaces

$$R_t = \pi^{-1}(t), t \in B,$$

where S is a 2-dimensional analytic space, B is a disk in $\mathbf{C}$ and R_t is irreducible.

Suppose that S is unramified domain over $B \times \mathbf{C}$ with smooth boundary.

Let a defining function

$$\Phi(t,z)$$

be a real valued C^2 function in a neighborhood of ∂S such that it is positive inside of S, negative outside of S and $\dfrac{\partial\Phi(t,z)}{\partial z}$ doesn't vanish on the boundary of S.

For $(t,z) \in \partial S$ set

$$k_1(t,z) = \frac{\partial\Phi(t,z)}{\partial t} / \left|\frac{\partial\Phi(t,z)}{\partial z}\right|,$$

$$k_2(t,z) = \frac{\partial^2\Phi}{\partial t\partial\bar{t}}\left|\frac{\partial\Phi}{\partial z}\right|^2 - 2Re\left\{\frac{\partial^2\Phi}{\partial\bar{t}\partial z}\frac{\partial\Phi}{\partial t}\frac{\partial\Phi}{\partial\bar{z}}\right\} + \left|\frac{\partial\Phi}{\partial t}\right|^2\frac{\partial^2\Phi}{\partial z\partial\bar{z}} \Big/ \left|\frac{\partial\Phi}{\partial z}\right|^3.$$

These do not depend on the choice of defining function $\Phi(t,z)$. If S is pseudoconvex, then $k_2(t,z) \geq 0$.

3 Behavior space

Let

$$\Lambda = \Lambda(R)$$

be a real Hilbert space which consists of square integrable complex differentials on a Riemann surface R. Its inner product is given as follows:

$$< \omega, \sigma >= Real\ part\ of \iint_R \omega \wedge {}^*\bar{\sigma} = \Re(\omega, \sigma),$$

where $\bar{\sigma}$ is the complex conjugate differential of σ, ${}^*\bar{\sigma}$ is the conjugate differential of $\bar{\sigma}$, (ω, σ) denotes the integral itself of above second expression which means the complex inner product and $\Re(\omega, \sigma)$ means its real part. Typical subspaces of Λ are the following:

$\Lambda_h = \{\lambda \in \Lambda : \lambda$ is harmonic$\}$,
$\Lambda_{eo} = \{\lambda \in \Lambda : \lambda$ is orthogonal to Λ_h and closed differential$\}$,
$\Lambda_{hse} = \{\lambda \in \Lambda_h : \int_\gamma \lambda = 0$ for any dividing cycle $\gamma\}$,
$\Lambda_{he} = \{\lambda \in \Lambda_h : \lambda$ is exact$\}$, $\Lambda_{ho} = {}^*\Lambda_{he}^\perp$, $\Lambda_{hm} = {}^*\Lambda_{hse}^\perp$,
where $\Lambda_\chi^\perp$ is the orthogonal complement of Λ_χ in Λ_h and ${}^*\Lambda_\chi = \{{}^*\omega : \omega \in \Lambda_\chi\}$. Note the following orthogonal decompositions:

$$\Lambda_h = \Lambda_{he} \oplus {}^*\Lambda_{ho} = \Lambda_{hse} \oplus {}^*\Lambda_{hm} = {}^*\Lambda_{he} \oplus \Lambda_{ho} = {}^*\Lambda_{hse} \oplus \Lambda_{hm}.$$

Let

$$\Gamma_h = \{\lambda \in \Lambda_h : \lambda \text{ is real}\},\ \Gamma_{hse} = \Gamma_h \cap \Lambda_{hse},$$

$$\Gamma_{he} = \Gamma_h \cap \Lambda_{he},\ \Gamma_{ho} = {}^*\Gamma_{he}^{\perp},\ \Gamma_{hm} = {}^*\Gamma_{hse}^{\perp},$$

where $\Gamma_\chi^\perp$ is the orthogonal complement of Γ_χ in Γ_h and ${}^*\Gamma_\chi = \{{}^*\omega; \omega \in \Gamma_\chi\}$.

We say a subspace

$$\Lambda_x = \Gamma_x + i\,{}^*\Gamma_x^\perp$$

behavior space, where Γ_x is a subspace of Γ_h and ${}^*\Gamma_x^\perp$ consists of conjugate differentials of orthogonal complement of Γ_x in Γ_h. We assume that

$$\Lambda_x(R_t) \circ h_t \subseteq \Lambda_x(R_0) + \Lambda_{eo}(R_0)$$

For example, $\Gamma_{he} + i\Gamma_{ho}$, $\Gamma_{hm} + i\Gamma_{hse}$, $i\Gamma_h$ and Γ_h are behavior spaces.

4 Elementary differentials

Suppose that there is a point p excluded the support of μ_t, take a local disk $V = \{z : |z - \zeta| < 1\}$ around $\zeta = p$ which does not meet the support of μ_t.

There exist meromorphic differentials φ_n^t, ψ_n^t on R_t with pole only at $h_t(p)$ such that

(i) $\varphi_n^t - \dfrac{dz}{(z-\zeta)^{n+1}},\ \psi_n^t - \dfrac{dz}{(z-\zeta)^{n+1}}$
is holomorphic on $V_t = h_t(V)$,
where z is a local variable on V_t.

(ii) φ_n^t (ψ_n^t) coincides with a differential
in $i\Gamma_h + \Lambda_{eo}$ $(\Gamma_h + \Lambda_{eo})$ on $R_t - V_t$.
$\varphi_n^t + \overline{\varphi_n^t}$ and $\psi_n^t - \overline{\psi_n^t}$ coincide with differentials in Λ_{eo}.

Set
$K_n^t = n!(\varphi_n^t - \psi_n^t)/4\pi = \hat{K}_n^t(z,\zeta)dz$,
$L_n^t = n!(\varphi_n^t + \psi_n^t)/4\pi = \hat{L}_n^t(z,\zeta)dz$.
Let write φ^t for φ_1^t and ψ^t for ψ_1^t.

Theorem 1

$$(\omega, K_{n+1}^t) = \frac{d^n\hat{\omega}}{dz^n}(\zeta)$$

for a square integrable holomorphic differential $\omega = \hat{\omega}(z)dz$ *on* R_t.

$K^t = K_1^t$ is called a Bergman kernel and $\hat{K}^t(\zeta,\zeta)d\zeta \wedge d\bar{\zeta}$ is the Bergman metric.

Since $\varphi_n^t + \overline{\varphi_n^t} - \psi_n^t + \overline{\psi_n^t} \in \Lambda_{eo}$ has a singularity $\dfrac{2d\bar{z}}{\overline{(z-\zeta)}^{n+1}}$, there exists a potential $P_n^t(z,\zeta)$ with a singularity $\dfrac{-1}{n\overline{(z-\zeta)}^n}$ such that

$$dP_n^t = \frac{1}{2}(\varphi_n^t - \psi_n^t + \overline{\varphi_n^t} + \overline{\psi_n^t}) = \frac{2\pi}{n!}(K_n^t + \overline{L_n^t}).$$

Let write P^t for P_1^t.

Let $G^t = G^t(z,\zeta)$ be the Green function with pole at $\zeta = h_t(p)$ on R_t. Set $\varphi_0^t = dG^t + i * dG^t$. We say

$$\gamma^t(\zeta) = \frac{1}{2\pi i}\int_{|z-\zeta|=\epsilon} G^t(z,\zeta)\frac{dz}{z-\zeta}$$

a Robin's constant at $h_t(p)$. The following theorem shows that the Bergman metric closely relate to the Robin constant.

Theorem 2 **(Suita's theorem)**

$$\hat{K}^t(\zeta, \zeta) = -\frac{1}{2\pi}\frac{\partial^2 \gamma^t(\zeta)}{\partial\zeta\partial\bar{\zeta}}.$$

5 Variation of a meromorphic differential

The elementary meromorphic differentials φ_n^t, ψ_n^t may behave smoothly in our circumstance.

Theorem 3

Suppose that R_t has quasiconformally holomorphic movement. Let a meromorphic differential ϕ^t on R_t satisfy that

$$\phi^t \circ h_t - \phi^0 \in \Lambda_x + \Lambda_{eo}$$

where the pole of ϕ^0 is excluded the support of μ_t.

There exist differentials $\phi_u^t, \phi_v^t \in \Lambda_x + \Lambda_{eo}$ on R_t such that

$$\lim_{u \to 0} \| \frac{\phi^{t+u} \circ h_{t+u} \circ h_t^{-1} - \phi^t}{u} - \phi_u^t \| = 0,$$

$$\lim_{v \to 0} \| \frac{\phi^{t+iv} \circ h_{t+iv} \circ h_t^{-1} - \phi^t}{v} - \phi_v^t \| = 0.$$

And $\phi_u^t + i\phi_v^t = i * (\phi_u^t + i\phi_v^t)$ is a holomorphic differential.

Set

$$\frac{\partial \phi^t}{\partial t} = \frac{1}{2}(\phi_u^t - i\phi_v^t), \; \frac{\partial \phi^t}{\partial \bar{t}} = \frac{1}{2}(\phi_u^t + i\phi_v^t).$$

6 Variation of Robin's constant

In the one view we have the first and second variational formulas of Robin's constant.

Theorem 4

Suppose that R_t has quasiconformally holomorphic movement.

$$\frac{\partial\gamma^t(\zeta)}{\partial t} = \frac{1}{4\pi}(\frac{\partial\varphi_0^t}{\partial t}, \overline{\varphi_0^t})$$
$$= \frac{i}{\pi}\iint_{R_0}(\frac{\partial G^t}{\partial w}\frac{\partial w}{\partial z})^2\frac{\partial\mu}{\partial t}(h_t)_z^2 dz d\bar{z},$$
$$\frac{\partial^2\gamma^t(\zeta)}{\partial\bar{t}\partial t} = -\frac{1}{2\pi}(\frac{\partial\varphi_0^t}{\partial\bar{t}}, \frac{\partial\varphi_0^t}{\partial\bar{t}})$$
$$= \frac{i}{\pi}\iint_{R_0}(\frac{\partial\varphi_0^t}{\partial\bar{t}}\frac{\partial G^t}{\partial w}\frac{\partial w}{\partial z})^2\frac{\partial\mu}{\partial t}(h_t)_z^2 dz d\bar{z} \le 0.$$

In the other view they are given by the following forms.

Theorem 5

Suppose that S is unramified domain over $B \times \mathbf{C}$ with smooth boundary.

$$\frac{\partial\gamma^t(\zeta)}{\partial t} = \frac{-1}{\pi}\int_{\partial R_t} k_1(t,z)|\frac{\partial G^t(z,\zeta)}{\partial z}|^2 ds_z,$$
$$\frac{\partial^2\gamma^t(\zeta)}{\partial\bar{t}\partial t} = \frac{-1}{\pi}\int_{\partial R_t} k_2(t,z)|\frac{\partial G^t(z,\zeta)}{\partial z}|^2 ds_z$$
$$-\frac{2i}{\pi}\iint_{R_t}|\frac{\partial^2 G^t(z,\zeta)}{\partial\bar{t}\partial z}|^2 dz d\bar{z}.$$

Hence $\frac{\partial^2\gamma^t(\zeta)}{\partial\bar{t}\partial t} \le 0$ *if S is pseudoconvex.*

The first formula may be regarded as Hadamard's variational formula.

7 Variation of Bergman metrics

In the one view we have the first and second variational formulas of Bergman metrics.

Theorem 6

Suppose that R_t has quasiconformally holomorphic movement.

$$\frac{\partial \hat{K}^t(\zeta,\zeta)}{\partial t} = (K^t, \frac{\partial K^t}{\partial \bar{t}}),$$

$$\frac{\partial^2 \hat{K}^t(\zeta,\zeta)}{\partial \bar{t}\partial t} = (\frac{\partial K^t}{\partial \bar{t}}, \frac{\partial K^t}{\partial \bar{t}}) + (\frac{\partial L^t}{\partial \bar{t}}, \frac{\partial L^t}{\partial \bar{t}}) \geq 0,$$

$$\frac{\partial^2 \log \hat{K}^t(\zeta,\zeta)}{\partial \bar{t}\partial t} = \frac{1}{\hat{K}^t(\zeta,\zeta)}\{(\frac{\partial K^t}{\partial \bar{t}}, \frac{\partial K^t}{\partial \bar{t}}) + (\frac{\partial L^t}{\partial \bar{t}}, \frac{\partial L^t}{\partial \bar{t}})\}$$

$$-\frac{1}{\hat{K}^t(\zeta,\zeta)^2}|(\frac{\partial K^t}{\partial \bar{t}}, K^t)|^2 \geq 0.$$

In the other view they are given by the following forms.

Theorem 7

Suppose that S is unramified domain over $B \times \mathbf{C}$ with smooth boundary.

$$\frac{\partial \hat{K}^t(\zeta,\zeta)}{\partial t} = -i \iint_{\partial R_t} \hat{K}^t(z,\zeta) \overline{\frac{\partial \hat{K}^t(z,\zeta)}{\partial t}} dz d\bar{z},$$

$$\frac{\partial^2 \hat{K}^t(\zeta,\zeta)}{\partial \bar{t}\partial t} = \frac{1}{2}\int_{\partial R_t} k_2(t,z)(|\hat{K}^t(z,\zeta)|^2 + |\hat{L}^t(z,\zeta)|^2) ds_z$$

$$+i \iint_{R_t} (|\frac{\partial^2 \hat{K}^t(z,\zeta)}{\partial \bar{t}}|^2 + |\frac{\partial^2 \hat{L}^t(z,\zeta)}{\partial \bar{t}}|^2) dz d\bar{z}.$$

Hence $\frac{\partial^2 \hat{K}^t(\zeta,\zeta)}{\partial \bar{t}\partial t} \geq 0$, *if S is pseudoconvex.*

8 Application

Using the above variational formulas we have several applications.

Theorem 8 (Lewittes)

Let R be a non-planar Riemann surface. If the Gaussian curvature of the Bergman metric has zero, then R is an (ultra) hyper-elliptic Riemann surface of parabolic type. Conversely, if R is an (ultra) hyper-elliptic Riemann surface of parabolic type, then the branch points coincide with the zeros of the Gaussian curvature of the Bergman metric.

Theorem 9

Let a compact bordered Riemann surface R_t of genus g with m (> 0) boundary component have a quasiconformally holomorphic movement. If $2g + m + 1$ Robin's constants $\gamma^t(p_i)$ are harmonic with respect to t, all R_t are conformally equivalent.

Theorem 10

Let a compact bordered Riemann surface R_t of finite type have quasiconformally holomorphic movement. If $\log \hat{K}^t(\zeta, \zeta)$ is harmonic with respect to t. All R_t are conformally equivalent.

Theorem 11

Let S be an unramified pseudoconvex domain over $B \times \mathbf{C}$ with smooth boundary. Then $\log \hat{K}^t(\zeta, \zeta)$ is plurisubharmonic on S. Further, if, for each $t \in B$, ∂R has at least one strictly pseudoconvex point $(t, \alpha(t))$, then $\log \hat{K}^t(\zeta, \zeta)$ is a strictly plurisubharmonic function on S.

Theorem 12 *(S.Hamano and H.Yamaguchi)*

Let S be an unramified pseudoconvex domain over $B\times\mathbf{C}$ with smooth boundary. If there exists a holomorphic section $\zeta=\zeta(t)$ on $\mathbf{B}$ of S such that $\log\hat{K}^t(\zeta,\zeta)$ is harmonic on $\mathbf{B}$, then S is biholomorphic to the product $\mathbf{B}\times R_0$ by the transformation of the form $t=t,\ w=f(t,z)$.

9 Variational formulas of Rauch type

Let R_t be compact and represented as a covering surface on $\hat{\mathbf{C}}$. Let $\{\zeta_j(t)\}_{j=1}^{\nu}$ be the branch points of order $\{k_j\}$ and the local parameters $\{w_j=(z-\zeta_j(t))^{1/k_j}\}$.

Theorem 13

$$\frac{\partial\hat{K}^t(\zeta,\zeta)}{\partial t}=\frac{1}{2\pi}\sum_{j=1}^{\nu}\frac{\zeta_j'(t)}{k_j(k_j-2)!}\Big[\frac{\partial^{k_j-2}}{\partial w_j^{k_j-2}}\Big(\frac{\partial P^t(w_j,\zeta)}{\partial w_j}\frac{\partial\overline{P^t(w_j,\zeta)}}{\partial w_j}\Big)\Big]_{w_j=0}$$

$$\frac{\partial^2\hat{K}^t(\zeta,\zeta)}{\partial\bar{t}\partial t}=\frac{1}{4\pi^2}\Big(\Big\|\sum_{j=1}^{\nu}\frac{\overline{\zeta_j'(t)}}{k_j(k_j-2)!}\frac{\partial\Omega_j^t(z)}{\partial z}dz\Big\|_{R_t}^2$$

$$+\Big\|\sum_{j=1}^{\nu}\frac{\overline{\zeta_j'(t)}}{k_j(k_j-2)!}\frac{\partial\Xi_j^t(z)}{\partial z}dz\Big\|_{R_t}^2\Big)\geq 0,$$

where

$$\Omega_j^t(z)=\Big[\frac{\partial^{k_j-2}}{\partial\overline{w_j}^{k_j-2}}\Big(\frac{\partial P^t(w_j,\zeta)}{\partial\overline{w_j}}P^t(z,w_j)\Big)\Big]_{w_j=0},$$

$$\Xi_j^t(z)=\Big[\frac{\partial^{k_j-2}}{\partial\overline{w_j}^{k_j-2}}\Big(\frac{\partial\overline{P^t(w_j,\zeta)}}{\partial\overline{w_j}}P^t(z,w_j)\Big)\Big]_{w_j=0}.$$

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