Title	Slow switching in globally coupled oscillator: robustness and occurrence through delayed coupling
Author(s)	Kori, H; Kuramoto, Y
Citation	PHYSICAL REVIEW E (2001), 63(4)
Issue Date	2001-04
URL	http://hdl.handle.net/2433/49875
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Type	Journal Article
Textversion	publisher

Universality and scaling for the breakup of phase synchronization at the onset of chaos in a periodically driven Rössler oscillator

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(Received 4 May 2001; published 24 September 2001)

Universal behavior discovered earlier in two-dimensional noninvertible maps is found numerically in a periodically driven Rössler system. The critical behavior is associated with the limit of a period-doubling cascade at the edge of the Arnold tongue, and may be reached by variation of two control parameters. The corresponding scaling regularities, distinct from those of the Feigenbaum cascade, are demonstrated. Presence of a critical quasiattractor, an infinite set of stable periodic orbits of quadrupled periods, is outlined. As argued, this type of critical behavior may occur in a wide class of periodically driven period-doubling systems.

DOI: 10.1103/PhysRevE.64.046214 PACS number(s): 05.45.Df, 05.10.Cc

I. INTRODUCTION

Classic synchronization, or mode locking, consists in a natural adjustment of frequencies of periodic self-oscillations in nonlinear dissipative systems due to their interaction, or due to presence of an external periodic force. This phenomenon was extensively studied in classic theory of oscillators [1–4]. It is observed in physical, technical, and biological systems and has numerous and important applications [3].

In modern interpretation, the concept of synchronization is of much broader meaning (see [5–8]). In particular, it takes into account a possibility of different combinations of dynamical behaviors in the interacting systems (periodic-quasiperiodic, periodic-chaotic, chaotic-chaotic, etc.). Many such situations are of interest both for theory and for novel applications in secure communication, control of chaos, studies of living systems (e.g., [8–11]).

One of the outlined phenomena is phase synchronization of low-dimensional chaotic dissipative systems [12,13,5]. This notion relates to a situation when some phase variable can be attributed to a system with a strange attractor. In the process of time evolution this variable appears to be locked and retains a certain relation with the phase of the periodic external force, although the dynamics of the amplitude is chaotic.

It is natural to consider a system depending on some parameters, which could manifest either periodic or chaotic autonomous dynamics in corresponding regions of its parameter space. In the presence of an external force, we can observe a classic version of synchronization in the first case and the phase synchronization of chaos in the second. It is interesting to analyze possible bifurcation scenarios associated with such situations.

Detailed bifurcational analysis for the classic synchronization was developed by many authors, e.g., [1–4,15–19]. In this article we turn to synchronization in a system, which demonstrates Feigenbaum's period-doubling scenario of the onset of chaos [20,21]. As a concrete example we select the Rössler model, which is a convenient object in the studies of phase synchronization [12,13]. Dynamical behavior of this system, associated with the spiral-type strange attractor, is very typical for many low-dimensional chaotic systems [14].

Let us assume that we observe the period-doubling cascade in the forced system and move along the chaos border in the parameter space—the frequency of the external force is tuned to reach the edge of the synchronization zone (the edge of the Arnold tongue). In this way we arrive at a critical point of special kind, which is the subject of our study.

Our numerical results indicate that this critical point relates to the universality class introduced earlier in the context of two-dimensional noninvertible iterative maps (the so-called C type of criticality) [22–25]. At the critical point, the forced Rössler system displays all the attributes of the universality class, including scaling regularities associated with definite universal constants distinct from those of Feigenbaum. One of the intriguing properties of the dynamics at the critical point is the presence of an infinite countable set of coexisting attractive periodic orbits of quadrupled period that we refer to as the critical quasiattractor.

In Sec. II we discuss some relevant details of the arrangement of the parameter space for the forced Rössler oscillator. In Sec. III we shortly recall some basic properties of the critical behavior for a model map that plays the same role for our universality class as the logistic map for the Feigenbaum criticality. In Sec. IV we present numerical data, which give evidence of the existence of the critical point. We locate this point with high accuracy at some particular fixed amplitude of the external force. Phase portraits are demonstrated for several stable limit cycles—representatives of the infinite family of attractors coexisting at the critical point. In conclusion we briefly discuss the significance of our results for the theory of phase synchronization and a possibility of experimental observation of the C-type critical behavior.

II. DRIVEN RÖSSLER MODEL AND ITS PARAMETER SPACE

Let us consider the Rössler equations with an additional term corresponding to the external periodic force,

$$\dot{x} = -x - z + A \sin 2\pi\Omega t,$$

$$\dot{y} = x + py,$$

$$\dot{z} = q + z(x - r), \tag{1}$$

where x, y, and z are dynamical variables and p, q, and r are internal parameters of the Rössler oscillator; A and Ω are the

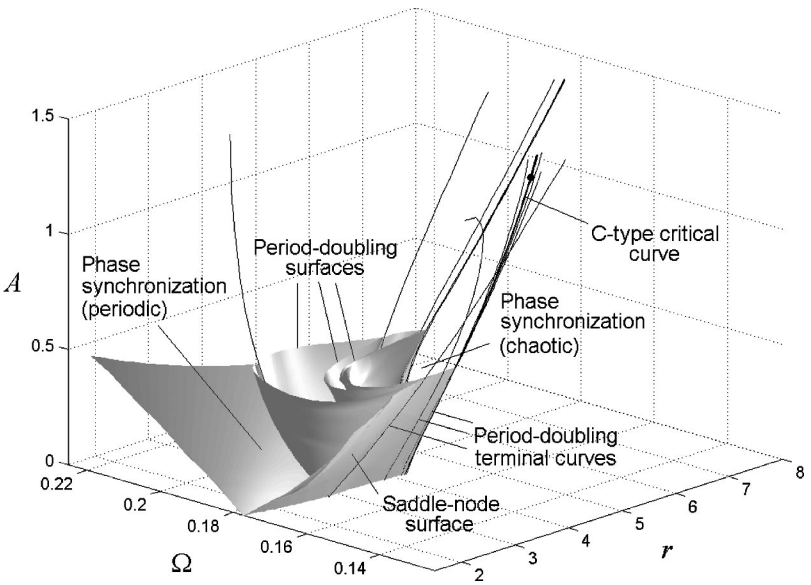

FIG. 1. Three-dimensional view of the parameter space (Ω,r,A) for the periodically driven Rössler oscillator. The inscriptions explain the nature of the regimes and bifurcations. For clarity of the diagram the bifurcational surfaces are drawn only partially. A point on the critical curve selected for detailed study is marked by a bullet.

amplitude and frequency of the external driving. We fix $p=q=0.2$ and consider r as the main internal control parameter. As known, in the autonomous system increase of r gives rise to the period-doubling bifurcation cascade and a subsequent transition to chaos [4]. According to the normalization accepted in Eq. (1), the period of the external force is $T=1/\Omega$. Thus, the periods of regular synchronized states must be integer multiples of this value.

In the numerical computations we used a Runge-Kutta method with adaptive step-size control [26]. The algorithm was supplemented by simultaneous integration of the linearized Rössler equations, the Newton-Raphson procedure to search for periodic orbits, and by calculation of the Floquet eigenvalues. In our forced three-dimensional system a periodic orbit has three Floquet eigenvalues, which are also referred to as multipliers. Usually one of the multipliers is small in modulus and does not influence the bifurcation events, while two others may be of relevance. As known [4], at the threshold of stability loss, one of the three following situations can occur generically: (i) one of the multipliers becomes equal to -1 (the period-doubling bifurcation), (ii) it becomes equal to 1 (the saddle-node or tangent bifurcation), and (iii) two complex-conjugate multipliers are at the unit circle (the Neimark bifurcation).

Figure 1 shows the main synchronization zone, or Arnold tongue, for the periodically driven Rössler oscillator in the three-dimensional parameter space (Ω,r,A). At small values of r the classic synchronization of the periodic limit-cycle dynamics occurs. For larger r the synchronous regime undergoes the Feigenbaum period-doubling cascade. The chaotic regimes inside the tongue correspond to the phase synchronization on the Rössler attractor.

Figure 2 displays a two-dimensional diagram in the parameter plane (Ω,r), which represents a cross section of the previous figure by the horizontal surface $A=0.1$. The middle part of the plot is the synchronization zone. Light gray regions correspond to quasiperiodic regimes, the dark gray area at the top corresponds to phase synchronization of chaos, and black areas correspond to asynchronous chaotic dynamics. Moving in the synchronization zone upwards, i.e., increasing r, one can observe a sequence of period-doubling bifurcations of the synchronous regime.

The outer borders of the synchronization tongue consist mainly of the saddle-node bifurcation lines. More accurate analysis reveals nontrivial bifurcation structures near the points, where the period-doubling curves approach the border of the synchronization zone. These structures include segments of Neimark and subcritical period-doubling bifurcation lines as shown by the insets in Fig. 2. Note the presence of intersecting bifurcation curves indicating occurrence of distinct coexisting attractors, hard transitions, and hysteresis. At larger amplitudes of the driving these areas become relatively more pronounced.

III. MODEL MAP AND THE CRITICAL BEHAVIOR OF *C* TYPE

Roughly speaking, near the onset of chaos at the edge of the synchronization zone in the forced Rössler oscillator, variation of one control parameter leads to period doubling, and variation of another one leads to the saddle-node (tangent) bifurcation.

The following two-dimensional map suggested in [24,25] appears to be an appropriate model for exploration of this situation:

$$X'=a-X^2+bY,\quad Y'=-X^2+dY. \tag{2}$$

Here a prime marks the dynamical variables X and Y relating to the next step of discrete time; a, b, and d are parameters. It is convenient to keep b constant and analyze dynamics by varying the two other parameters, a and d. Increase of a at small or moderate values of d gives rise to the period-doubling cascade. The period-doubling bifurcation curves on the parameter plane (a,d) converge to the Feigenbaum critical line. On the other hand, increasing d one can find that this line is terminated at some critical point. For a particular fixed value of $b=-0.6663$ it is located at

$$a=0.249\,902\,800\ldots,\quad d=0.452\,902\,880\ldots. \tag{3}$$

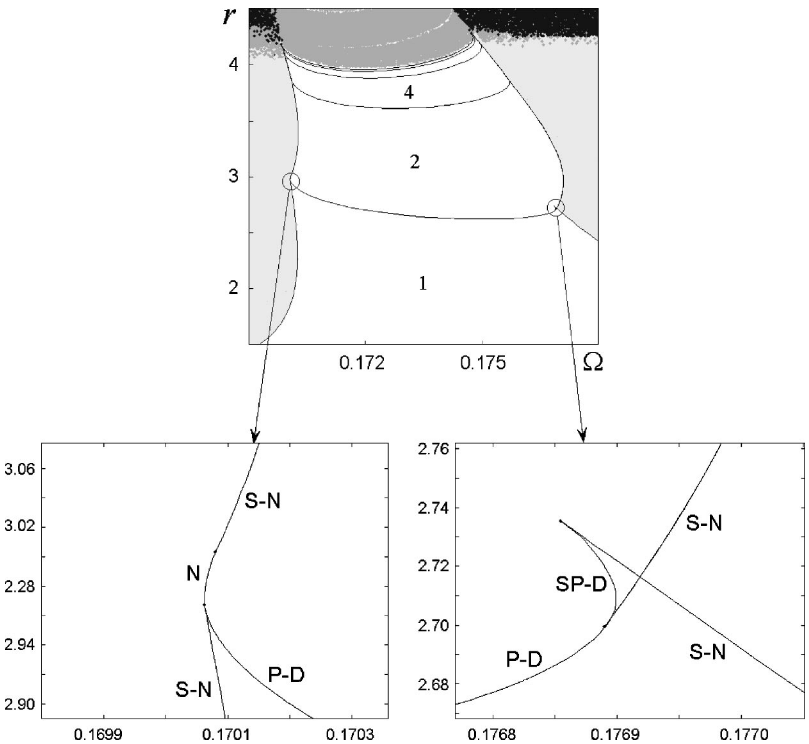

FIG. 2. Bifurcation diagram for the periodically driven Rössler oscillator on the parameter plane (Ω,r), at $A=0.1$. The white area in the middle is the synchronization zone. Light gray areas correspond to quasiperiodic regimes, dark gray areas at the top to phase synchronization of chaos, and black areas to asynchronous chaotic dynamics. Insets show details of the bifurcational structure near the terminal points of the period-doubling curves marked as P-D. The subcritical period-doubling bifurcation line is designated as SP-D, the bifurcation lines of saddle-node (tangent) bifurcations are designated as S-N, and those of Neimark bifurcations as N.

Details of dynamics at the critical point and in its neighborhood were studied by means of the renormalization-group (RG) approach. It is based on a two-dimensional version of the doubling transformation of Feigenbaum-Cvitanović (see [20,21]). It appears that the critical behavior is associated with a period-2 saddle orbit of the RG transformation. For this reason this type of criticality is referred to as C-type, where the letter ''C'' stands for the word ''cycle.'' Alternatively, one can speak of a fixed point of the *quadrupling* transformation. In contrast to the case of Feigenbaum's criticality, all manifestations of scaling correspond to the quadrupling, rather than the doubling of time scales.

The numerical solution for the fixed point of the quadrupling RG transformation yields universal functions (see the numerical data for coefficients of polynomial expansions in [22,23,25]) and a pair of universal constants, the phase space scaling factors

$$\alpha_1=6.565\,350\ldots \quad \text{and} \quad \alpha_2=22.120\,227\ldots. \tag{4}$$

The quadrupling RG transformation linearized at the fixed point gives rise to an eigenvalue problem. This has been solved numerically and appears to have two relevant eigenvalues [22,23,25]

$$\delta_1=92.431\,263\,48\ldots \quad \text{and} \quad \delta_2=4.192\,444\,18\ldots. \tag{5}$$

These are the factors responsible for the scaling properties of the parameter space near the C-type critical point. The presence of two relevant eigenvalues means that the discussed type of critical behavior is of codimension 2, i.e., one needs to adjust *two* control parameters to reach the critical situation.

Unfortunately, the spectrum of the linearized RG equation contains an eigenvalue close to 1, namely, $\delta_3\simeq 0.93$. Although formally irrelevant, it is associated with an eigenmode, which decays very slowly from level to level of the period quadrupling. Due to this circumstance, the clearly expressed C-type universality becomes observable, in general, only at high levels. To avoid this problem and optimize the convergence, one can exploit the presence of the third control parameter b in the model map (2): its value may be selected in such a way that the amplitude of the slow mode vanishes.

In the three-dimensional parameter space, the C-type criticality occurs at some curve, but only a particular point on this curve is well suited for practical observation of the universal scaling regularities. The abovementioned value of $b=-0.6663$ was selected for this reason.

A remarkable property of the dynamics at the critical point (3) is coexistence of an infinite set of attractors—stable cycles of periods 4^k, $k=0,1,\ldots$ [24,25]. The Floquet eigenvalues (multipliers) for these cycles are equal (asymptotically) to universal values found from the RG analysis,

$$\mu_1^c=-0.725\,255\ldots, \quad \mu_2^c=0.847\,450\ldots. \tag{6}$$

The ''intermediate'' cycles of periods 2×4^n are unstable, and their universal multipliers are[1]

$$\tilde{\mu}_1^c=-0.848\,865\ldots, \quad \tilde{\mu}_2^c=1.174\,459\ldots. \tag{7}$$

[1]It is also possible that in asymptotics of large n, the cycles of period 2×4^n are stable with multipliers (6), while the cycles of period 4^n are unstable with multipliers (7). For the map (2) this is the case for positive b.

TABLE I. Period-doubling terminal points at $A=0.1$.

N	x	y	z	r	Ω
1	2.5759048090	2.7458248531	3.2766571344	2.9668861906	0.17006220271
2	3.3463735559	2.6148283377	8.2361397123	3.8618621573	0.17009194241
4	3.8011617532	2.9208207926	7.3514962256	4.1109005901	0.16989964755
8	3.908794885	2.8172715016	7.9993329812	4.1667635657	0.16985207870
16	3.9230764444	2.8638499524	7.8052490605	4.1790083430	0.16984155604
32	3.9308229446	2.8458704049	7.8969364703	4.1815343711	0.16983937313
64	3.9301626916	2.8533802216	7.8632883002	4.1821300905	0.1698388587
128	3.9311583394	2.8503551802	7.8774107247	4.1822297423	0.16983877260
256	3.9308108246	2.8516062593	7.8718942407	4.1822653227	0.16983874186

As known, any type of critical dynamics allowing application of RG analysis gives rise to a class of quantitative universality that may include many systems of different nature [20,27–29,23,30]. The same must be true with respect to the C-type critical behavior. As we argue in this article, the driven Rössler oscillator at the period-doubling accumulation point on the edge of Arnold tongue is a representative of the corresponding universality class.

IV. *C*-TYPE CRITICAL BEHAVIOR IN THE DRIVEN RÖSSLER OSCILLATOR

We now return to the periodically forced Rössler model. Exploring the edge of the synchronization zone, we observe a sequence of codimension-2 bifurcation points, at which the period-doubling curves meet the saddle-node lines (see Fig. 2). We refer to these points as the *period-doubling terminal points*. At such points the synchronized periodic orbit appears to be at the threshold of instability in respect to the two distinct modes of perturbation, and two of the multipliers are $\mu_1=1$ and $\mu_2=-1$.

In Table I we present coordinates of the points terminating the period-doubling bifurcation curves on one edge of the synchronization zone up to period 256. (Hereafter the periods are measured in units of a period of the external force.) One more period-doubling terminal sequence and the respective critical point are located at the opposite side of the Arnold tongue.

Inside the Arnold tongue, where the Feigenbaum bifurcation cascade occurs, a sequence of the period-doubling bifurcations is known to be infinite. Hence, it is natural to conjecture that the sequence of the terminal points is infinite too, and it converges to some critical point. The numerical data of Table I support this assertion.

The same computations may be performed as well at other values of the amplitude A. In three-dimensional parameter space it yields a sequence of ‘‘terminal curves’’ (see Fig. 1), which accumulate to some limit curve. We argue that the system demonstrates at this curve the critical behavior of C type discussed in Sec. II.

As explained, convergence to the asymptotically universal scaling behavior of C type may be very slow due to the presence of the eigenvalue slightly less than 1. For this reason, to obtain convincing numerical data revealing the nature of the critical behavior, it is worth selecting a particular point on the critical curve, which corresponds to zero amplitude of the slow eigenmode in the solution of the RG equation.

Let us suppose that we have computed the sequence of terminal points and estimated the limit for some value of A. Then we search for the limit cycles of period 1,4,16, . . . and compare their multipliers with the universal values (6). Next, we change the amplitude A gradually and trace the variation of the multipliers for several coexisting cycles. The closer to the optimal point, the closer are all the multipliers to the values (6). On the final stage of the procedure we simultaneously adjust the two parameters r, Ω to satisfy equalities $\mu_1(4^k)=\mu_1^c$, $\mu_2(4^k)=\mu_2^c$, and tune the third parameter A to fulfill the additional condition $\mu_2(4^{k-1})=\mu_2^c$.

The greater k is, the better estimate for the optimal critical

TABLE II. A sequence of the period-doubling terminal points at the optimal value of amplitude $A=1.35$ and estimates for the parameter-space scaling factor δ_2.

N	r	Ω	$\delta_{2,r}$	$\delta_{2,\Omega}$
1	5.732239864848575	0.1510840937189741		
2	2.977715775644518	0.1383526282404057		
4	5.123248075904194	0.1488441036460531		
8	4.718523541397686	0.1475413488683935	6.805923	9.772726
16	4.983120254420295	0.1484037045629827	8.108688	1.216606
32	4.887643400101043	0.1480995675987042	4.238980	4.283447
64	4.947202062137271	0.1482900572326032	4.442623	4.527047
128	4.924430060892722	0.1482175502249565	4.192730	4.194587

TABLE III. Initial points (at $t=0$) and multipliers for cycles of period $N=2^k$ at the critical point (8).

N	x_0	y_0	z_0	μ_1	μ_2
1	4.506416552153	3.567399962374	1.816412780933	0.777162	−0.600727
2	4.877551206560	2.987654298236	1.111795695017	1.229085	−0.911129
4	5.327317412038	2.742405627024	1.524162104533	0.858520	−0.685901
8	5.281240682500	2.727878439530	1.389046629011	1.180212	−0.860374
16	5.336876017064	2.70046225872	1.453022915555	0.85021	−0.71942
32	5.324414524360	2.704014865671	1.432967779067	1.17467	−0.84945
64	5.332097301574	2.700664355370	1.442868136652	0.84756	−0.73033
128	5.329978026968	2.701455341398	1.439851023798	1.1724	−0.8342
256	5.331100238443	2.700990282432	1.441351778370	0.850	−0.721

point could be obtained. Practically, due to restrictions imposed by limited numerical precision in our computations of the unstable cycles of large periods (and their multipliers), we have to stop at $k=8$.

In two first columns of Table II we present coordinates for the sequence of the terminal point at the selected optimal amplitude of the external force $A=1.35$. The limit point of this sequence is the critical point located in accordance with the best estimate at

$$r_c=4.935\,701\,677\,387\,16,$$

$$\Omega_c=0.148\,253\,488\,114\,690\,6,\quad A_c=1.35. \tag{8}$$

In Fig. 1 this point is marked by a bullet.

In Table III the data for the periodic orbits (stable and unstable) at the critical point are presented. The coordinates (x_0,y_0,z_0) relate to the starting point of the periodic orbit at $t=0$. Observe that the multipliers of the periodic orbits are remarkably close to their expected universal values (6) and (7).

Using the values of (x_0,y_0,z_0) for subsequent cycles of periods $N=4^k$ and $N_1=4^{k-1}$ from Table II one can estimate the smaller phase space scaling factor α_1 [see Eq. (4)] by ratios $\alpha_{1,x}=\Delta x_{N_1}/\Delta x_N$, $\alpha_{1,y}=\Delta y_{N_1}/\Delta y_N$, $\alpha_{1,z}=\Delta z_{N_1}/\Delta z_N$, where $\Delta x_N,\Delta y_N,\Delta z_N$ designate the distances in the *x*, *y*, and *z* directions between two points of the orbit of period *N*, which are separated by half the period and have the largest spacing (Table IV). Obviously, the results are in good correspondence with the value $\alpha_1=6.565\,35\ldots$ obtained from the RG approach.

The next test for the *C*-type universality is an estimate of the parameter-space scaling factor from the numerical data for the terminal points (Table II). Composing the ratios with subsequent terms of the parameter sequences

TABLE IV. Estimates for the phase space scaling factor α_1.

N	$\alpha_{1,x}$	$\alpha_{1,y}$	$\alpha_{1,z}$
16	5.976519	5.528877	6.920036
32	6.410982	5.730828	6.421471
64	6.442042	6.244803	6.658738
256	6.552378	6.409687	6.580197

$$\delta_{2,r}=[r(N_1)-r(N_1/2)]/[r(N)-r(N/2)],$$

$$\delta_{2,\Omega}=[\Omega(N_1)-\Omega(N_1/2)]/[\Omega(N)-\Omega(N/2)],$$

we obtain estimates for the smaller parameter-space scaling factor, as seen in the last two columns of Table II. They are in good correspondence with the value $\delta_2=4.192\,44\ldots$ found from the RG analysis.

Unfortunately, dealing with a restricted number of period-doubling levels we could not obtain convincing empirical data for the rest of the (larger) constants α_2 and δ_1. Nevertheless, the observed agreement of the multipliers and of the scaling factors with those found for the model map (2), and with those obtained from the RG analysis, strongly supports the assertion that the observed critical behavior in the forced Rössler oscillator actually is of the type discussed in Sec. III. This means that the system must exhibit all the attributes of the *C*-type universality class, including the critical quasiattractor—the infinite countable set of stable periodic orbits of quadrupled period. From Table III we observe that the cycles of period 1, 4, 16, 64, 256 indeed are stable. They have to be regarded as the first representatives of the infinite set of stable cycles at the critical point of *C* type. In Fig. 3 we show phase portraits for the attractive cycle of periods 1, 4, 16, 64. It is worth noting here that the *C*-type critical point on the opposite edge of the Arnold tongue is of another kind: the stable orbits are of periods 2×4^n.

V. CONCLUSION

Analyzing the situation of breakup of synchronization in a periodically driven Rössler oscillator for the cycles belonging to the period-doubling sequence we find a specific type of universal behavior at the so-called *C*-type critical point. Earlier it was known only in context of abstract, artificially constructed model maps. Now we have an example, which is much closer to physically realizable objects.

As known, the Rössler oscillator manifests the dynamical behavior typical for a wide class of low-dimensional dissipative chaotic systems, namely, the period-doubling cascade leading to the onset of chaotic attractor of the spiral type. We believe that the dynamical properties analogous to those found in the forced Rössler oscillator will occur also in other systems of this class under external periodic driving.

A remarkable feature of the critical point is the presence

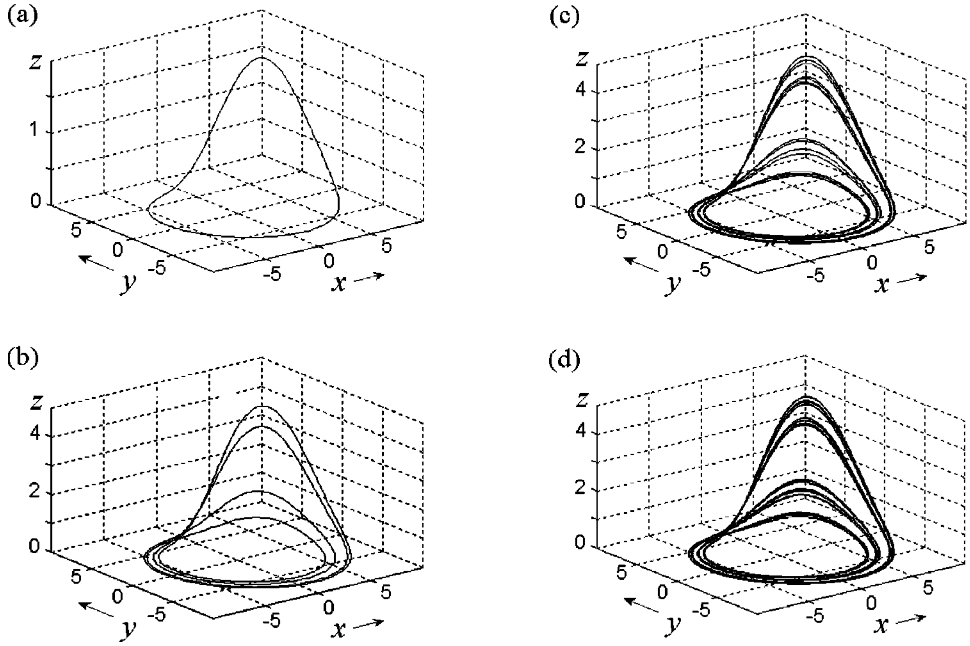

FIG. 3. Attractive limit cycles of period 1 (a), 4 (b), 16 (c), and 64 (d) (measured in units of the period of external force) for the periodically driven Rössler oscillator at the critical point (8). These are four representatives of an infinite set of stable periodic orbits constituting the critical quasiattractor.

of an infinite set of coexisting attractive periodic orbits. As may be conjectured, this is a universal attribute of the synchronization breakup corresponding to the limit of period doubling at the edge of the Arnold tongue.

It may be expected that the critical behavior of *C* type could be observed in carefully organized experiments on synchronization of period doubling dissipative systems (e.g., convective systems, electronic oscillators, etc.). The known experiments on the Feigenbaum universal behavior show that the universality and scaling regularities in accurately controlled conditions may be traced through not more than four to five levels of period doubling [21]. Hence, it is important to stress again that the amplitude of the external force should be selected properly to ensure a possibility of observation of the expected universal regularities on the low levels of the period-doubling cascade, just as we have done for the Rössler system in this study.

ACKNOWLEDGMENTS

This work was supported by RFBR (Grant No. 00-02-17509) and by CRDF (Grant No. REC-006).

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