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# Categories of holomorphic line bundles on higher dimensional noncommutative complex tori

Hiroshige Kajiura[a)]
*Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan*

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We construct explicitly noncommutative deformations of categories of holomorphic line bundles over higher dimensional tori. Our basic tools are Heisenberg modules over noncommutative tori and complex/holomorphic structures on them introduced by Schwarz ["Theta functions on noncommutative tori," Lett. Math. Phys. **58**, 81–90 (2001)]. We obtain differential graded (DG) categories as full subcategories of curved DG categories of Heisenberg modules over the complex noncommutative tori. Also, we present the explicit composition formula of morphisms, which, in fact, depends on the noncommutativity. © *2007 American Institute of Physics.*
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## I. INTRODUCTION

In this paper, we propose a way to construct differential graded (DG) categories of finitely generated projective modules over higher dimensional noncommutative complex tori. Also, we give explicit examples of this construction as noncommutative deformations of the DG categories of holomorphic line bundles over higher dimensional complex tori.

The motivation of the present paper is to construct an explicit example of a noncommutative deformation of a complex manifold which may be thought of as one of the extended deformations of a complex manifold proposed by Barannikov-Kontsevich.[1] For an $n$-dimensional complex or a Calabi-Yau manifold $M$, the extended deformation is defined by a deformation $\epsilon \in \mathfrak{g}^1$ of the Dolbeault operator $\bar{\partial}: \mathfrak{g}^k \to \mathfrak{g}^{k+1}$ such that $(\bar{\partial}+\epsilon)^2=0$, where $\mathfrak{g} := \oplus_{k=0}^n \mathfrak{g}^k$ is the graded vector space given by $\mathfrak{g}^k := \oplus_{k=p+q} \Gamma(M, \wedge^p TM \otimes \wedge^q \bar{T}M^*)$. The degree 1 graded piece controlling the extended deformation consists of $\mathfrak{g}^1 = \Gamma(M, \wedge^2 TM) \oplus \Gamma(M, TM \otimes \bar{T}M^*) \oplus \Gamma(M, \wedge^2 \bar{T}M^*)$. Namely, it defines a generalization of the usual complex structure deformation $\epsilon \in \Gamma(M, TM \otimes \bar{T}M^*)$. In particular, the deformation corresponding to $\epsilon \in \Gamma(M, \wedge^2 TM)$ is called the *noncommutative deformation* of the complex manifold $M$. On the other hand, examples of the models of noncommutative deformation should be constructed so that we can see how it is noncommutative, as in the case of the deformation quantization of Kontsevich.[27] A candidate might be to consider an algebra deformation of $M$. Let $V := \oplus_{k=0}^n V^k$ be a graded vector space given by $V^k := \Gamma(M, \wedge^k \bar{T}M)$. This $V$ has a natural graded commutative product $\cdot : V^k \otimes V^l \to V^{k+l}$, together with a differential given by the Dolbeault operator $\bar{\partial}: V^k \to V^{k+1}$. Then, $(V, \bar{\partial}, \cdot)$ forms a DG algebra. A deformation of this DG algebraic structure may describe the noncommutative deformation corresponding to $\epsilon \in \Gamma(M, \wedge^2 TM)$. We can also replace this DG algebra with the DG category of holomorphic vector bundles or coherent sheaves on $M$, where the DG algebra $V$ should be included in the DG category as the endomorphism algebra of the structure sheaf on $M$. This algebraic or categorical approach can be thought of as in the spirit of *homological mirror symmetry* by Kontsevich.[26]

Actually, (topological) string theory suggests considering the complexes $(\mathfrak{g}, \bar{\partial})$ and $(V, \bar{\partial})$; the algebraic structures on their cohomologies are defined in terms of the *closed*[46] *and open*[47] *string amplitudes*, respectively, in the B-twisted topological string theory (B model). The DG category

a)Electronic mail. kajiura@yukawa.kyoto-u.ac.jp

 

above corresponds to physically what is called a *D-brane category* (see Ref. 29); the objects are the D-branes, the morphisms are the open strings between the D-branes in the B model. Thus, the approach above is physically to construct an open string model instead of the closed string model.

It is natural from the viewpoints of both string theory (see Refs. 13, 15, 28, and 30) and deformation theory (see Refs. 41 and 10 and references therein) that DG categories constructed as above should be treated in the category of $A_\infty$ categories, where equivalence between $A_\infty$ categories should be defined by homotopy equivalence.

Now, let us concentrate on an $n$-dimensional complex torus $(M:=T^{2n},\bar{\partial})$. It would be easy if its noncommutative deformation corresponding to $\epsilon\in\Gamma(T^{2n},\wedge^2 TM)$ could be described by the DG algebra $(V,\bar{\partial},*)$, where $*:V^k\otimes V^l\to V^{k+l}$ is the natural extension of the Moyal product on $V^0$, the space of functions on $T^{2n}$, defined by the Poisson bivector $\epsilon\in\Gamma(T^{2n},\wedge^2 TM)$. However, as far as one identifies homotopy equivalent DG algebras with each other, all these DG algebras turn out to be equivalent, being independent of the noncommutative parameter $\epsilon$. In fact, one can easily show that the DG algebra $(V,\bar{\partial},*)$ is formal, i.e., homotopy equivalent to a graded algebra on the cohomology $H(V,\bar{\partial})$, and, in particular, the product on the cohomology $H(V,\bar{\partial})$ is independent of $\epsilon$. These results follow from the fact that one can take a Hodge-Kodaira decomposition of the complex $(V,\bar{\partial})$ so that the space of harmonic forms is closed with respect to the product $*$.

Therefore, in the same way as in the complex one-tori (=real two-tori) case,[14,33,22,16] we should include nontrivial vector bundles which are compatible with the complex structure in some sense. In the real two-tori case, one can construct a DG category of holomorphic vector bundles, in the sense of Ref. 39, over a noncommutative two torus,[33,16] where holomorphic vector bundles are described by DG modules. In particular, the derived category of the DG category is independent of the noncommutativity parameter $\theta\in\mathbb{R}$.[33] Though noncommutative deformation of complex tori in this approach is relatively well understood for the complex one-tori case, its higher dimensional extension is quite nontrivial and interesting especially from the viewpoint of extended deformations.[1,19,11,2] However, in this higher dimensional case, a different problem will arise. Even though we start with a DG module of a holomorphic vector bundle, its noncommutative deformation might not be described by a DG module. There may be several ways to resolve this problem. Our idea in this paper is that we treat the deformed holomorphic vector bundles as *curved differential graded (CDG)* modules over $V$. The important point is that even though the deformed ones are not DG modules, the space of morphisms may be equipped with a differential. Namely, in the context of DG categories, the cohomology should be defined not on the objects but on the *morphisms* between the objects. Thus, one may be able to extract finite dimensional graded vector spaces as the cohomologies of the morphisms.

According to such a spirit, we construct DG categories consisting of some of these CDG modules of deformed holomorphic vector bundles on higher dimensional noncommutative tori. We remark that this procedure is just the same as the DG categories of the B-twisted topological Landau-Ginzburg model in Refs. 20, 21, and 43 and also similar to the procedure by Fukaya *et al.*[10] in the mirror dual A-model side. It would also be interesting to construct a triangulated category via the twisted complexes as is done in Refs. 43, 44, and 4.

Our starting point is based on Schwarz's framework of noncommutative supergeometry[40] and noncommutative complex tori.[39,5] In Ref. 40, a CDG algebra[35] is restudied and applied to noncommutative geometry under the name *Q algebra*, where modules over a Q algebra are discussed. On the other hand, in Ref. 39, a complex structure is introduced on a real $2n$-dimensional noncommutative torus $T_\theta^{2n}$, and a holomorphic structure on the *Heisenberg modules*, noncommutative analogs of vector bundles, over $T_\theta^{2n}$ is defined. Then, our setup can be thought of as an application of the noncommutative supergeometry[40] to the theory of holomorphic Heisenberg modules.[39] This setup provides us with explicit descriptions of noncommutative models. Though one of our motivation comes from Fukaya's noncommutative model of Lagrangian foliations on symplectic tori[9] and their mirror dual, our approach in this paper is different from the one since we deal with the Heisenberg modules which are *finitely generated* projective modules over noncommutative tori. For recent papers, see Ref. 3 for another approach to noncommutative complex tori and the setup in Ref. 4 which should be closer to ours.

The construction of this paper is as follows. In Sec. II, we recall the definitions of CDG algebras,[35] CDG modules, and CDG categories. The notion of module over a Q algebra is more general than that of the CDG modules over a CDG algebra. However, for our purpose, it is enough to consider CDG modules since we begin with Heisenberg modules with *constant* curvature connections. In Sec. III, we construct CDG categories of Heisenberg modules over noncommutative tori with complex structures. In particular, we propose a way to obtain DG categories as full subcategories of the CDG categories. Following the general strategy in Sec. III, we construct CDG categories of holomorphic line bundles over noncommutative complex tori and the DG categories as their full subcategories in Sec. IV. In Sec. IV A, we construct the CDG category on a commutative complex tori. In this case, the CDG category is exactly a DG category. In Sec. IV B, we consider three types of noncommutative deformations of the DG category as CDG categories. Then, we obtain DG categories as the full subcategories of the CDG categories. Furthermore, we present the composition formula of the zeroth cohomologies of the DG categories explicitly. The structure constants of the compositions, in fact, depend on the noncommutative parameters, which implies that the DG categories or the triangulated/derived categories of them depend on the noncommutative parameters. These results can be thought of as generalizations of complex one-dimensional case[14,33,22,16] and also a complex two-tori case[22,23] (in the case that the structure constant of the composition is not deformed by the noncommutative parameter). From a string theory or homotopy algebraic point of view, these deformations should correspond to deformations of an $A_\infty$ structure as weak $A_\infty$ algebras discussed in the context of open-closed homotopy algebras[18] (OCHAs) (see also Ref. 12). We would like to study explicitly this correspondence also elsewhere.

In this paper, any (graded) vector space is over the field $k=\mathbb{C}$. We use indices $i,j,\ldots$ for both the ones which run over $1,\ldots,d=2n$ and the ones which run over $1,\ldots,n$, where $n$ and $d=2n$ are the complex and the real dimension of a noncommutative torus.

## II. CURVED DIFFERENTIAL GRADED ALGEBRAS, CURVED DIFFERENTIAL GRADED MODULES, AND CURVED DIFFERENTIAL GRADED CATEGORIES

**Definition 2.1:** [*(Cyclic) (CDG) algebra*[35]] A *CDG algebra* $(V,f,d,m)$ consists of a $\mathbb{Z}$ (or $\mathbb{Z}_2$) graded vector space $V=\oplus_{k\in\mathbb{Z}}V^k$, where $V^k$ is the degree $k$ graded piece, equipped with a degree element $f\in V^2$, a degree 1 differential $d:V^k\to V^{k+1}$, and a degree preserving (=degree 0) bilinear map $m:V^k\otimes V^l\to V^{k+l}$, satisfying the following relations:

$$d(f)=0, \tag{1}$$

$$(d)^2(v)=m(f,v)-m(v,f), \tag{2}$$

$$dm(v,v')=m(d(v),v')+(-1)^{|v|}m(v,d(v')), \tag{3}$$

$$m(m(v,v'),v'')=m(v,m(v',v'')), \tag{4}$$

where $|v|$ is the degree of $v$, that is, $v\in V^{|v|}$.

Suppose that we have in addition a nondegenerate symmetric inner product

$$\eta: V^k\otimes V^l\to\mathbb{C}$$

of fixed degree $|\eta|\in\mathbb{Z}$ on $V$. Namely, the $\eta$ is nondegenerate, nonzero only if $k+l+|\eta|=0$, and satisfies $\eta(v,v')=(-1)^{kl}\eta(v',v)$ for $v\in V^k$ and $v'\in V^l$. Then, we call $(V,f,\eta,d,m)$ a *cyclic CDG algebra* if the following conditions hold:

$$\eta(d(v),v')+(-1)^{|v|}\eta(v,d(v'))=0,\qquad \eta(m(v,v'),v'')=(-1)^{|v|(|v'|+|v''|)}\eta(m(v',v''),v).$$

**Remark 2.1:** A CDG algebra is identified with a weak $A_\infty$ algebra $(V,\{m_k:V^{\otimes k}\to V\}_{k\geq 0})$ with $m_0=f$, $m_1=d$, $m_2=\cdot$, and $m_3=m_4=\cdots=0$. This algebraic structure is what is called a Q algebra introduced in the framework of noncommutative supergeometry in Ref. 39. Also, a CDG algebra $(V,f,d,\cdot)$ with $f=0$ is a DG algebra, which is a (strict) $A_\infty$ algebra $(V,\{m_k\}_{k\geqslant 1})$[42] with $m_3=m_4=\cdots=0$.

**Definition 2.2:** (*CDG module*) A *right CDG module* $(\mathcal{E},d^{\mathcal{E}},m^{\mathcal{E}})$ over a CDG algebra $(V,-f,d,m)$ is a $\mathbb{Z}$-graded vector space $\mathcal{E}$ equipped with a degree 1 linear map $d^{\mathcal{E}}:\mathcal{E}\to\mathcal{E}$ and a right action $m^{\mathcal{E}}:\mathcal{E}\otimes V\to\mathcal{E}$, satisfying the following condition: for any $v,v'\in V$ and $v^{\mathcal{E}}\in\mathcal{E}$,

$$(d^{\mathcal{E}})^2(v^{\mathcal{E}})=m^{\mathcal{E}}(v^{\mathcal{E}},f),$$

$$d^{\mathcal{E}}m^{\mathcal{E}}(v^{\mathcal{E}},v)=m^{\mathcal{E}}(d^{\mathcal{E}}(v^{\mathcal{E}}),v)+(-1)^{|\xi|}m^{\mathcal{E}}(v^{\mathcal{E}},d(v)),$$

$$m^{\mathcal{E}}(v^{\mathcal{E}},m(v,v'))=m^{\mathcal{E}}(m^{\mathcal{E}}(v^{\mathcal{E}},v),v').$$

In particular, if $f=0$, then $(\mathcal{E},d^{\mathcal{E}},m^{\mathcal{E}})$ is called a *DG module* over the DG algebra $(V,d,m)$. The third condition is nothing but the condition that $\mathcal{E}$ is a (graded) right module over $V$.

A CDG category is a generalization of a CDG algebra, where morphisms in a CDG category correspond to elements of a CDG algebra (see Remark 2.2 below).

**Definition 2.3:** [*(Cyclic) CDG category*] A *CDG category* $\mathcal{C}$ consists of a set of objects $\mathrm{Ob}(\mathcal{C})=\{a,b,\ldots\}$, a $\mathbb{Z}$-graded vector space $V_{ab}=\oplus_{k\in\mathbb{Z}}V_{ab}^k$ for each two objects $a$, $b$ and the grading $k\in\mathbb{Z}$, $f_a:\mathbb{C}\to V_{aa}^2$ for each $a$, a differential $d:V_{ab}^k\to V_{ab}^{k+1}$, and a composition of morphisms $m:V_{bc}^k\otimes V_{ab}^l\to V_{ac}^{k+l}$, satisfying the following relations:

$$d(f_a)=0, \tag{5}$$

$$(d)^2(v_{ab})=m(f_b,v_{ab})-m(v_{ab},f_a), \tag{6}$$

$$dm(v_{bc},v_{ab})=m(d(v_{bc}),v_{ab})+(-1)^{|v_{bc}|}m(v_{bc},d(v_{ab})), \tag{7}$$

$$m(m(v_{cd},v_{bc}),v_{ab})=m(v_{cd},m(v_{bc},v_{ab})), \tag{8}$$

where $|v_{ab}|$ is the $\mathbb{Z}$ grading of $v_{ab}$, that is, $v_{ab}\in V_{ab}^{|v_{ab}|}$.

Let $\eta$ be a nondegenerate symmetric inner product of fixed degree $|\eta|\in\mathbb{Z}$ on $V:=\oplus_{a,b}V_{ab}$. Namely, for each $a$ and $b$,

$$\eta:V_{ba}^k\otimes V_{ab}^l\to\mathbb{C} \tag{9}$$

is nondegenerate, nonzero only if $k+l+|\eta|=0$, and satisfies $\eta(V_{ba}^k,V_{ab}^l)=(-1)^{kl}\eta(V_{ab}^l,V_{ba}^k)$. In this situation, we call a CDG category with inner product $\eta$ a *cyclic CDG category* $\mathcal{C}$ if the following conditions hold:

$$\eta(dv_{ab},v_{ab})+(-1)^{|v_{ab}|}\eta(v_{ab},dv_{ab})=0, \tag{10}$$

$$\eta(m(v_{bc}\otimes v_{ab}),v_{ca})=(-1)^{|v_{bc}|(|v_{ab}|+|v_{ca}|)}\eta(m(v_{ab}\otimes v_{ca}),v_{bc}). \tag{11}$$

Also, we call a cyclic CDG category $\mathcal{C}$ a *cyclic DG category* if $f_a=0$ for any $a\in\mathrm{Ob}(\mathcal{C})$.

**Remark 2.2:** A CDG category $\mathcal{C}$ consisting of one object only is a CDG algebra. Similarly, for a fixed object $a\in\mathrm{Ob}(\mathcal{C})$, the CDG category structure of $\mathcal{C}$ reduces to a CDG algebra $(V_{aa},f_a,d,m)$. On the other hand, if the space of morphisms $\mathcal{V}=\oplus_{a,b}V_{ab}$ is thought of as a $\mathbb{Z}$-graded vector space, $(\mathcal{V},\eta,\oplus_{a\in\mathrm{Ob}(\mathcal{C})}f_a,d,m)$ can be regarded as a cyclic CDG algebra (see also Ref. 46).

Suppose that a CDG category $\mathcal{C}$ has an object $o\in\mathrm{Ob}(\mathcal{C})$ such that $f_o=0$ and for any object $a\in\mathrm{Ob}(\mathcal{C})$, associated with $f_a\in V_{aa}^2$, there exists a central element $\hat{f}_a$ in $V_{oo}$ such that

$$m(f_a,v_{oa}) = m(v_{oa},\hat{f}_a).$$

Then, $(V_{oa} =: \mathcal{E}_a, d, m)$ can be regarded as a CDG module over the cyclic CDG algebra $(V_{oo}, \eta, -\hat{f}_a, d, m)$.

## III. CURVED DIFFERENTIAL GRADED MODULES AND CURVED DIFFERENTIAL GRADED CATEGORIES ON NONCOMMUTATIVE TORI

### A. Higher dimensional noncommutative tori

Let us consider an algebra generated by $U_i$, $i=1,\dots,d$, with relations

$$U_jU_k = e^{-2\pi\sqrt{-1}\theta^{jk}}U_kU_j, \quad j,k=1,\dots,d, \tag{12}$$

for an antisymmetric $d\times d$ matrix $\theta := \{\theta^{jk}\}$. Namely, any element of the algebra is a linear combination over $\mathbb{C}$ of elements $U_{\mathbf{m}}$, $\mathbf{m}=(m_1,\dots,m_d)\in\mathbb{Z}^d$, defined by

$$U_{\mathbf{m}} := U_1^{m_1}U_2^{m_2}\cdots U_d^{m_d}e^{\pi\sqrt{-1}\sum_{j<k}m_jm_k\theta^{jk}},$$

and the relation between $U_{\mathbf{m}}$ and $U_{\mathbf{m}'}$ becomes

$$U_{\mathbf{m}}U_{\mathbf{m}'} = e^{-\pi\sqrt{-1}\sum_{j,k}m_j\theta^{jk}m'_k}U_{\mathbf{m}+\mathbf{m}'}. \tag{13}$$

One can represent any element of this algebra as a formal sum,

$$u = \sum_{\mathbf{m}\in\mathbb{Z}^d} u_{\mathbf{m}}U_{\mathbf{m}}, \quad u_{\mathbf{m}}\in\mathbb{C}.$$

For any element $u$ represented as above, an involution $*$ is defined by

$$u^* := \sum_{\mathbf{m}\in\mathbb{Z}^d} u^*_{\mathbf{m}}U^*_{\mathbf{m}},$$

where $u^*_{\mathbf{m}}$ is the complex conjugate of $u_{\mathbf{m}}$ and $U^*_{\mathbf{m}} := U_{-\mathbf{m}}$. One can consider a subalgebra $T^d_\theta$ such that any element, again represented as $u=\sum_{\mathbf{m}}u_{\mathbf{m}}U_{\mathbf{m}}$, belongs to the Schwartz space $\mathcal{S}(\mathbb{Z}^d)$, that is, the coefficients $\{u_{\mathbf{m}}\}$ as a function on $\mathbb{Z}^d$ tend to zero faster than any power of $\|\mathbf{m}\|$. This algebra $T^d_\theta$ is, in fact, a $C^*$ algebra and is called (the smooth version of) a *noncommutative torus*.[36,25]

There is a canonical normalized *trace* on $T^d_\theta$ specified by the rule

$$\mathrm{Tr}(u) = u_{\mathbf{m}=0}, \quad u = \sum_{\mathbf{m}} u_{\mathbf{m}}U_{\mathbf{m}}. \tag{14}$$

For $\theta=0$ we can realize the algebra $T^d_\theta$ as an algebra of functions on a $d$-dimensional torus $T^d$. Then trace (14) corresponds to an integral over $T^d$ provided that volume of $T^d$ is 1.

In order to define a connection on a module $E$ over a noncommutative torus $T^d_\theta$, we shall first define a natural Lie algebra of shifts $\mathcal{L}_\theta$ acting on $T^d_\theta$. The shortest way to define this Lie algebra is by specifying a basis consisting of derivations $\delta_j: T^d_\theta \to T^d_\theta$, $j=1,\dots,d$, satisfying

$$\delta_j(U_{\mathbf{m}}) = 2\pi\sqrt{-1}m_jU_{\mathbf{m}}. \tag{15}$$

For the multiplicative generators $U_j$ the above relation reads as

$$\delta_jU_k = 2\pi\sqrt{-1}\,\delta_{jk}U_k.$$

These derivations then span a $d$-dimensional abelian Lie algebra (over $\mathbb{C}$) that we denote $\mathcal{L}_\theta$.

A *connection* on a (right) module $E$ over $T^d_\theta$ is a set of operators $\nabla_X: E\to E$, $X\in\mathcal{L}_\theta$ depending linearly on $X$ and satisfying

$$\nabla_X(\xi\cdot u) = \nabla_X(\xi)\cdot u + \xi\cdot X(u)$$

for any $\xi \in E$ and $u \in T_\theta^d$. A connection $\nabla$ is called a *constant curvature connection* if the curvature of the connection is of the following form: for $\nabla_i := \nabla_{\delta_i}$, $i=1,\ldots,d$,

$$[\nabla_i, \nabla_j] = -2\pi\sqrt{-1}F_{ij} \cdot \mathbf{1}_{\mathrm{End}_{T_\theta^d}(E)}, \quad F_{ij} = -F_{ji} \in \mathbb{R}. \tag{16}$$

(Namely, $F_{ij}$ is a constant.) On a noncommutative torus, one can construct a class of finitely generated projective modules called *Heisenberg modules* (see Refs. 36 and 25). In fact, on any Heisenberg module there exists a constant curvature connection. They play a special role. It was shown by Rieffel[36] that if the matrix $\theta^{ij}$ is irrational in the sense that at least one of its entries is irrational, then any projective module over $T_\theta^d$ is isomorphic to a direct sum of Heisenberg modules.

Heisenberg modules are applied to discuss the Morita equivalence of noncommutative tori. Let $\mathrm{SO}(d,d,\mathbb{Z})$ be the group defined by

$$\mathrm{SO}(d,d,\mathbb{Z}) := \{g \in \mathrm{Mat}_{2n}(\mathbb{Z}) | g^t J g = J\}, \quad J := \begin{pmatrix} \mathbf{0}_n & \mathbf{1}_n \\ \mathbf{1}_n & \mathbf{0}_n \end{pmatrix}.$$

An $\mathrm{SO}(d,d,\mathbb{Z})$ action on a generic skew symmetric matrix in $\mathrm{Mat}_d(\mathbb{R})$ is defined by

$$g(\theta) := (\mathcal{A}\theta + \mathcal{B})(\mathcal{C}\theta + \mathcal{D})^{-1}, \quad g := \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \in \mathrm{SO}(d,d,\mathbb{Z}).$$

In fact, $g(\theta)$ is again a skew symmetric matrix in $\mathrm{Mat}_d(\mathbb{R})$ due to the condition $g \in \mathrm{SO}(d,d,\mathbb{Z})$. Then, it is known that a noncommutative tori $T_\theta^d$ is Morita equivalent to $T_{\theta'}^d$ if[37] and only if[40] they are related by $\theta' = g(\theta)$, $g \in \mathrm{SO}(d,d,\mathbb{Z})$ (for more recent papers, see Refs. 45, 31, and 7). To establish this Morita equivalence, one may construct a $T_\theta^d$–$T_{g(\theta)}^d$ *Morita equivalence bimodule*, denoted by $P_{\theta-g(\theta)}$ (see Refs. 36 and 37). One can, in fact, construct the Morita equivalence bimodule $P_{\theta-g(\theta)}$ for any $g \in \mathrm{SO}(d,d,\mathbb{Z})$ as a left Heisenberg module $E$ over $T_\theta^d$. In this case, the algebra $\mathrm{End}_{T_\theta^d}(E)$, the algebra of endomorphisms of $E$ which commute with the left action of $T_\theta^d$, coincides with the noncommutative torus $T_{g(\theta)}^d$. This implies that one can construct a $T_{g(\theta)}^d$–$T_\theta^d$ Morita equivalence bimodule $P_{g(\theta)-\theta}$ as a right Heisenberg module over $T_\theta^d$. We denote it by $E_{g,\theta}$; we have $\mathrm{End}_{T_\theta^d}(E_{g,\theta}) \simeq T_{g(\theta)}^d$. Also, the $T_\theta^d$–$T_{g(\theta)}^d$ Morita equivalence bimodule is given by the right Heisenberg module $E_{g^{-1},g(\theta)}$. On $T_{g(\theta)}^d$, a trace $\mathrm{Tr}_{T_{g(\theta)}^d}: T_{g(\theta)}^d \to \mathbb{C}$ and *derivations* $\delta_i: T_{g(\theta)}^d \to T_{g(\theta)}^d$, $i=1,\ldots,d$, are defined by appropriate rescaling of those for $T_\theta^d$ as

$$\mathrm{Tr}_{T_{g(\theta)}^d}(u) = \sqrt{|\det(\mathcal{C}\theta + \mathcal{D})|}\, u_{\mathbf{m}=0}, \quad u := \sum_{\mathbf{m} \in \mathbb{Z}^n} u_{\mathbf{m}} Z_{\mathbf{m}} \in T_{g(\theta)}^d \tag{17}$$

and

$$\delta_j(Z_{\mathbf{m}}) = \frac{2\pi\sqrt{-1}m_j}{\sqrt{|\det(\mathcal{C}\theta + \mathcal{D})|}} Z_{\mathbf{m}},$$

where $g = \left(\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right)$ and $Z_1, \ldots, Z_d$ are the generators of $T_{g(\theta)}^d$ with relations $Z_j Z_k = e^{-2\pi\sqrt{-1}(g\theta)^{jk}} Z_k Z_j$, $j,k=1,\ldots,d$. For $X \in \mathcal{L}_\theta$, a linear map $\nabla: T_{g(\theta)}^d \otimes \mathcal{L}_\theta \to T_{g(\theta)}^d$ is defined by extending linearly

$$\nabla_{\delta_i}(u) := \delta_i(u), \quad i=1,\ldots,d, \quad u \in T_{g(\theta)}^d. \tag{18}$$

When we have a $T_\theta^d$–$T_{\theta'}^d$ Morita equivalence bimodule, one can consider the following *tensor product* (see Refs. 38 and 40):

$$E_{g_a,\theta}\otimes_{T^d_\theta}P_{\theta-g(\theta)}\simeq E_{g_a g^{-1},g(\theta)} \tag{19}$$

for a Heisenberg module $E_{g_a,\theta}$ with any $g_a\in SO(d,d,\mathbb{Z})$, where the tensor product $\otimes_{T^d_\theta}$ is defined by the standard tensor product $\otimes$ over $\mathbb{C}$ with the identification $(\xi_a\cdot u)\otimes p\sim\xi_a\otimes(u\cdot p)$ for any $\xi_a\in E_{g_a,\theta}$, $u\in T^d_\theta$ and $p\in P_{\theta-g(\theta)}$. Let us denote $\theta_a:=g_a(\theta)$. For a right Heisenberg module $E_{g_a,\theta}$ and a Morita equivalence bimodule $P_{\theta_b-\theta_a}$ with an $SO(d,d,\mathbb{Z})$ element $g_b$, the existence of tensor product (19) implies that we have the following tensor product:

$$P_{\theta_b-\theta_a}\otimes_{T^d_{\theta_a}}E_{g_a,\theta}\simeq E_{g_b,\theta}.$$

Thus, we may identify $P_{\theta_b-\theta_a}$ with the space of homomorphisms from $E_{g_a,\theta}$ to $E_{g_b,\theta}$. Hereafter, we write

$$\mathrm{Hom}(E_{g_b,\theta},E_{g_a,\theta}):=P_{g_b(\theta)-g_a(\theta)}.$$

On a Morita equivalence $T^d_{\theta_b}-T^d_{\theta_a}$ bimodule $\mathrm{Hom}(E_{g_a,\theta},E_{g_b,\theta})$, we define a connection $\nabla:\mathrm{Hom}(E_{g_a,\theta},E_{g_b,\theta})\otimes\mathcal{L}_\theta\to\mathrm{Hom}(E_{g_a,\theta},E_{g_b,\theta})$ as a linear map, satisfying

$$\nabla_X(u_b\cdot\xi)=\nabla_X(u_b)\cdot\xi+u_b\cdot\nabla_X(\xi),\quad \nabla_X(\xi\cdot u_a)=\nabla_X(\xi)\cdot u_a+\xi\cdot\nabla_X(u_a)$$

for any $u_a\in T^d_{\theta_a}$ and $u_b\in T^d_{\theta_b}$, where $\nabla_X(u_b)$ and $\nabla_X(u_a)$ are defined by Eq. (18). Since these Morita equivalence bimodules are Heisenberg modules, they can be equipped with constant curvature connections.

A Heisenberg module $E_{g,\theta}$ attached to an element $g\in SO(d,d,\mathbb{Z})$ as above is called a *basic module*. Since any Heisenberg module is constructed as a direct sum of basic modules, in this paper we concentrate on categories of basic modules. (Of course, we can consider the corresponding "additive" categories and further (pre)triangulated categories in the sense of Refs. 43 and 44 through twisted complexes.)

Let $\mathrm{Ob}:=\{a,b,\ldots\}$ be a collection of labels and consider a map $g:\mathrm{Ob}\to SO(d,d,\mathbb{Z})$, $g(a):=g_a\in SO(d,d,\mathbb{Z})$ for $a\in\mathrm{Ob}$. For the collection of Heisenberg modules $\{E_{g_a,\theta}|a\in\mathrm{Ob}\}$, assume that we have an associative product

$$m:\mathrm{Hom}(E_{g_b,\theta},E_{g_c,\theta})\otimes\mathrm{Hom}(E_{g_a,\theta},E_{g_b,\theta})\to\mathrm{Hom}(E_{g_a,\theta},E_{g_c,\theta})$$

for any $a,b,c\in\mathrm{Ob}$. Namely, for any $a,b,c,d\in\mathrm{Ob}$ and $\xi_{ab}\in\mathrm{Hom}(E_{g_a,\theta},E_{g_b,\theta})$, $\xi_{bc}\in\mathrm{Hom}(E_{g_b,\theta},E_{g_c,\theta})$, $\xi_{cd}\in\mathrm{Hom}(E_{g_c,\theta},E_{g_d,\theta})$, we assume that the product $m$ satisfies

$$m(m(\xi_{cd},\xi_{bc}),\xi_{ab})=m(\xi_{cd},m(\xi_{bc},\xi_{ab})). \tag{20}$$

Such a product $m:\mathrm{Hom}(E_{g_b,\theta},E_{g_c,\theta})\otimes\mathrm{Hom}(E_{g_a,\theta},E_{g_b,\theta})\to\mathrm{Hom}(E_{g_a,\theta},E_{g_c,\theta})$ is obtained essentially by the tensor product; $m$ is constructed by fixing a map inducing the isomorphism

$$\mathrm{Hom}(E_{g_b,\theta},E_{g_c,\theta})\otimes_{T^d_{\theta_b}}\mathrm{Hom}(E_{g_a,\theta},E_{g_b,\theta})\simeq\mathrm{Hom}(E_{g_a,\theta},E_{g_c,\theta}).$$

There exists a choice of the map so that the associativity holds (see Ref. 40).

For $a\in\mathrm{Ob}$, suppose that a constant curvature connection $\nabla_a$ is defined on $E_{g_a,\theta}$. Also, for $b\in\mathrm{Ob}$, $a\neq b$, define a constant curvature connection $\nabla:\mathrm{Hom}(E_{g_a,\theta},E_{g_b,\theta})\otimes\mathcal{L}_\theta\to\mathrm{Hom}(E_{g_a,\theta},E_{g_b,\theta})$ whose constant curvature $F_{ab}:=\{F_{ab,ij}\}_{i,j=1,\ldots,d}$ is defined by

$$F_{ab,ij}\cdot\xi_{ab}:=\frac{\sqrt{-1}}{2\pi}[\nabla_i,\nabla_j](\xi_{ab}),\quad F_{ab,ij}=-F_{ab,ji}\in\mathbb{R}$$

for any $\xi\in\mathrm{Hom}(E_{g_a,\theta},E_{g_b,\theta})$. Then, a constant curvature connection $\nabla_b:E_{g_b,\theta}\otimes\mathcal{L}_\theta\to E_{g_b,\theta}$ can be induced as follows:[38,40]

$$\nabla_b(m(\xi_{ab},\xi_a)) := m(\nabla(\xi_{ab}),\xi_a) + m(\xi_{ab},\nabla_a(\xi_a))$$

for any $\xi_a \in E_{g_a,\theta}$ and $\xi_{ab} \in \mathrm{Hom}(E_{g_a,\theta},E_{g_b,\theta})$, where the relation between the curvatures of $E_{g_a,\theta}$, $E_{g_b,\theta}$ and that of $\mathrm{Hom}(E_{g_a,\theta},E_{g_b,\theta})$ is given by

$$F_b - F_a = F_{ab}.$$

Thus, repeating this procedure leads to the following category $\mathcal{C}^{\mathrm{pre}}_{\theta,E}$:

**Definition 3.1:** For a noncommutative torus $T^d_\theta$, let $\mathrm{Ob} := \{a,b,\ldots\}$ be a collection of labels and $E$ a map from Ob to the space of basic modules with constant curvature connections; for $a \in \mathrm{Ob}$, we denote $E(a)=(E_{g_a,\theta},\nabla_a) =: E_a$. A *category* $\mathcal{C}^{\mathrm{pre}}_{\theta,E}$ is defined by the following data.

- The collection of objects is

$$\mathrm{Ob}(\mathcal{C}^{\mathrm{pre}}_{\theta,E}) := \mathrm{Ob}.$$

Each object $a \in \mathrm{Ob}$ is associated with a basic module with a constant curvature connection $E_a$ whose constant curvature is denoted by a skew symmetric matrix $F_a \in \mathrm{Mat}_{2n}(\mathbb{R})$.

- For any $a,b \in \mathrm{Ob}(\mathcal{C}^{\mathrm{pre}}_{\theta,E})$, the space of morphisms is

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta,E}}(a,b) := \mathrm{Hom}(E_{g_a,\theta},E_{g_b,\theta}),$$

which is equipped with a constant curvature connection $\nabla : \mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta,E}}(a,b) \otimes \mathcal{L}_\theta \to \mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta,E}}(a,b)$ with its constant curvature $F_{ab}=F_b-F_a$.

- For any $a,b,c \in \mathrm{Ob}$, there exists an associative product [Eq. (20)]

$$m:\mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta,E}}(b,c) \otimes \mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta,E}}(a,b) \to \mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta,E}}(a,c).$$

- For any $a,b,c \in \mathrm{Ob}$ and $\xi_{ab} \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta,E}}(a,b)$, $\xi_{bc} \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta,E}}(b,c)$, the Leibniz rule holds,

$$\nabla m(\xi_{bc},\xi_{ab}) = m(\nabla(\xi_{bc}),\xi_{ab}) + m(\xi_{bc},\nabla(\xi_{ab})). \tag{21}$$

- For any $a \in \mathrm{Ob}$, a trace $\mathrm{Tr}_a : \mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta,E}}(a,a) \to \mathbb{C}$ is given by Eq. (17),

$$\mathrm{Tr}_a(u) = \sqrt{|\det(\mathcal{C}_a\theta + \mathcal{D}_a)|}\,u_{\mathbf{m}=0}, \quad u := \sum_{\mathbf{m}\in\mathbb{Z}^n} u_{\mathbf{m}} Z_{\mathbf{m}} \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta,E}}(a,a),$$

for $g_a = \left(\begin{smallmatrix}\mathcal{A}_a & \mathcal{B}_a \\ \mathcal{C}_a & \mathcal{D}_a\end{smallmatrix}\right)$. In particular, for any $a,b \in \mathrm{Ob}$, $\mathrm{Tr}_a\, m : \mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta,E}}(b,a) \otimes \mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta,E}}(a,b) \to \mathbb{C}$ gives a nondegenerate bilinear map such that

$$\mathrm{Tr}_a\, m(\xi_{ba},\xi_{ab}) = \mathrm{Tr}_b\, m(\xi_{ab},\xi_{ba}), \quad \xi_{ab} \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta,E}}(a,b), \quad \xi_{ba} \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta,E}}(b,a). \tag{22}$$

The last identity [Eq. (22)], together with the nondegeneracy of $\mathrm{Tr}_a\, m$, is a typical property of Morita equivalence bimodules (see Refs. 36 and 25, and for the noncommutative two-tori case, see Ref. 15).

### B. Noncommutative complex tori and curved differential graded structures on them

Let us consider a complex structure on the noncommutative torus $T^{2n}_\theta$ as introduced by Schwarz.[39] We take a different notation which fits our arguments, though it is equivalent to the one in Ref. 39. When we define a complex structure on a commutative torus $T^{2n}$, we may take a $\mathbb{C}$-valued $n\times n$ matrix $\tau=\{\tau^j_i\}$, $i,j=1,\ldots,n$, whose imaginary part $\tau_I := \mathrm{Im}(\tau)$ is positive definite. A commutative complex torus is then described by $\mathbb{C}^n/(\mathbb{Z}^n + \tau^t\mathbb{Z}^n)$, where $\tau^t$ is the transpose of $\tau$. The complex coordinates of $\mathbb{C}^n$ are given by $(z_1,\ldots,z_n)$, $z^i = x^i + \sum_j y^j \tau^i_j$, $i=1,\ldots,n$. The corresponding Dolbeault operator $\bar\partial$ is given by

$$\bar{\partial} = \sum_{i=1}^{n} d\bar{z}^i \frac{\partial}{\partial \bar{z}^i}, \quad \frac{\partial}{\partial \bar{z}^i} := \frac{1}{2\sqrt{-1}} \sum_{j=1}^{n} \left( ((\tau_I)^{-1}\tau)_i^j \frac{\partial}{\partial x^j} - ((\tau_I)^{-1})_i^j \frac{\partial}{\partial y^j} \right),$$

where we denote $\mathrm{Im}(\tau) =: \tau_I$ which is by definition positive definite.

Based on these formulas, for a noncommutative torus $T_\theta^{2n}$ and a fixed complex structure $\tau$, let us define $\bar{\partial}_i \in \mathcal{L}_\theta$, $i=1,\ldots,n$, by

$$\bar{\partial}_i := \frac{1}{2\sqrt{-1}} \sum_{j=1}^{n} (((\tau_I)^{-1}\tau)_i^j \delta_j - ((\tau_I)^{-1})_i^j \delta_{n+j}).$$

Also, for $E_a := (E_{g_a,\theta}, \nabla_a)$, a Heisenberg module $E_{g_a,\theta}$ over $T_\theta^{2n}$ with a constant curvature connection $\nabla_{a,i}$, $i=1,\ldots,2n$, define a *holomorphic structure* $\bar{\nabla}_{a,i}: E_{g_a,\theta} \to E_{g_a,\theta}$, $i=1,\ldots,n$, by

$$\bar{\nabla}_{a,i} := \frac{1}{2\sqrt{-1}} \sum_{j=1}^{n} (((\tau_I)^{-1}\tau)_i^j \nabla_{a,j} - ((\tau_I)^{-1})_i^j \nabla_{a,n+j}). \tag{23}$$

For each pair $(E_a, E_b)$, we define a *holomorphic structure* $\bar{\nabla}_i: \mathrm{Hom}(E_{g_a,\theta}, E_{g_b,\theta}) \to \mathrm{Hom}(E_{g_a,\theta}, E_{g_b,\theta})$, $i=1,\ldots,n$, by the same formula,

$$\bar{\nabla}_i := \frac{1}{2\sqrt{-1}} \sum_{j=1}^{n} (((\tau_I)^{-1}\tau)_i^j \nabla_j - ((\tau_I)^{-1})_i^j \nabla_{n+j}). \tag{24}$$

Let $\Lambda$ be the Grassmann algebra generated by $d\bar{z}^1, \ldots, d\bar{z}^n$ of degree 1. Namely, they satisfy $d\bar{z}^i d\bar{z}^j = -d\bar{z}^j d\bar{z}^i$ for any $i,j=1,\ldots,n$, so, in particular, $(d\bar{z}^i)^2=0$. These generators are thought of as a formal basis of the antiholomorphic one forms on the complex noncommutative torus $T_\theta^{2n}$. By $\Lambda^k$ we denote the degree $k$ graded piece of $\Lambda$. The graded vector space $V := T_\theta^{2n} \otimes \Lambda$ is then thought of as the space of smooth antiholomorphic forms on the complex noncommutative torus $T_\theta^{2n}$, which also has the the graded decomposition,

$$V = \oplus_{k=0}^{n} V^k.$$

Any element in $V^k$ can be written as

$$v = \sum_{\mathbf{m} \in \mathbb{Z}^n} \sum_{i_1,\ldots,i_k} v_{\mathbf{m};i_1\cdots i_k} U_{\mathbf{m}} \cdot (d\bar{z}^{i_1} \cdots d\bar{z}^{i_k}),$$

where $v_{\mathbf{m};i_1\cdots i_k} \in \mathbb{C}$ is skew symmetric with respect to the indices $i_1 \cdots i_k$. A product $m: V^k \otimes V^l \to V^{k+l}$ is defined naturally by combining the product on $T_\theta^{2n}$ with the one on the Grassmann algebra $\Lambda$, and then $(V,m)$ forms a graded algebra. One can define a differential $d: V^k \to V^{k+1}$,

$$d := \sum_{i=1}^{n} d\bar{z}^i \cdot \bar{\partial}_i,$$

which satisfies the Leibniz rule with respect to the product $m$.

An inner product $\eta: V^k \otimes V^l \to \mathbb{C}$ of degree $-n$ is defined by the composition of the product $m$ with a trace $\int_{T_\theta^{2n}}: V \to \mathbb{C}$,

$$\eta = \int_{T_\theta^{2n}} m, \quad \int_{T_\theta^{2n}} v = v_{\mathbf{m}=0;i_1\cdots i_k} \epsilon_{1\cdots n}^{i_1\cdots i_k}.$$

Here $\epsilon$ is defined by

$$\epsilon^{i_1\cdots i_k}_{1\cdots n} = \begin{cases} 0, & k \neq n, \\ \sum_{\sigma\in\mathfrak{S}_n} \epsilon(\sigma)\delta^{i_1}_{\sigma(1)}\cdots\delta^{i_n}_{\sigma(n)}, & k=n, \end{cases}$$

where $\epsilon(\sigma)$ is the signature of the permutation of $n$ elements $\sigma\in\mathfrak{S}_n$. Namely, $\int_{T^d_\theta}: V^k\to\mathbb{C}$ is thought of as the integration of differential forms over $T^{2n}_\theta$, as an extension of the trace map $\mathrm{Tr}: T^{2n}_\theta\to\mathbb{C}$ in Eq. (14), and hence gives a nonzero map only if $k=n$.

**Lemma 3.1:** *$(V,\eta,d,m)$ forms a cyclic DG algebra.* ■

For $E_a:=(E_{g_a,\theta},\nabla_a)$ a Heisenberg module over $T^{2n}_\theta$ with a constant curvature connection, we lift $E_a$ to a $\mathbb{Z}$-graded right module $\mathcal{E}_a:=E_{g_a,\theta}\otimes\Lambda$ over $V$. We denote by $m_a:\mathcal{E}_a\otimes V\to\mathcal{E}_a$ the right action of $V$ on $\mathcal{E}_a$. The connection $\nabla_a: E_{g_a,\theta}\otimes\mathcal{L}_\theta\to E_{g_a,\theta}$ is then lifted to a degree 1 linear map $d_a:\mathcal{E}_a\to\mathcal{E}_a$ defined by

$$d_a := \sum_{i=1}^{n} \mathrm{d}\bar{z}^i\cdot\bar{\nabla}_{a,i},$$

where $\bar{\nabla}_{a,i}$ is holomorphic structure (23). This $d_a$ is not a differential in general. Namely, the graded module has its curvature,

$$(d_a)^2 v^{\mathcal{E}_a} = \hat{f}_a v^{\mathcal{E}_a}, \quad \hat{f}_a := -\pi\sqrt{-1}(\mathrm{d}\bar{z}^t\tau_I^{-1})(\tau-\mathbf{1}_n)F_a\begin{pmatrix}\tau^t\\ -\mathbf{1}_n\end{pmatrix}(\tau_I^{t,-1}\mathrm{d}\bar{z}) \in \Lambda^2$$

for any $v^{\mathcal{E}_a}\in\mathcal{E}_a$, where $\mathrm{d}\bar{z}^t:=(\mathrm{d}\bar{z}^1\cdots\mathrm{d}\bar{z}^n)$. This $d_a$ defines a differential on $\mathcal{E}_a$, that is, $\hat{f}_a=0$ if and only if

$$(\tau-\mathbf{1}_n)F_a\begin{pmatrix}\tau^t\\ -\mathbf{1}_n\end{pmatrix}=0.$$

In this case, $(\mathcal{E}_a,d_a,m_a)$ forms a DG module over $V$. In the commutative case ($\theta=0$), this condition on $F_a$ is nothing but the condition that the corresponding (line) bundle is holomorphic, i.e., the curvature is a (1,1) form with respect to the complex structure defined by $\tau$. However, for general $\theta$, $\hat{f}_a$ may not be zero even if it is zero when $\theta$ is set to be zero.

On the other hand, since $\hat{f}_a\in\Lambda^2\subset V^2$ is a central element in $V$ with respect to the product $m$, $(d)^2(v)=m(\hat{f}_a,v)-m(v,\hat{f}_a)$ $(=0)$ holds and then $(V,\eta,-\hat{f}_a,d,m)$ forms a cyclic CDG algebra. Thus, the following lemma.

**Lemma 3.2:** *$(\mathcal{E}_a,\hat{f}_a,d_a,m_a)$ forms a CDG module over the cyclic CDG algebra $(V,\eta,-\hat{f}_a,d,m)$.* ■

We call this $\hat{f}_a\in V^2$ the *potential two form* of $\mathcal{E}_a$.

Now, for a category $\mathcal{C}^{\mathrm{pre}}_{\theta,E}$ given in Definition 3.1, we construct a CDG category $\mathcal{C}_{\theta,\tau,E}=:\mathcal{C}$.

**Definition 3.2:** For a fixed category $\mathcal{C}^{\mathrm{pre}}_{\theta,E}$, a *category* $\mathcal{C}$ is defined as follows.

- The collection of objects is

$$\mathrm{Ob}(\mathcal{C}) := \mathrm{Ob},$$

  where any object $a\in\mathrm{Ob}$ is associated with a CDG module $(\mathcal{E}_a,\hat{f}_a,d_a,m_a)$ over the CDG algebra $(V,\eta,-\hat{f}_a,d,m)$ corresponding to $E_a$ as in Lemma 3.2.
- For any $a,b\in\mathrm{Ob}(\mathcal{C})$, the space of morphisms is the graded vector space

$$\mathrm{Hom}_{\mathcal{C}}(a,b) := \mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta,E}}(a,b)\otimes\Lambda =: V_{ab} = \oplus_{k=1}^n V^k_{ab},$$

  which is equipped with a degree 1 linear map $d: V^k_{ab}\to V^{k+1}_{ab}$,

$$d := \sum_{i=1}^{n} \mathrm{d}\bar{z}^i \bar{\nabla}_i,$$

where $\bar{\nabla}_i$ is holomorphic structure (24) corresponding to the constant curvature connection $\nabla : \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,b) \otimes \mathcal{L}_\theta \to \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,b)$.

- For any $a,b,c \in \mathrm{Ob}(\mathcal{C})$, an associative product $m: V_{bc}^l \otimes V_{ab}^k \to V_{ac}^{k+l}$ is given by the lift of the product $m$ on $\mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(*,*)$.
- For any two objects $a,b \in \mathrm{Ob}(\mathcal{C})$, a nondegenerate graded symmetric inner product $\eta : V_{ba} \otimes V_{ab} \to \mathbb{C}$ of degree $-n$ is defined by

$$\eta = \int_{T_{\theta_a}^{2n}} m, \quad m: V_{ba} \otimes V_{ab} \to V_{aa}.$$

Here $\int_{T_{\theta_a}^{2n}} : V_{aa} \to \mathbb{C}$ is defined by

$$\int_{T_{\theta_a}^{2n}} v = \sqrt{|\det(\mathcal{C}_a \theta + \mathcal{D}_a)|}\, v_{\mathbf{m}=0;i_1\cdots i_k} \epsilon_{1\cdots n}^{i_1\cdots i_k}, \quad v := \sum_{\mathbf{m}\in\mathbb{Z}^n} \sum_{i_1,\ldots,i_k} v_{\mathbf{m};i_1\cdots i_k} U_{\mathbf{m}} \cdot \mathrm{d}\bar{z}^{i_1} \cdots \mathrm{d}\bar{z}^{i_k} \in V_{aa},$$

for $g_a = \begin{pmatrix} \mathcal{A}_a & \mathcal{B}_a \\ \mathcal{C}_a & \mathcal{D}_a \end{pmatrix}$ as an extension of the trace map $\mathrm{Tr}_a \to \mathbb{C}$.

Due to Leibniz rule (21), it is clear that $d: V_{ab}^k \to V_{ab}^{k+1}$ is a derivation,

$$dm(v_{bc} \otimes v_{ab}) = m(d(v_{bc}) \otimes v_{ab}) + (-1)^{|v_{bc}|} m(v_{bc} \otimes d(v_{ab})). \tag{25}$$

Let us define $\hat{f}_{ab} \in \Lambda^2$ by

$$\hat{f}_{ab} := d^2, \quad d: V_{ab}^k \to V_{ab}^{k+1}.$$

Then, Leibniz rule (25) and $F_{ab} = F_b - F_a$ imply that $\hat{f}_{ab} = \hat{f}_b - \hat{f}_a$. For each $a \in \mathrm{Ob}(\mathcal{C})$, let $f_a := \hat{f}_a \cdot \mathbf{1}_a \in V_{aa}$, where $\mathbf{1}_a$ is the identity in $T_{\theta_a}^{2n}$. The following is the main claim of this paper.

**Proposition 3.1:** *(Cyclic DG category of holomorphic vector bundles)*

*(i)* *For a given category $\mathcal{C}_{\theta,E}^{\mathrm{pre}}$ in Definition 3.1, $\mathcal{C} := \mathcal{C}_{\theta,\tau,E}$ forms a cyclic CDG category.*

*(ii)* *Let $\mathcal{C}^{\hat{f}}$ be the full subcategory of $\mathcal{C}$ such that any $a \in \mathrm{Ob}(\mathcal{C}^{\hat{f}}) \subset \mathrm{Ob}(\mathcal{C})$ is the set of CDG modules over a cyclic CDG algebra $(V, \eta, -\hat{f}, d, m)$ for a fixed $\hat{f} \in \Lambda^2 \subset V^2$. Then $\mathcal{C}^{\hat{f}}$ forms a cyclic DG category.*

■

This implies that one can construct a DG category of holomorphic vector bundles over a noncommutative tori.

## IV. THREE EXAMPLES OF NONCOMMUTATIVE DEFORMATIONS

Now, we construct examples of various noncommutative deformations of the DG categories of Heisenberg modules described by CDG modules over a cyclic CDG algebra of a noncommutative torus, where we treat Heisenberg modules corresponding to noncommutative deformations of holomorphic line bundles. The setup given in the previous subsection allows us to deform both the complex structure $\tau$ and the noncommutativity $\theta$ or either of them. In this paper, starting from a commutative ($\theta=0$) $n$-dimensional complex torus with the standard complex structure $\tau = \sqrt{-1}\mathbf{1}_n$ in Sec. IV A, we deform the noncommutative parameter $\theta$ with preserving the standard complex structure in Sec. IV B. Also, we give the composition formula on the zeroth cohomologies of the DG categories explicitly. We show that the structure constants of the compositions, in fact, depend on $\theta$.

### A. Holomorphic line bundles: The commutative case

Let us begin with the commutative torus $T^{2n}:=T^{2n}_{\theta=0}$. The generators $U_1,\ldots,U_n,U_{\bar{1}}:=U_{n+1},\ldots,U_{\bar{n}}:=U_{2n}$ then commute with each other. The arguments in Sec. III B show that it is enough to construct a category $\mathcal{C}^{\mathrm{pre}}_{\theta=0,E}$ in order to construct a cyclic CDG category $\mathcal{C}$.

A category $\mathcal{C}^{\mathrm{pre}}_{\theta=0,E}$ is constructed as follows. Any object $a\in\mathrm{Ob}(\mathcal{C}^{\mathrm{pre}}_{\theta=0,E})$ is associated with a pair $E_a:=(E_{g_a,\theta=0},\nabla_a)$ of a basic module $E_{g_a,\theta=0}$ with a constant curvature connection $\nabla_a$. The basic module is defined by

$$E_{g_a,\theta=0}:=\mathcal{S}(\mathbb{R}^n\times(\mathbb{Z}^n/A_a\mathbb{Z}^n))$$

for a fixed nondegenerate symmetric matrix $A_a\in\mathrm{Mat}_n(\mathbb{Z})$, where $g_a\in\mathrm{SO}(d,d,\mathbb{Z})$ is given by

$$g_a=\begin{pmatrix}\mathbf{1}_{2n} & \mathbf{0}_{2n}\\ F_a & \mathbf{1}_{2n}\end{pmatrix},\quad F_a:=\begin{pmatrix}\mathbf{0}_n & A_a\\ -A_a & \mathbf{0}_n\end{pmatrix}.$$

The right action of $T^{2n}$ on $E_{g_a,\theta=0}$ is defined by specifying the right action of each generator; for $\xi_a\in E_{g_a,\theta=0}$, it is given by

$$(\xi_aU_i)(x;\mu)=\xi_a(x;\mu)e^{2\pi\sqrt{-1}(x_i+(A_a^{-1}\mu)_i)},$$

$$(\xi_aU_{\bar{i}})(x;\mu)=\xi_a(x+A_a^{-1}t_i;\mu-t_i),\quad i=1,\ldots,n,$$

where $x:=(x_1\cdots x_n)^t\in\mathbb{R}^n$, $\mu\in\mathbb{Z}^n/A_a\mathbb{Z}^n$, and $t_i\in\mathbb{R}^n$ is defined by $(t_1\cdots t_n)=\mathbf{1}_n$. A constant curvature connection $\nabla_a:E_{g_a,\theta=0}\otimes\mathcal{L}_\theta\to E_{g_a,\theta=0}$ is given by

$$(\nabla_{a,1}\cdots\nabla_{a,2n})^t=\begin{pmatrix}\mathbf{1}_n & \\ & -A_a\end{pmatrix}\begin{pmatrix}\partial_x\\ 2\pi\sqrt{-1}x\end{pmatrix},\tag{26}$$

where $\partial_x:=(\partial/\partial x_1\cdots\partial/\partial x_n)^t$, and the curvature [defined by Eq. (16)] is $F_a$ above. The generators of the endomorphism algebra are the same as $U_i,U_{\bar{i}}$,

$$(Z_i\xi_a)(x;\mu)=e^{2\pi\sqrt{-1}(x_i+(A_a^{-1}\mu)_i)}\xi_a(x;\mu),$$

$$(Z_{\bar{i}}\xi_a)(x;\mu)=\xi_a(x+A_a^{-1}t_i;\mu-t_i),\quad i=1,\ldots,n.$$

Namely, the endomorphism algebra also forms a commutative torus $T^{2n}$.

This $E_a:=(E_{g_a},\nabla_a)$ is lifted to a CDG module $(\mathcal{E}_a,\hat{f}_a,d_a,m_a)$ over the cyclic CDG algebra $(V=T^{2n}\otimes\Lambda,\eta,-\hat{f}_a,d,m)$ by the procedure in the previous subsection, where the complex structure is taken to be the standard one: $\tau=\sqrt{-1}\mathbf{1}_n$. Then, one obtains $(d_a)^2=\hat{f}_a=0$, that is, $\mathcal{E}_a$, in fact, forms a DG module corresponding to a *holomorphic* vector bundle. In particular, $\mathcal{E}_a$ is regarded as a holomorphic *line* bundle on commutative torus $T^{2n}$, as explained briefly in the beginning of the next subsection (see also Ref. 17).

For any $a,b\in\mathrm{Ob}(\mathcal{C}^{\mathrm{pre}}_{\theta=0,E})$, the space $\mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta=0,E}}(a,b)$ is defined as follows. If $A_{ab}:=A_b-A_a$ is nondegenerate, then it is again the Schwartz space $\mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta=0,E}}(a,b):=\mathcal{S}(\mathbb{R}^n\times(\mathbb{Z}^n/A_{ab}\mathbb{Z}^n))$. For $\xi_{ab}\in\mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta=0,E}}(a,b)$, the right action of $T^{2n}$, generated by $U_i$ and $U_{\bar{i}}$, and the left action of $T^{2n}$, generated by $Z_i$ and $Z_{\bar{i}}$, are defined by

$$(\xi_{ab}U_i)(x;\mu)=\xi_{ab}(x;\mu)e^{2\pi\sqrt{-1}(x_i+(A_{ab}^{-1}\mu)_i)},$$

$$(\xi_{ab}U_{\bar{i}})(x;\mu)=\xi_{ab}(x+A_{ab}^{-1}t_i;\mu-t_i),$$

$$(Z_i\xi_{ab})(x;\mu)=e^{2\pi\sqrt{-1}(x_i+(A_{ab}^{-1}\mu)_i)}\xi_{ab}(x;\mu),$$

$$(Z_{\bar{i}}\xi_{ab})(x;\mu) = \xi_{ab}(x + A_{ab}^{-1}t_i;\mu - t_i)$$

for $i=1,\ldots,n$, where $\mu \in \mathbb{Z}^n/A_{ab}\mathbb{Z}^n$. In fact, all these generators $U_i$, $U_{\bar{i}}$, $Z_i$, and $Z_{\bar{i}}$ commute with each other.

On the other hand, if $A_a=A_b$, we define $\mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta=0,E}}(a,b) := T^{2n}$, on which the left and right actions of $T^{2n}$ are defined just by the commutative product on $T^{2n}$.

In general, the way of constructing the space $\mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta,E}}(a,b)=\mathrm{Hom}(E_{g_a,\theta=0},E_{g_b,\theta=0})$ depends on the rank of $A_{ab}:=A_b-A_a$. In rank $n$ case (nondegenerate case), one has $\mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta=0,E}}(a,b) := \mathcal{S}(\mathbb{R}^n \times (\mathbb{Z}^n/A_{ab}\mathbb{Z}^n))$ and in rank 0 case, one has $\mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta=0,E}}(a,b) := T^{2n}$ as above. In rank $1<r<n$ case, we should combine these two constructions with each other appropriately. In order to avoid such case-by-case arguments, in this paper we assume that $A_{ab}$ is nondegenerate for any $a,b \in \mathrm{Ob}(\mathcal{C}^{\mathrm{pre}}_{\theta=0,E})$ such that $a \neq b$.

The constant curvature connection $\nabla_i : \mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta=0,E}}(a,b) \to \mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta=0,E}}(a,b)$, $i=1,\ldots,2n$, is given by

$$(\nabla_1 \cdots \nabla_{2n})^t := \begin{pmatrix} \mathbf{1}_n & \\ & -A_{ab} \end{pmatrix} \begin{pmatrix} \partial_x \\ 2\pi\sqrt{-1}x \end{pmatrix}$$

if $a \neq b$, and if $a=b$, it is defined by the derivation $\nabla$ on the noncommutative torus $T^{2n}_{\theta_a}=T^{2n}_{\theta_b}$ in Eq. (18) with $\theta_a=\theta_b=0$.

For $a,b,c \in \mathrm{Ob}(\mathcal{C}^{\mathrm{pre}}_{\theta=0,E})$ and $\xi_{ab} \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta=0,E}}(a,b)$, $\xi_{bc} \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta=0,E}}(b,c)$, the product $m: \mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta=0,E}}(b,c) \otimes \mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta=0,E}}(a,b) \to \mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta=0,E}}(a,c)$ is given as follows:

- For $a=b$, it is the right action of $T^{2n}$ on $\mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta=0,E}}(b,c)$.
- For $b=c$, it is the left action of $T^{2n}$ on $\mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta=0,E}}(a,b)$.
- For $a=c$, the product $m: \mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta=0,E}}(b,a) \otimes \mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta=0,E}}(a,b) \to T^{2n}$ is given by

$$m(\xi_{ba},\xi_{ab})(x,\rho) = \sum_{\mathbf{m}\in\mathbb{Z}^n} \sum_{\mu\in\mathbb{Z}^n/A_{ab}\mathbb{Z}^n} U_{\mathbf{m}} \int_{\mathbb{R}^n} \mathrm{d}x^n \xi_{ba}(x,\mu)(\xi_{ab}(x,-\mu)U_{-\mathbf{m}})$$

for $\xi_{ab} \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta=0,E}}(a,b)$ and $\xi_{ba} \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta=0,E}}(b,a)$, where $U_{\mathbf{m}} \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta=0,E}}(a,a)=T^{2n}$. For the remaining general case, it is given by

$$m(\xi_{bc},\xi_{ab})(x,\rho) = \sum_{u\in\mathbb{Z}^n} \xi_{bc}(x + A_{bc}^{-1}(u - A_{ab}A_{ac}^{-1}\rho), -u+\rho)\xi_{ab}(x - A_{ab}^{-1}(u - A_{ab}A_{ac}^{-1}\rho),u). \tag{27}$$

These structures, together with the trace map, defined by Eq. (14) form a category $\mathcal{C}^{\mathrm{pre}}_{\theta=0,E}$ and the corresponding cyclic CDG category $\mathcal{C}_{\theta=0}$. In particular, we have $d^2=0$ for $d: V_{ab} \to V_{ab}$ with any pair $a,b \in \mathrm{Ob}(\mathcal{C}_{\theta=0})$. Thus, $\mathcal{C}_{\theta=0}$ is a cyclic DG category.

For any $a,b \in \mathrm{Ob}(\mathcal{C}_{\theta=0})$, $a \neq b$, the bases of the zeroth cohomology of $V_{ab}$ are given by Gaussians[39] (see also Ref. 6) and called *theta vectors*, though here we are discussing the $\theta=0$ case. We shall give examples of these theta vectors in the noncommutative case $\theta \neq 0$ in the next subsection. The mirror dual $\hat{T}^{2n}$ of this complex torus $T^{2n} := \mathbb{C}^n/(\mathbb{Z}^n \oplus \sqrt{-1}\mathbb{Z}^n)$ is the real $2n$-dimensional torus with a symplectic structure $\omega := \begin{pmatrix} \mathbf{0}_n & -\mathbf{1}_n \\ \mathbf{1}_n & \mathbf{0}_n \end{pmatrix}$. In this mirror dual torus $\hat{T}^{2n}$, a line bundle specified by $A_a$ corresponds to an affine Lagrangian submanifold $L_a$. Then, the intersection of $L_a$ and $L_b$ is a point $\hat{v}_{ab}$ on $\hat{T}^{2n}$, which defines the set $\tilde{V}_{ab}$ of the infinite copies of the points on the covering space $\mathbb{C}^n$. The structure constant $C^{\mu\nu}_{abc,\rho} \in \mathbb{C}$ can be identified with the sum of the exponentials of the symplectic areas of the triangles $\tilde{v}_{ab}\tilde{v}_{bc}\tilde{v}_{ac}$ for any $\tilde{v}_{ab} \in \tilde{V}_{ab}$, $\tilde{v}_{bc} \in \tilde{V}_{bc}$, and $\tilde{v}_{ac} \in \tilde{V}_{ac}$ with respect to $\omega$, where the triangles related by parallel translations on the covering space are identified with each other and not overcounted (see Ref. 34 (two-tori case), Refs. 8 and 17).

### B. Noncommutative deformations of holomorphic line bundles

Let us consider a real $2n$-dimensional noncommutative torus $T_\theta^{2n}$ with its generators $U_1,\ldots,U_{2n}$ with the following relation:

$$U_iU_j = e^{-2\pi\sqrt{-1}\theta^{ij}}U_jU_i, \quad \theta := \begin{pmatrix} \theta_1 & -\theta_2 \\ \theta_2^t & \theta_3 \end{pmatrix}.$$

Since $\theta \in \mathrm{Mat}_{2n}(\mathbb{R})$ is antisymmetric, $\theta_1, \theta_3 \in \mathrm{Mat}_n(\mathbb{R})$ are antisymmetric and $\theta_2 \in \mathrm{Mat}_n(\mathbb{R})$ can be an arbitrary $n\times n$ matrix.

A Heisenberg module $E_{g,\theta}$, $g \in \mathrm{SO}(2n,2n,\mathbb{Z})$, on this noncommutative torus $T_\theta^{2n}$ is associated with two notions, the $K_0$ group element and the Chern character (see Ref. 25). The Chern character of $E_{g,\theta}$ is defined by its constant curvature $F$, a skew symmetric $2n\times 2n$ matrix with entries in $\mathbb{R}$. On the other hand, the $K_0$ group element of $E_{g,\theta}$ is defined by $F_0$, the constant curvature of $E_{g,\theta}$ when we set $\theta=0$. Thus, the $K_0$ group element is independent of the noncommutativity $\theta$. A Heisenberg module $E_{g,\theta}$ is thought of as a noncommutative analog of a line bundle if $g \in \mathrm{SO}(2n,2n,\mathbb{Z})$ is of the form

$$g = \begin{pmatrix} \mathbf{1}_{2n} & \mathbf{0}_{2n} \\ F_0 & \mathbf{1}_{2n} \end{pmatrix}$$

for a skew symmetric matrix $F_0 \in \mathrm{Mat}_{2n}(\mathbb{Z})$. In fact, $F_0$ corresponds to the first Chern character of a line bundle if $\theta=0$. Since we shall discuss noncommutative deformations of the line bundles, discussed in the previous subsection, let us consider, in particular, the case that $F_0$ is of the following form:

$$F_0 = \begin{pmatrix} \mathbf{0}_n & A \\ -A & \mathbf{0}_n \end{pmatrix},$$

where $A \in \mathrm{Mat}_n(\mathbb{Z})$ is a nondegenerate symmetric matrix. For the Heisenberg modules $E_{g,\theta}$ with $g$ given as above, we shall consider noncommutative tori in the following three cases: type $\theta_1$: $\theta_2=\theta_3=0$; type $\theta_2$: $\theta_1=\theta_3=0$; type $\theta_3$: $\theta_1=\theta_2=0$. In each case, the endomorphism algebra, $T_{\theta'}$, $\theta' := (\mathbf{1}_n\theta+\mathbf{0}_n)(F_0\theta+\mathbf{1}_n)^{-1}$, turns out to be as follows. In the type $\theta_1$ case and $\theta_3$ case,

$$\theta = \begin{pmatrix} \theta_1 & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{0}_n \end{pmatrix}, \quad \theta = \begin{pmatrix} \mathbf{0}_n & \mathbf{0}_n \\ \mathbf{0}_n & \theta_3 \end{pmatrix},$$

we have $\theta'=\theta$. However, in the type $\theta_2$ case, one obtains

$$\theta' = \begin{pmatrix} \mathbf{0}_n & -\theta_2(\mathbf{1}_n + A\theta_2)^{-1} \\ \theta_2^t(\mathbf{1}_n + A\theta_2^t)^{-1} & \mathbf{0}_n \end{pmatrix}, \quad \theta := \begin{pmatrix} \mathbf{0}_n & -\theta_2 \\ \theta_2^t & \mathbf{0}_n \end{pmatrix}. \tag{28}$$

Now, for the cyclic DG category $\mathcal{C}_{\theta=0}$ in the previous subsection, we construct its noncommutative deformations of each of the three types above explicitly as CDG categories. Namely, we construct noncommutative deformations of $\mathcal{C}_{\theta=0,E}^{\mathrm{pre}}$ in the previous subsection.

*Type* $\theta_1$. A category $\mathcal{C}_{\theta,E}^{\mathrm{pre}}$ is constructed as follows. Any object $a \in \mathrm{Ob}(\mathcal{C}_{\theta,E}^{\mathrm{pre}})$ is associated with a pair $E_a := (E_{g_a,\theta}, \nabla_a)$. The basic module $E_{g_a,\theta}$ is defined by

$$E_{g_a,\theta} = \mathcal{S}(\mathbb{R}^n \times (\mathbb{Z}^n/A_a\mathbb{Z}^n))$$

for a nondegenerate symmetric matrix $A_a \in \mathrm{Mat}_n(\mathbb{Z})$, where

$$g_a = \begin{pmatrix} \mathbf{1}_{2n} & \mathbf{0}_{2n} \\ F_{a,0} & \mathbf{1}_{2n} \end{pmatrix}, \quad F_{a,0} := \begin{pmatrix} \mathbf{0}_n & A_a \\ -A_a & \mathbf{0}_n \end{pmatrix}.$$

For $\xi_a \in E_{g_a,\theta}$, the action of each generator is defined by

$$(\xi_a U_i)(x;\mu) = \xi_a(x;\mu) * e^{2\pi\sqrt{-1}(x_i + (A_a^{-1}\mu)_i))},$$

$$(\xi_a U_{\bar{i}})(x;\mu) = \xi_a(x + A_a^{-1}t_i;\mu - t_i), \quad i = 1,\ldots,n.$$

Here $*: C^\infty(\mathbb{R}^n)\otimes C^\infty(\mathbb{R}^n)\to C^\infty(\mathbb{R}^n)$ is the Moyal star product[32] defined by

$$(f*g)(x) := f(x)e^{-(\sqrt{-1}/4\pi)\overleftarrow{\partial}_x\theta_1\vec{\partial}_x}g(x),$$

where $\overleftarrow{\partial}_x\theta_1\vec{\partial}_x := \Sigma_{p,q=1}^n\overleftarrow{\partial}_{x_p}\theta_1^{pq}\vec{\partial}_{x_q}$. The generator of the endomorphism is then given by

$$(Z_i\xi_a)(x;\mu) = e^{2\pi\sqrt{-1}(x_i + (A_a^{-1}\mu)_i))} * \xi_a(x;\mu),$$

$$(Z_{\bar{i}}\xi_a)(x;\mu) = \xi_a(x + A_a^{-1}t_i;\mu - t_i).$$

A constant curvature connection $\nabla_a: E_{g_a,\theta}\otimes\mathcal{L}_\theta\to E_{g_a,\theta}$ is given as

$$(\nabla_{a,1}\cdots\nabla_{a,n})^t = \begin{pmatrix} \mathbf{1}_n & \\ -(1/2)A_a\theta_1 & -A_a\end{pmatrix}\begin{pmatrix}\partial_x \\ 2\pi\sqrt{-1}x\end{pmatrix},$$

whose the constant curvature is

$$F_a = \begin{pmatrix}\mathbf{0}_n & A_a \\ -A_a & A_a\theta_1 A_a\end{pmatrix}.$$

We assume that $A_{ab}$ is nondegenerate for any $a,b\in\mathrm{Ob}(\mathcal{C}_{\theta,E}^{\mathrm{pre}})$, $a\neq b$. For any $a,b\in\mathrm{Ob}(\mathcal{C}_{\theta,E}^{\mathrm{pre}})$, the space $\mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,b)$ is defined as follows. If $a\neq b$, $\mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,b) := \mathrm{Hom}(E_{g_a,\theta},E_{g_b,\theta}) = \mathcal{S}(\mathbb{R}^n\times(\mathbb{Z}^n/A_{ab}\mathbb{Z}^n))$; the right action of $T_{\theta_a}$, generated by $U_i$ and $U_{\bar{i}}$, and the left action of $T_{\theta_b}$, generated by $Z_i$ and $Z_{\bar{i}}$, are defined by

$$(\xi_{ab}U_i)(x;\mu) = \xi_{ab}(x;\mu) * e^{2\pi\sqrt{-1}\,(x_i + (A_{ab}^{-1}\mu)_i))},$$

$$(\xi_{ab}U_{\bar{i}})(x;\mu) = \xi_{ab}(x + A_{ab}^{-1}t_i;\mu - t_i),$$

$$(Z_i\xi_{ab})(x;\mu) = e^{2\pi\sqrt{-1}\,(x_i + (A_{ab}^{-1}\mu)_i))} * \xi_{ab}(x;\mu),$$

$$(Z_{\bar{i}}\xi_{ab})(x;\mu) = \xi_{ab}(x + A_{ab}^{-1}t_i;\mu - t_i).$$

If $a=b$, then $\mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,b) = T_{\theta_a} = T_{\theta_b}$ and these actions are defined by the usual product of the noncommutative torus $T_{\theta_a} = T_{\theta_b}$. The constant curvature connection $\nabla: \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,b)\otimes\mathcal{L}_\theta \to \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,b)$ is given by

$$(\nabla_1\cdots\nabla_{2n})^t := \begin{pmatrix}\mathbf{1}_n & \\ -(1/2)A_{ab}^+\theta_1 & -A_{ab}\end{pmatrix}\begin{pmatrix}\partial_x \\ 2\pi\sqrt{-1}x\end{pmatrix}, \quad A_{ab}^+ := A_a + A_b$$

if $a\neq b$, and if $a=b$, it is defined by the derivation $\nabla$ of the noncommutative torus $T_{\theta_a} = T_{\theta_b}$ in Eq. (18).

For any $a,b,c\in\mathrm{Ob}(\mathcal{C}_{\theta,E}^{\mathrm{pre}})$ and $\xi_{ab}\in\mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,b)$, $\xi_{bc}\in\mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(b,c)$, the product $m: \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(b,c)\otimes\mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,b)\to\mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,c)$ is given as follows:

- For $a=b$, it is the right action of $T_{\theta_a} = T_{\theta_b}$ on $\mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(b,c)$.
- For $b=c$, it is the left action of $T_{\theta_b} = T_{\theta_c}$ on $\mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,b)$.
- For $a=c$, the product $m: \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(b,a)\otimes\mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,b)\to T_{\theta_a}$ is given by

$$m(\xi_{ba},\xi_{ab})(x,\rho) = \sum_{\mathbf{m}\in\mathbb{Z}^n}\sum_{\mu\in\mathbb{Z}^n/A_{ab}\mathbb{Z}^n} U_{\mathbf{m}}\int_{\mathbb{R}^n} \mathrm{d}x^n \xi_{ba}(x,\mu) * (\xi_{ab}(x,-\mu)U_{-\mathbf{m}}).$$

For the remaining general case, it is given by

$$m(\xi_{bc},\xi_{ab})(x,\rho) = \sum_{u\in\mathbb{Z}^n} \xi_{bc}(x + A_{bc}^{-1}(u - A_{ab}A_{ac}^{-1}\rho), -u+\rho) * \xi_{ab}(x - A_{ab}^{-1}(u - A_{ab}A_{ac}^{-1}\rho), u).$$

The trace map is then given by Eq. (17).

*Type* $\theta_2$. A category $\mathcal{C}_{\theta,E}^{\mathrm{pre}}$ is constructed as follows. An object $a\in \mathrm{Ob}(\mathcal{C}_{\theta,E}^{\mathrm{pre}})$ is associated with a pair $E_a := (E_{g_a,\theta},\nabla_a)$, where we assume that $\det(\mathbf{1}_n + \theta_2 A_a)\neq 0$, which is always satisfied if one of the entries of $\theta_2$ is irrational. The basic module $E_{g_a,\theta}$ is defined by

$$E_{g_a,\theta} = \mathcal{S}(\mathbb{R}^n \times (\mathbb{Z}^n/A_a\mathbb{Z}^n))$$

for a nondegenerate symmetric matrix $A_a \in \mathrm{Mat}_n(\mathbb{Z})$, where

$$g_a = \begin{pmatrix} \mathbf{1}_{2n} & \mathbf{0}_{2n} \\ F_{a,0} & \mathbf{1}_{2n}\end{pmatrix}, \quad F_{a,0} = \begin{pmatrix} \mathbf{0}_n & A_a \\ -A_a & \mathbf{0}_n\end{pmatrix}.$$

For $\xi_a \in E_{g_a,\theta}$, the action of each generator is defined by

$$(\xi_a U_i)(x;\mu) = \xi_a(x;\mu)e^{2\pi\sqrt{-1}\ \ x_i + (A_a^{-1}\mu)_i)},$$

$$(\xi_a U_{\bar{i}})(x;\mu) = \xi_a(x + (\mathbf{1}_n + \theta_2 A_a)A_a^{-1}t_i;\mu - t_i).$$

The action of the generators of the endomorphism is then given by

$$(Z_i\xi_a)(x;\mu) = e^{2\pi\sqrt{-1}\ (((\mathbf{1}_n + \theta_2 A_a)^{-1}x)_i + (A_a^{-1}\mu)_i))}\xi_a(x;\mu),$$

$$(Z_{\bar{i}}\xi_a)(x;\mu) = \xi_a(x + A_a^{-1}t_i;\mu - t_i),$$

where the relation is

$$Z_iZ_j = e^{-2\pi\sqrt{-1}\theta_a}Z_jZ_i, \quad \theta_a = \begin{pmatrix} \mathbf{0}_n & -(\mathbf{1}_n + \theta_2 A_a)^{-1}\theta_2 \\ \theta_2^t(\mathbf{1}_n + A_a\theta_2^t)^{-1} & \mathbf{0}_n\end{pmatrix}.$$

A constant curvature connection $\nabla_a : E_{g_a,\theta}\otimes\mathcal{L}_\theta \to E_{g_a,\theta}$ is given as

$$(\nabla_{a,1}\cdots\nabla_{a,2n})^t = \begin{pmatrix} \mathbf{1}_n & \mathbf{0}_n \\ \mathbf{0}_n & -(A_a^{-1}+\theta_2)^{-1}\end{pmatrix}\begin{pmatrix}\partial_x \\ 2\pi\sqrt{-1}x\end{pmatrix},$$

with its curvature

$$F_a = \begin{pmatrix} \mathbf{0}_n & (A_a^{-1}+\theta_2^t)^{-1} \\ -(A_a^{-1}+\theta_2)^{-1} & \mathbf{0}_n\end{pmatrix}.$$

We assume that $A_{ab}$ is nondegenerate for any $a,b\in\mathrm{Ob}(\mathcal{C}_{\theta,E}^{\mathrm{pre}})$, $a\neq b$.

For any $a,b\in\mathrm{Ob}(\mathcal{C}_{\theta,E}^{\mathrm{pre}})$, the space $\mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,b)$ is defined as follows. If $a\neq b$, $\mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,b) := \mathrm{Hom}(E_{g_a,\theta},E_{g_b,\theta}) = \mathcal{S}(\mathbb{R}^n\times(\mathbb{Z}^n/A_{ab}\mathbb{Z}^n))$; the right action of $T_{\theta_a}$, generated by $U_i$ and $U_{\bar{i}}$, and the left action of $T_{\theta_b}$, generated by $Z_i$ and $Z_{\bar{i}}$, are defined by

$$(\xi_{ab}U_i)(x;\mu) = \xi_{ab}(x;\mu)e^{2\pi\sqrt{-1}\ \ ((\mathbf{1}_n + \theta_2 A_a)^{-1}x)_i + (A_{ab}^{-1}\mu)_i))},$$

$$(\xi_{ab}U_{\bar{i}})(x;\mu) = \xi_{ab}(x + (\mathbf{1}_n + \theta_2 A_b)A_{ab}^{-1}t_i;\mu - t_i),$$

$$(Z_i\xi_{ab})(x;\mu) = e^{2\pi\sqrt{-1}\ (((\mathbf{1}_n + \theta_2 A_b)^{-1}x)_i + (A_{ab}^{-1}\mu)_i))}\xi_{ab}(x;\mu),$$

$$(Z_{\bar{i}}\xi_{ab})(x;\mu) = \xi_{ab}(x + (\mathbf{1}_n + \theta_2 A_a)A_{ab}^{-1}t_i;\mu - t_i).$$

If $a=b$, then $\mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,b) \coloneqq T_{\theta_a}=T_{\theta_b}$ and these actions are defined by the usual product of noncommutative torus $T_{\theta_a}=T_{\theta_b}$. Note that $\mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,b)=\mathrm{Hom}(E_{g_a,\theta},E_{g_b,\theta})$ is isomorphic to $E_{g_b g_a^{-1},\theta}$; for $a\neq b$, an element $\xi_{ab}\in \mathrm{Hom}(E_{g_a,\theta},E_{g_b,\theta})$ is identified with $\xi'_{ab}\in E_{g_b g_a^{-1},\theta}$ by the following relation:

$$\xi'_{ab}(x',\mu) = \xi'_{ab}((\mathbf{1}_n + \theta_2 A_a)^{-1}x,\mu) = \xi_{ab}(x,\mu).$$

A constant curvature connection $\nabla:\mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,b)\otimes \mathcal{L}_\theta \to \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,b)$ is given by

$$(\nabla_1 \cdots \nabla_{2n})^t = \begin{pmatrix} \mathbf{1}_n & \mathbf{0}_n \\ \mathbf{0}_n & -(A_b^{-1}+\theta_2)^{-1} + (A_a^{-1}+\theta_2) \end{pmatrix}\begin{pmatrix} \partial_x \\ 2\pi\sqrt{-1}x \end{pmatrix}$$

if $a\neq b$, and if $a=b$, it is defined by the derivation $\nabla$ of the noncommutative torus $T_{\theta_a}=T_{\theta_b}$ in Eq. (18).

For any $a,b,c\in \mathrm{Ob}(\mathcal{C}_{\theta,E}^{\mathrm{pre}})$ and $\xi_{ab}\in \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,b)$, $\xi_{bc}\in \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(b,c)$, the product $m:\mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(b,c)\otimes \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,b)\to \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,c)$ is given as follows:

- For $a=b$, it is the right action of $T_{\theta_a}=T_{\theta_b}$ on $\mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(b,c)$.
- For $b=c$, it is the left action of $T_{\theta_b}=T_{\theta_c}$ on $\mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,b)$.
- For $a=c$, the product $m:\mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(b,a)\otimes \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,b)\to T_{\theta_a}$ is given by

$$m(\xi_{ba},\xi_{ab})(x,\rho) = \sum_{\mathbf{m}\in\mathbb{Z}^n}\ \sum_{\mu\in\mathbb{Z}^n/A_{ab}\mathbb{Z}^n} U_{\mathbf{m}}\int_{\mathbb{R}^n}\mathrm{d}x^n\,\xi_{ba}(x,\mu)(\xi_{ab}(x,-\mu)U_{-\mathbf{m}}).$$

For the remaining general case, it is given by

$$m(\xi_{bc},\xi_{ab})(x,\rho) = \sum_{u\in\mathbb{Z}^n}\xi_{bc}(x + (\mathbf{1}_n + \theta_2 A_c)A_{bc}^{-1}(u - A_{ab}A_{ac}^{-1}\rho), -u+\rho)$$
$$\times\,\xi_{ab}(x - (\mathbf{1}_n + \theta_2 A_a)A_{ab}^{-1}(u - A_{ab}A_{ac}^{-1}\rho),u).$$

The trace map is then given by Eq. (17).

*Type* $\theta_3$. A category $\mathcal{C}_{\theta,E}^{\mathrm{pre}}$ is constructed as follows. Any object $a\in \mathrm{Ob}(\mathcal{C}_{\theta,E}^{\mathrm{pre}})$ is associated with a pair $E_a \coloneqq (E_{g_a,\theta},\nabla_a)$. The basic module $E_{g_a,\theta}$ is defined by

$$E_{g_a,\theta} = \mathcal{S}(\mathbb{R}^n \times (\mathbb{Z}^n/A_a\mathbb{Z}^n))$$

for a nondegenerate symmetric matrix $A_a\in \mathrm{Mat}_n(\mathbb{Z})$, where

$$g_a = \begin{pmatrix} \mathbf{1}_{2n} & \mathbf{0}_{2n} \\ F_{a,0} & \mathbf{1}_{2n} \end{pmatrix},\quad F_{a,0} = \begin{pmatrix} \mathbf{0}_n & A_a \\ -A_a & \mathbf{0}_n \end{pmatrix}.$$

For $\xi_a\in E_{g_a,\theta}$, the action of each generator is defined by

$$(\xi_a U_i)(x;\mu) = \xi_a(x;\mu)e^{2\pi\sqrt{-1}\ (x_i + (A_a^{-1}\mu)_i))},$$

$$(\xi_a U_{\bar{i}})(x;\mu) = \xi_a(x + A_a^{-1}t_i;\mu - t_i)e^{\pi\sqrt{-1}x^t A_a \theta_3 t_i}, \quad i = 1, \ldots, n,$$

and the endomorphisms are generated by

$$(Z_i \xi_a)(x;\mu) = e^{2\pi\sqrt{-1}\ (x_i + (A_a^{-1}\mu)_i))}\xi_a(x;\mu),$$

$$(Z_{\bar{i}}\xi_a)(x;\mu) = e^{-\pi\sqrt{-1}x^t A_a \theta_3 t_i}\xi_a(x + A_a^{-1}t_i;\mu - t_i).$$

A constant curvature connection $\nabla_a : E_{g_a,\theta} \otimes \mathcal{L}_\theta \to E_{g_a,\theta}$ is given as

$$(\nabla_{a,1} \cdots \nabla_{a,2n})^t = \begin{pmatrix} \mathbf{1}_n & (1/2)A_a\theta_3 A_a \\ \mathbf{0}_n & -A_a \end{pmatrix}\begin{pmatrix} \partial_x \\ 2\pi\sqrt{-1}x \end{pmatrix},$$

with its curvature

$$F_a = \begin{pmatrix} A_a\theta_3 A_a & A_a \\ -A_a & \mathbf{0}_n \end{pmatrix}.$$

We assume that $A_{ab}$ is nondegenerate for any $a,b \in \mathrm{Ob}(\mathcal{C}_{\theta,E}^{\mathrm{pre}})$, $a \neq b$.

For any $a,b \in \mathrm{Ob}(\mathcal{C}_{\theta,E}^{\mathrm{pre}})$, the space $\mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,b)$ is defined as follows. If $a \neq b$, $\mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,b) \coloneqq \mathrm{Hom}(E_{g_a,\theta},E_{g_b,\theta}) = \mathcal{S}(\mathbb{R}^n \times (\mathbb{Z}^n/A_{ab}\mathbb{Z}^n))$; the right action of $T_{\theta_a}$, generated by $U_i$ and $U_{\bar{i}}$, and the left action of $T_{\theta_b}$, generated by $Z_i$ and $Z_{\bar{i}}$, are defined by

$$(\xi_{ab}U_i)(x;\mu) = \xi_{ab}(x;\mu)e^{2\pi\sqrt{-1}\ (x_i + (A_{ab}^{-1}\mu)_i))},$$

$$(\xi_{ab}U_{\bar{i}})(x;\mu) = \xi_{ab}(x + A_{ab}^{-1}t_i;\mu - t_i)e^{\pi\sqrt{-1}x^t A_{ab}\theta_3 t_i},$$

$$(Z_i\xi_{ab})(x;\mu) = e^{2\pi\sqrt{-1}\ (x_i + (A_{ab}^{-1}\mu)_i))}\xi_{ab}(x;\mu),$$

$$(Z_{\bar{i}}\xi_{ab})(x;\mu) = e^{-\pi\sqrt{-1}x^t A_{ab}\theta_3 t_i}\xi_{ab}(x + A_{ab}^{-1}t_i;\mu - t_i).$$

If $a=b$, then $\mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,b) \coloneqq T_{\theta_a} = T_{\theta_b}$ and the left and right actions on it are defined by the usual product of noncommutative torus $T_{\theta_a} = T_{\theta_b}$.

A constant curvature connection $\nabla_a : \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,b) \otimes \mathcal{L}_\theta \to \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,b)$ is given by

$$(\nabla_1 \cdots \nabla_{2n})^t = \begin{pmatrix} \mathbf{1}_n & (1/2)A_{ab}^{+}\theta_3 A_{ab} \\ \mathbf{0}_n & -A_{ab} \end{pmatrix}\begin{pmatrix} \partial_x \\ 2\pi\sqrt{-1}x \end{pmatrix}$$

if $a \neq b$, and if $a=b$, it is defined by the usual derivation $\nabla$ of noncommutative torus $T_{\theta_a} = T_{\theta_b}$ [Eq. (18)].

For any $a,b,c \in \mathrm{Ob}(\mathcal{C}_{\theta,E}^{\mathrm{pre}})$ and $\xi_{ab} \in \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,b)$, $\xi_{bc} \in \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(b,c)$, the product $m : \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(b,c) \otimes \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,b) \to \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,c)$ is given as follows:

- For $a=b$, it is the right action of $T_{\theta_a} = T_{\theta_b}$ on $\mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(b,c)$.
- For $b=c$, it is the left action of $T_{\theta_b} = T_{\theta_c}$ on $\mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,b)$.
- For $a=c$, the product $m : \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(b,a) \otimes \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,b) \to T_{\theta_a}$ is given by

$$m(\xi_{ba},\xi_{ab})(x,\rho) = \sum_{\mathbf{m}\in\mathbb{Z}^n}\ \sum_{\mu\in\mathbb{Z}^n/A_{ab}\mathbb{Z}^n} U_{\mathbf{m}}\int_{\mathbb{R}^n} \mathrm{d}x^n \xi_{ba}(x,\mu)(\xi_{ab}(x,-\mu)U_{-\mathbf{m}}).$$

For the remaining general case, it is given by

$$m(\xi_{bc},\xi_{ab})(x,\rho)=\sum_{u\in\mathbb{Z}^n}\xi_{bc}(x',-u+\rho)\exp(\pi\sqrt{-1}x'^tA_{bc}\theta_3A_{ab}x'')\xi_{ab}(x'',u),$$

where $x':=x+A_{bc}^{-1}(u-A_{ab}A_{ac}^{-1}\rho)$ and $x'':=x-A_{ab}^{-1}(u-A_{ab}A_{ac}^{-1}\rho)$. The trace map is then given by Eq. (17). By direct calculations, one obtains the following.

**Lemma 4.1:** *For a fixed noncommutative parameter of type $\theta_1$, $\theta_2$, or $\theta_3$, the composition $m$ of morphisms is associative (associativity) and the constant curvature connections on morphisms satisfy the Leibniz rule (Leibniz rule).* ■

Then, in any case of type $\theta_s$, $s=1,2,3$, $\mathcal{C}_{\theta,E}^{\mathrm{pre}}$ forms a category in Definition 3.1 and then the corresponding category $\mathcal{C}$ forms a cyclic CDG category.

In particular, by looking at the condition that $d:V_{ab}\to V_{ab}$ satisfies $d^2=0$ explicitly, one can see the following:

**Proposition 4.1**: *For type $\theta_1$, type $\theta_2$ such that $\theta_2^t=-\theta_2$, and type $\theta_3$, two objects $a,b\in\mathrm{Ob}(\mathcal{C})$ in the CDG category $\mathcal{C}$ together form a full sub-DG category $\mathcal{C}^{\hat{f}}$, defined in Proposition 3.1 (ii), of $\mathcal{C}$ for some $\hat{f}\in\Lambda^2\subset V^2$ if and only if*

$$A_a\theta_sA_a=A_b\theta_sA_b,\quad s=1,2,3, \tag{29}$$

*holds*. ■

Thus, we have obtained the cyclic DG categories $\mathcal{C}^{\hat{f}}$ on noncommutative tori with three types of noncommutativities $\theta_1$, $\theta_2$, and $\theta_3$.

Let us calculate the compositions of the morphisms of the zeroth cohomologies $H^0(\mathcal{V}):=\oplus_{a,b\in\mathrm{Ob}(\mathcal{C}^{\hat{f}})}H^0(V_{ab})$ of these DG categories $\mathcal{C}^{\hat{f}}$. They define ring structures on $H^0(\mathcal{V})$, which are subrings of the full cohomologies $H^*(\mathcal{V})$. We shall observe that the ring $H^0(\mathcal{V})$ actually depends on the noncommutative parameter $\theta_s$, as opposed to the complex one-tori case.[14,22,33,16] We remark that, from a homotopy algebraic viewpoint, a DG category is homotopy equivalent to a *minimal* $A_\infty$ category. Then, by forgetting the higher compositions of the minimal $A_\infty$ category, one obtains the ring $H^*(\mathcal{V})$. Thus, if at least the subring $H^0(\mathcal{V})$ is deformed, the minimal $A_\infty$ structure is also deformed.

Recall that $H^0(V_{ab})$ is given by $\mathrm{Ker}(d:V_{ab}^0\to V_{ab}^1)=\cap_{i=1}^n\mathrm{Ker}(\bar{\nabla}_i:V_{ab}^0\to V_{ab}^0)$.

*Type* $\theta_1$. For $a,b\in\mathrm{Ob}(\mathcal{C}_{\theta,E}^{\mathrm{pre}})$, $a\neq b$, the holomorphic structure $\bar{\nabla}_i:\mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,b)\to\mathrm{Hom}_{\mathcal{C}_{\theta,E}^{\mathrm{pre}}}(a,b)$ is given by

$$(\bar{\nabla}_1\cdots\bar{\nabla}_n)^t=\left(\mathbf{1}_n-\frac{\sqrt{-1}}{2}A_{ab}^+\theta_1\right)\partial_x+2\pi A_{ab}x.$$

The cohomology $H^0(V_{ab})$ is spanned by the basis elements of the form

$$e_{ab}^{\mu}(x;\rho):=C_{ab}\delta_{[A_{ab}]_\rho^\mu}\exp(-\pi(x^tM_{ab}x)),\quad \mu\in\mathbb{Z}^n/A_{ab}\mathbb{Z}^n, \tag{30}$$

where $C_{ab}\in\mathbb{C}$ is an appropriate rescaling and $M_{ab}\in\mathrm{Mat}_n(\mathbb{C})$ should be a symmetric matrix, satisfying

$$(\bar{\nabla}_1\cdots\bar{\nabla}_n)(\exp(-\pi(x^tM_{ab}x)))=0. \tag{31}$$

Condition (31) turns out to be

$$-\left(\mathbf{1}_n-\frac{\sqrt{-1}}{2}A_{ab}^+\theta_1\right)M_{ab}+A_{ab}=0.$$

This $M_{ab}$ is symmetric if and only if condition (29) holds,

$$A_a\theta_1A_a=A_b\theta_1A_b,$$

and then the explicit form of $M_{ab}$ is given by

$$M_{ab} = A_{ab}\left(A_{ab} + \frac{\sqrt{-1}}{2}(A_b\theta_1 A_a - A_a\theta_1 A_b)\right)^{-1} A_{ab}.$$

Here the real part of $M_{ab} \in \mathrm{Mat}_n(\mathbb{C})$ should be positive definite in order for $e^{\mu}_{ab}$ to exist in $H^0(V_{ab})$. Note that the real part of $M_{ab}$ is positive definite if and only if $A_{ab}$ is positive definite. This $e^{\mu}_{ab} \in H^0(V_{ab})$ in Eq. (30) is a theta vector. Thus, for $a,b \in \mathrm{Ob}(\mathcal{C})$ such that $A_a\theta_1 A_a = A_b\theta_1 A_b$ and $A_{ab}$ is positive definite, $\dim(H^0(V_{ab})) = \sharp(\mathbb{Z}^n/A_{ab}\mathbb{Z}^n) = \det(A_{ab})$. For the rescaling $C_{ab}$ in Eq. (30), we set

$$C_{ab} := \frac{\det(\mathbf{1}_n - \sqrt{-1}A_a\theta_1)^{1/4}\det(\mathbf{1}_n - \sqrt{-1}A_b\theta_1)^{1/4}}{\det(\mathbf{1}_n - (\sqrt{-1}/2)A^{+}_{ab}\theta_1)^{1/2}}.$$

Now, for $a,b,c \in \mathrm{Ob}(\mathcal{C})$ such that $A_a\theta_1 A_a = A_b\theta_1 A_b = A_c\theta_1 A_c$, assume that $A_{ab}$ and $A_{bc}$ are positive definite. Then we get the product formula

$$m(e^{\mu}_{ab}, e^{\nu}_{bc}) = \sum_{\rho \in \mathbb{Z}^n/A_{ac}\mathbb{Z}^n}\sum_{u \in \mathbb{Z}^n} \delta_{[A_{ab}]^{\mu}_{-u+\rho}}\delta_{[A_{bc}]^{\nu}_{u}}\exp(-\pi(u - A_{bc}A^{-1}_{ac}\rho)^t((A^{-1}_{ab} + A^{-1}_{bc})(\mathbf{1}_n - \sqrt{-1}A_b\theta_1)^{-1})(u - A_{bc}A^{-1}_{ac}\rho)) \cdot e^{\rho}_{ac}.$$

Note that the $n\times n$ matrix $(A^{-1}_{ab} + A^{-1}_{bc})(\mathbf{1}_n - \sqrt{-1}A_b\theta_1)^{-1}$ is automatically symmetric due to the condition $A_a\theta_1 A_a = A_b\theta_1 A_b = A_c\theta_1 A_c$.

In this type $\theta_1$ case, these theta vectors $\{e^{\mu}_{ab}\}$ can be described by theta functions, where the product of two theta vectors just corresponds to the Moyal product of two theta functions.[17]

*Type* $\theta_2$. For $a,b \in \mathrm{Ob}(\mathcal{C}^{\mathrm{pre}}_{\theta,E})$, $a \neq b$, the holomorphic structure $\bar{\nabla}_i : \mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta,E}}(a,b) \to \mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta,E}}(a,b)$ is given by

$$(\bar{\nabla}_1 \cdots \bar{\nabla}_n)^t = \partial_x + 2\pi((A^{-1}_b + \theta_2)^{-1} - (A^{-1}_a + \theta_2)^{-1})x.$$

The theta vectors are of the form

$$e^{\mu}_{ab}(x,\rho) = \delta_{[A_{ab}]^{\mu}_{\rho}}\exp(-\pi x^t M_{ab} x), \quad \mu \in \mathbb{Z}^n/A_{ab}\mathbb{Z}^n,$$

where $M_{ab} \in \mathrm{Mat}_n(\mathbb{R}) \subset \mathrm{Mat}_n(\mathbb{C})$ is given by

$$M_{ab} := (A^{-1}_b + \theta_2)^{-1} - (A^{-1}_a + \theta_2)^{-1} = (\mathbf{1}_n + A_b\theta_2)^{-1}A_{ab}(\mathbf{1}_n + \theta_2 A_a)^{-1}, \tag{32}$$

which should be a symmetric positive definite matrix. Then, one has $\dim(H^0(V_{ab})) = \det(A_{ab})$.

Now, for $a,b,c \in \mathrm{Ob}(\mathcal{C})$ such that $M_{ab}$ and $M_{bc}$ are symmetric, assume that $M_{ab}$ and $M_{bc}$ are positive definite. Then, $M_{ac}$ is also a symmetric positive definite matrix. The product of $e^{\mu}_{ab}$ with $e^{\nu}_{bc}$ is then

$$m(e^{\mu}_{ab}, e^{\nu}_{bc}) = \sum_{\rho \in \mathbb{Z}^n/A_{ac}\mathbb{Z}^n}\sum_{u \in \mathbb{Z}^n} \delta_{[A_{ab}]^{\mu}_{-u+\rho}}\delta_{[A_{bc}]^{\nu}_{u}}\exp(-\pi(u - A_{bc}A_{ac}\rho)^t((A^{-1}_{ab} + A^{-1}_{bc}) \times (\mathbf{1}_n + A_b\theta^t_2)(\mathbf{1}_n + A_b\theta_2)^{-1})(u - A_{bc}A_{ac}\rho)) \cdot e^{\rho}_{ac},$$

where the $n\times n$ matrix $(A^{-1}_{ab} + A^{-1}_{bc})(\mathbf{1}_n + A_b\theta^t_2)(\mathbf{1}_n + A_b\theta_2)^{-1} \in \mathrm{Mat}_n(\mathbb{C})$ is automatically symmetric. In particular, one can see that the structure constants do not depend on $\theta_2$ if and only if $\theta_2$ is symmetric: $\theta^t_2 = \theta_2$. This gives the reason that the structure constant of the product does not depend on the noncommutative parameter in the case of noncommutative real two tori.[14,33,16] See also Refs. 24 and 23, where for a complex two tori with noncommutativity of type $\theta_2$ with symmetric $\theta_2$, such structure constants are computed and checked to be independent of the noncommutativity $\theta_2$.

On the other hand, if $\theta_2$ is antisymmetric, $M_{ab}$ in Eq. (32) is symmetric if and only if $A_a\theta_2 A_a = A_b\theta_2 A_b$.

*Type* $\theta_3$. For $a,b \in \mathrm{Ob}(\mathcal{C}^{\mathrm{pre}}_{\theta,E})$, $a \neq b$, the holomorphic structure $\bar{\nabla}_i : \mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta,E}}(a,b) \to \mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\theta,E}}(a,b)$ is given by

$$(\bar{\nabla}_1 \cdots \bar{\nabla}_n)^t = \partial_x + 2\pi\left(\mathbf{1}_n + \frac{\sqrt{-1}}{2}A^{+}_{ab}\theta_3\right)A_{ab}x.$$

The theta vectors are of the form

$$e^{\mu}_{ab}(x,\rho) = \delta_{[A_{ab}]^{\mu}_{\rho}} \exp(-\pi x^t M_{ab} x), \quad \mu \in \mathbb{Z}^n / A_{ab}\mathbb{Z}^n,$$

where $M_{ab} \in \mathrm{Mat}_n(\mathbb{C})$ is given by

$$M_{ab} := \left(\mathbf{1}_n - \frac{\sqrt{-1}}{2}A^{+}_{ab}\theta_3\right)A_{ab},$$

which should be a symmetric matrix whose real part is positive definite. Here, again, the real part of $M_{ab}$ is positive definite if and only if $A_{ab}$ is positive definite. Then, one has $\dim(H^0(V_{ab})) = \det(A_{ab})$. The condition that $M_{ab}$ above is symmetric is equal to

$$A^{+}_{ab}\theta_3 A_{ab} = (A^{+}_{ab}\theta_3 A_{ab})^t,$$

which is, in fact, equivalent to $A_a\theta_3 A_a = A_b\theta_3 A_b$.

Now, for $a,b,c \in \mathrm{Ob}(\mathcal{C})$ such that $A_a\theta_3 A_a = A_b\theta_3 A_b = A_c\theta_3 A_c$, assume that $A_{ab}$ and $A_{bc}$ are positive definite. The product of two theta vectors is given by

$$m(e^{\mu}_{ab}, e^{\nu}_{bc}) = \sum_{\rho \in \mathbb{Z}^n / A_{ac}\mathbb{Z}^n} \sum_{u \in \mathbb{Z}^n} \delta_{[A_{ab}]^{\mu}_{-u+\rho}} \delta_{[A_{bc}]^{\nu}_{u}} \exp(-\pi(u - A_{bc}A^{-1}_{ac}\rho)^t((A^{-1}_{ab} + A^{-1}_{bc}) \times(\mathbf{1}_n + \sqrt{-1}A_b\theta_3))(u - A_{bc}A^{-1}_{ac}\rho)) \cdot e^{\rho}_{ac}.$$

One can show that the matrix $(A^{-1}_{ab} + A^{-1}_{bc})(1_n + \sqrt{-1}A_b\theta_3)$ defining a quadratic form in the expression above is symmetric,

$$(A^{-1}_{ab} + A^{-1}_{bc})(\mathbf{1}_n + \sqrt{-1}A_b\theta_3) = (\mathbf{1}_n - \sqrt{-1}\,\theta_3 A_b)(A^{-1}_{ab} + A^{-1}_{bc}).$$

Thus, we have seen that the structure constants $C^{\mu\nu}_{abc,\rho}$ depend on the noncommutative parameter $\theta$ in all these three cases.

Though in this paper we have fixed a constant curvature connection for a Heisenberg module, we can also take all the constant curvature connections on a Heisenberg module into account in a similar way as in the noncommutative complex one-tori case.[14,16] The moduli space of the (constant curvature) connections on a Heisenberg module $E_{g,\theta}$ on $T^{2n}_{\theta}$ is known to form a commutative torus $T^{2n}$. It might also be interesting to investigate the details of various structures on the moduli space.

We end with showing an example for the case of a noncommutative complex two torus ($n = 2$). In this case, for any fixed $\theta_s, s=1,2,3$, condition (29) reduces to

$$\det(A_a) = \det(A_b).$$

Thus, for the objects of $\mathcal{C}^f$, one can in general have an infinite number of objects. For instance, diagonal matrices $A \in \mathrm{Mat}_n(\mathbb{Z})$ with $\det(A) = -4$ are

$$A_a = \begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix}, \quad A_b = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}, \quad A_c = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix},$$

and $A_{a'} := -A_a$, $A_{b'} := -A_b$, $A_{c'} := -A_c$. Since the zeroth cohomologies of morphisms between $\{a,b,c\}$ and $\{a',b',c'\}$ are absent, the algebra on the zeroth cohomologies is the direct sum of the one on $\{a,b,c\}$ and the one on $\{a',b',c'\}$. If we concentrate on the one side $\{a,b,c\}$, all $A_{ab}$, $A_{bc}$, and $A_{ac}$ are positive definite, and the dimensions of the zeroth cohomologies are

$$\dim(H^0(V_{ab}))=2, \quad \dim(H^0(V_{bc}))=2, \quad \dim(H^0(V_{ac}))=9.$$

However, there exist infinite symmetric matrices $A\in \mathrm{Mat}_n(\mathbb{Z})$ with $\det(A)=-4$ if nondiagonal ones also are taken into consideration, since the matrix $g^tAg$ has $\det(A)=-4$ for any $SL(2,\mathbb{Z})$ element $g$.

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[1] Barannikov, S., and Kontsevich, M., "Frobenius manifolds and formality of Lie algebras of polyvector fields," Int. Math. Res. Notices **1998**, 201–215.
[2] Ben-Bassat, O., "Mirror symmetry and generalized complex manifolds I," J. Geom. Phys. **56** 533–558 (2006); "Mirror symmetry and generalized complex manifolds II," J. Geom. Phys. **56**, 1096–1115 (2006); also available on e-print arXiv:math.AG/0405303;
[3] Ben-Bassat, O., Block, J., and Pantev, T., e-print arXiv:math.AGmath.AG/0509161.
[4] Block, J., e-print arXiv:math.QA/0509284.
[5] Dabrowski, L., Krajewski, T., and Landi, G., "Some properties of nonlinear sigma models in noncommutative geometry," Int. J. Mod. Phys. B **14**, 2367 (2000); also available on e-print arXiv:hep-th/0003099; "Nonlinear sigma-models in noncommutative geometry: Fields with values in finite spaces," Int. J. Mod. Phys. A **18**, 2371 (2003), e-print arXiv:math.QA/0309143.
[6] Dieng, M., and Schwarz, A., "Differential and complex geometry of two-dimensional noncommutative tori," Lett. Math. Phys. **61**, 263–270 (2002); also available on e-print arXiv:math.QA/0203160.
[7] Elliott, G. A., and Li, H., e-print arXiv:math.OA/0311502; e-print arXiv:math.OA/0501030.
[8] Fukaya, K., "Mirror symmetry of abelian varieties and multi-theta functions," J. Algeb. Geom. **11**, 393–512 (2002); Kyoto University report, 1998 (unpublished); available at http://www.kusm.kyoto-u.ac.jp/fukaya/~fukaya.html
[9] Fukaya, K., "Floer homology of Lagrangian foliation and noncommutative mirror symmetry," Kyoto University Report No. 98-08, 1998 (unpublished); available at http://www.kusm.kyoto-u.ac.jp/~fukaya/fukaya.html
[10] Fukaya, K., Oh, Y. G., Ohta, H., and Ono, K., "Lagrangian intersection Floer theory-anomaly and obstructon-," report, 2000 (unpublished); available at http://www.kusm.kyoto-u.ac.jp/~fukaya/fukaya.html
[11] Gualtieri, M., Ph.D. thesis, Oxford University, (2003); also available on e-print arXiv:math.DG/0401221.
[12] Herbst, M., Lazaroiu, C. I., and Lerche, W., "Superpotentials, A(infinity) relations and WDVV equations for open topological strings," J. High Energy Phys. 0502, 071 (2005); also available on e-print arXiv:hep-th/0402110.
[13] Kajiura, H., "Homotopy algebra morphism and geometry of classical string field theories," Nucl. Phys. B **630**, 361 (2002); also available on e-print arXiv:hep-th/0112228.
[14] Kajiura, H., "Kronecker foliation, D1-branes and Morita equivalence of noncommutative two-tori," J. High Energy Phys. 0208, 050 (2002); also available on e-print arXiv:hep-th/0207097.
[15] Kajiura, H., "Noncommutative homotopy algebras associated with open strings," Rev. Math. Phys. **19**, 1–99 (2007); also available on e-print arXiv:math.QA/0306332.
[16] Kajiura, H., e-print arXiv:hep-th/0406233.
[17] Kajiura, H., "Star product formula of theta functions," Lett. Math. Phys. **75**, 279 (2006); also available on e-print arXiv:math.QA/0510307.
[18] Kajiura, H., and Stasheff, J., "Homotopy algebras inspired by classical open-closed string field theory," Commun. Math. Phys. **263**, 553–581 (2006); also available on e-print arXiv:math.QA/0410291; "Open-closed homotopy algebras in mathematical physics," J. Math. Phys. **47**, 023506 (2006); also available on e-print arXiv:hep-th/0510118.
[19] Kapustin, A., "Topological strings on noncommutative manifolds," Int. J. Geom. Methods Mod. Phys. **1**, 49 (2004); also available on e-print arXiv:hep-th/0310057.
[20] Kapustin, A., and Li, Y., "D-branes in Landau-Ginzburg models and algebraic geometry," J. High Energy Phys. 0312, 005 (2003); also available on e-print arXiv:hep-th/0210296.
[21] Kapustin, A., and Li, Y., "Topological correlators in Landau-Ginzburg models with boundaries," Adv. Theor. Math. Phys. **7**, 727–749 (2003); also available on e-print arXiv:hep-th/0305136.
[22] Kim, E., and Kim, H., e-print arXiv:math.QA/0312228.
[23] Kim, E., and Kim, H., e-print arXiv:hep-th/0410108.
[24] Kim, H., and Lee, C. Y., "Theta functions on noncommutative T(4)," J. Math. Phys. **45**, 461 (2004); also available on e-print arXiv:hep-th/0303091.
[25] Konechny, A., and Schwarz, A., "Introduction to M(atrix) theory and noncommutative geometry," Phys. Rep. **360**, 353–465 (2002); also available on e-print arXiv:hep-th/0012145, and references therein.
[26] Kontsevich, M., *Proceedings of the International Congress of Mathematicians, Zürich, 1994* (Birkhäuser, Basel, 1995), Vol. 1 and 2, pp. 120–139.
[27] Kontsevich, M., "Deformation quantization of Poisson manifolds," Lett. Math. Phys. **66**, 157–216 (2003); also available on e-print arXiv:math.QA/9709040.

[28] Lazaroiu, C. I., "String field theory and brane superpotentials," J. High Energy Phys. 0110, 018 (2001); also available on e-print arXiv:hep-th/0107162.
[29] Lazaroiu, C. I., "D-brane categories," Int. J. Mod. Phys. A **18**, 5299 (2003); also available on e-print arXiv:hep-th/0305095.
[30] Lazaroiu, C. I., and Roiban, R., "Holomorphic potentials for graded D-branes," J. High Energy Phys. 0202, 038 (2002); also available on e-print arXiv:hep-th/0110288; Lazaroiu, C. I., and Roiban, R., "Gauge-fixing, semiclassical approximation and potentials for graded Chern-Simons theories," J. High Energy Phys. 0203, 022 (2002); also available on e-print arXiv:hep-th/0112029.
[31] Li, H., "Strong Morita equivalence of higher-dimensional noncommutative tori," J. Reine Angew. Math. **576**, 167–180 (2004); also available on e-print arXiv:math.OA/0303123.
[32] Moyal, J. E., "Quantum mechanics as a statistical theory," Proc. Cambridge Philos. Soc. **45**, 99–124 (1949).
[33] Polishchuk, A., and Schwarz, A., "Categories of holomorphic vector bundles on noncommutative two-tori," Commun. Math. Phys. **236**, 135 (2003); also available on e-print arXiv:math.QA/0211262.
[34] Polishchuk, A., and Zaslow, E., "Categorical mirror symmetry: The elliptic curve," Adv. Theor. Math. Phys. **2**, 443–470 (1998); also available on e-print arXiv:math.AG/9801119.
[35] Positsel'skii, L. E., "Nonhomogeneous quadratic duality and curvature," Funkc. Anal. Priloz. **27**, 57–66 (1993); [Funct. Anal. Appl. **27**, 197–204 (1993)].
[36] Rieffel, M., "Projective modules over higher-dimensional noncommutative tori," Can. J. Math. **40**, 257–338 (1988).
[37] Rieffel, M., and Schwarz, A., "Morita equivalence of multidimensional noncommutative tori," Int. J. Math. **10**, 289–299 (1999); also available on e-print arXiv:math.QA/9803057.
[38] Schwarz, A., "Morita equivalence and duality," Nucl. Phys. B **534**, 720 (1998); also available on e-print arXiv:hep-th/9805034.
[39] Schwarz, A., "Theta functions on noncommutative tori," Lett. Math. Phys. **58**, 81–90 (2001); also available on e-print arXiv:math.QA/0107186.
[40] Schwarz, A., "Noncommutative supergeometry, duality and deformations," Nucl. Phys. B **650**, 475–496 (2003); also available on e-print arXiv:hep-th/0210271; extended version of e-print arXiv:hep-th/9912212.
[41] Schlessinger, M., and Stasheff, J., "Deformation theory and rational homotopy type," U. of North Carolina report, 1979 (unpublished); "The Lie algebra structure of tangent cohomology and deformation theory," J. Pure Appl. Algebra **38**, 313–322 (1985).
[42] Stasheff, J. D., "On the homotopy associativity of *H*-spaces, I," Trans. Am. Math. Soc. **108**, 275 (1963); "On the homotopy associativity of *H*-spaces, II," Trans. Am. Math. Soc. **108**, 293 (1963).
[43] Takahashi, A., e-print arXiv:math.AG/0506347.
[44] Takahashi, A., "Homological algebra of weak differential graded algebras," (in preparation).
[45] Tang, X., and Weinstein, A., "Quantization and Morita equivalence for constant Dirac structures on tori," Ann. Inst. Fourier **54**, 1565–1580 (2004); also available on e-print arXiv:math.QA/0305413.
[46] Witten, E., "Mirror Manifolds and Topological Field Theory," *Essays on Mirror Manifolds*, Internat. Press, Hong Kong, pp. 120–158 (1992); also available on e-print arXiv:hep-th/9112056
[47] Witten, E., "Chern-Simons gauge theory as a string theory," Prog. Math. **133**, 637 (1995); also available on e-print arXiv:hep-th/9207094.