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## Correspondences between Eisenstein series of Jacobi forms and modular forms with quadratic primitive Dirichlet character

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**Abstract:** Let  $p$  be an odd prime and  $\chi_p$  the quadratic primitive Dirichlet character modulo  $p$ . In this paper, we construct Jacobi-Eisenstein series of weight  $k$  and index  $m$  on  $\Gamma_0(p) \times \mathbb{Z}^2$  with  $\chi_p$ , and Eisenstein series of elliptic modular forms of weight  $2k-2$  on  $\Gamma_0(mp)$  with character  $\chi_p^2$ . Moreover, considering Fourier coefficients of Eisenstein series we construct the correspondence between both Eisenstein series.

### 0. Introduction

By Shimura, Shintani, Niwa and Kohnen we know the relations between the spaces of modular forms of half-integral weight and of integral weight. Skoruppa considered the relation between modular forms of half-integral weight and Jacobi forms. Skoruppa and Zagier constructed in their paper [SZ] a correspondence between the spaces of Jacobi forms of weight  $k$  and index  $m$  and modular forms of weight  $2k-2$  and level  $m$ .

Arakawa([Ara]) and Horie([Hor1], [Hor2]) defined the Jacobi forms on the Jacobi group  $\Gamma_0(N) \times \mathbb{Z}^2$  with character.

Recently, Manickam and Ramakrishnan ([MR2]) constructed a new Jacobi-Eisenstein series of square level  $N$  with trivial character, and considered the correspondence between Jacobi-Eisenstein series of weight  $k$  and index 1 and Eisenstein series of weight  $2k-2$  and level  $N$ . Moreover, using the theory of new forms of Jacobi forms, they obtained the correspondence between the spaces of these Eisenstein series for arbitrary level with trivial character.

In this paper we consider correspondences between both Eisenstein series with quadratic primitive character.

### 1. Preparations.

a) **Fourier coefficients of Jacobi forms:** Let  $N$  be an odd positive integer,  $p$  an odd prime,  $\chi_N$  a primitive Dirichlet character modulo  $N$ ,  $\chi_p$  a quadratic primitive Dirichlet character modulo  $p$ . We recall Jacobi form  $\phi_{k,m}$  of weight  $k$  and index  $m$  on  $\Gamma_0(N) \times \mathbb{Z}^2$  with  $\chi_N$ .

**Definition.** We denote  $\Gamma = \Gamma_0(N)$ . For a holomorphic function  $\phi : \mathfrak{h} \times \mathbb{C} \rightarrow \mathbb{C}$  we define the actions for fixed positive integers  $k$  and  $m$

(1)

$$a) \quad (\phi|_{k,m}M)(\tau, z) := \frac{1}{(c\tau + d)^k} e^m \left( \frac{-cz^2}{c\tau + d} \right) \phi \left( \frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right)$$

$$\text{(with } M := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \text{ and } e^m(x) := e^{2\pi i m x}),$$

$$b) \quad (\phi|_m X)(\tau, z) := e^m(\lambda^2\tau + 2\lambda z)\phi(\tau, z + \lambda\tau + \mu) \quad (X = [\lambda, \mu] \in \mathbb{Z}^2)$$

and define an action of the semi-direct product  $\Gamma \ltimes \mathbb{Z}^2$  with group law  $(M, X)(M', X') = (MM', XM' + X')$ . This group  $\Gamma \ltimes \mathbb{Z}^2$  is called Jacobi group on  $\Gamma$ . A Jacobi form of weight  $k$  and index  $m$  ( $k, m \in \mathbb{N}$ ) on the group  $\Gamma$  with the character  $\chi_N$  is a holomorphic function  $\phi : \mathfrak{h} \times \mathbb{C} \rightarrow \mathbb{C}$  satisfying

$$i) \quad \phi|_{k,m}M = \chi_N(d)\phi \quad (M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma),$$

$$ii) \quad \phi|_m X = \phi \quad (X \in \mathbb{Z}^2),$$

iii)  $\phi$  is holomorphic at any cusp of  $\Gamma_0(N)$ , namely, for each  $M \in SL_2(\mathbb{Z})$ ,  $\phi|_{k,m}M$  has a Fourier expansion of the form

$$\sum_{\substack{n \in \mathbb{N}_0, r \in \mathbb{Z} \\ 4mn - r^2 n_M \geq 0}} c_M(n, r) q^{n/n_M} \zeta^r \quad (q := e(\tau), \zeta := e(z))$$

with a natural number  $n_M$  depending on  $M$  and with  $c(n, r) = 0$  unless  $n \geq r^2 n_M / 4m$ .

(If  $\phi$  satisfies the strong condition " $c(n, r) \neq 0 \Rightarrow n > r^2 n_M / 4m$ ", it is called a cusp form). The vector space of all such functions  $\phi$  is denoted by  $J_{k,m}(\Gamma, \chi_N)$ ; if it is not confused we write simply  $J_{k,m,N}$  for  $J_{k,m}(\Gamma, \chi_N)$  and  $J_{k,m,N}^{\text{cusp}}$  is the space of cusp forms.

**Remark.** As above, the Fourier coefficients  $c_M(n, r)$  are depending on  $M \in SL_2(\mathbb{Z})$ ,

$$D_M := r^2 n_M - 4mn \text{ and } r \pmod{2m}: \sum_{\substack{D_M, r \in \mathbb{Z}, D_M \leq 0 \\ D_M \equiv r^2 n_M \pmod{4m}}} c_M(D_M, r) q^{(r^2 n_M - D_M)/(4mn_M)} \zeta^r.$$

Since the calculations in this paper are analogueous for all  $n_M$ , we write down our calculations and results simply with the form

$$\phi_{k,m}(\tau, z) = \sum_{\substack{D, r \in \mathbb{Z} \\ D \leq 0 \\ D \equiv r^2 \pmod{4m}}} c(D, r) q^{(r^2 - D)/4m} \zeta^r.$$

b) **Operators  $V_l, U_l$ :** We now define Operators on  $J_{k,m,N}$ .

We have operators  $V_l : J_{k,m,N} \rightarrow J_{k,ml,N}$  and  $U_l : J_{k,m,N} \rightarrow J_{k,m,l^2,N}$  defined by

$$(2a) \quad (\phi|V_l)(\tau, z) := l^{k-1} \sum_{\substack{M \in \Gamma_0(N) \setminus M_2^*(N) \\ \det M = l}} \frac{\chi_N(a)}{(c\tau + d)^k} e\left(\frac{-lcz^2}{c\tau + d}\right) \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{lz}{c\tau + d}\right)$$

where  $M_2^*(N) = \{M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \det M \neq 0, N|c, (a, N) = 1\}$ , and

$$(2b) \quad (\phi|U_l)(\tau, z) := \phi(\tau, lz).$$

c) **Operator  $T_l$ :** For  $\phi_{k,m,N} = \sum_{\substack{D \leq 0 \\ r \in \mathbb{Z} \\ D \equiv r^2 (4m)}} c(D, r) q^{\frac{r^2 - D}{4m}} \zeta^r \in J_{k,m,N}$  and a positive integer  $l$  prime to  $mN$ , we define the Operator  $T_l : J_{k,m,N} \rightarrow J_{k,m,N}$  as follows:

$$\phi_{k,m}|T_l = \sum_{\substack{D \leq 0 \\ r \in \mathbb{Z} \\ D \equiv r^2 \pmod{4m}}} c^*(D, r) q^{\frac{r^2 - D}{4m}} \zeta^r$$

are related to  $c(\cdot, \cdot)$  by

$$(4) \quad c^*(D, r) = \sum_{\substack{a, r' \\ a|l^2, a^2|l^2 D \\ l^2 D/a^2 \equiv 0, 1 \pmod{4} \\ r'^2 \equiv l^2 D/a^2 \pmod{4m} \\ ar' \equiv lr \pmod{2m}}} \chi_N(a) \varepsilon_D(a) a^{k-2} c\left(\frac{l^2}{a^2} D, r'\right).$$

Here  $\varepsilon_D(\cdot)$  is as [EZ] p.50, i.e. defined by

$$\varepsilon_D(a) = \begin{cases} \varepsilon_{D_0}(a/g^2)g & \text{if } (a, D) = g^2 \text{ with } D/g^2 \equiv 0, 1 \pmod{4}, \\ 0 & \text{otherwise.} \end{cases}$$

For  $l$  and  $l'$  both prime to  $mN$  we have an equation

$$(6) \quad T_l \cdot T_{l'} = \sum_{d|(l, l')} d^{2k-3} T_{ll'/d^2}.$$

d) **Atkin-Lehner Involution on  $J_{k,m,N}$**  (cf. [EZ]p.60, [MR1]p.2613, [SZ]): For  $n|mN$  (i.e.  $n|mN$ , and  $n$  and  $mN/n$  are coprime), we have  $W_n$  with

$$(7) \quad \phi|W_n = \sum_{D, r} c(D, \lambda_n r) q^{\frac{r^2 - D}{4m}} \zeta^r$$

where  $\lambda_n$  is the modulo  $mN$  uniquely determined integer which satisfies  $\lambda_n \equiv -1 \pmod{2n}$  and  $\lambda_n \equiv +1 \pmod{2mN/n}$ . Thus the  $W_n$  form a group of involutions. Finally, note that the  $W_n$  and  $T_l$  commute, as is easily seen by (4) and (7).

**Remark:** 1)  $T_l$  and  $W_n$  are hermitian.

2)  $U_l, V_l$  commute with all  $T_{l'}$  ( $(l', lmN) = 1$ ), and we have

$$(7a) \quad \begin{aligned} U_l \circ W_n &= W_{(n, mN)} \circ U_l \quad (n \parallel mNl^2) \\ V_l \circ W_n &= W_{(n, mN)} \circ V_l \quad (n \parallel mNl). \end{aligned}$$

**2. Construction of Eisenstein series:** Let be  $k \geq 4$ . In this paragraph we construct Jacobi-Eisenstein series, and define their spaces. From now on we consider the case  $N = p$ .

a) **Eisenstein series**  $E_{k, m, \chi_p}^\kappa(\tau, z)$ : Using the condition  $4nm - r^2 \geq 0$  and the decomposition  $m = m_1 m_2^2$  ( $m_1$  : squarefree) we can construct  $m_2$  Jacobi-Eisenstein series  $E_{k, m, s, \chi_p}^\infty$  ( $s = 0, 1, \dots, m_2 - 1$ ) of the cusp  $\infty$  given by

$$(8) \quad E_{k, m, s, \chi_p}^\infty(\tau, z) = \sum_{\gamma \in \Gamma_\infty^J \setminus \Gamma^J} q^{m_1 s^2} \zeta^{2m_1 m_2 s} | \gamma \quad (s = 0, 1, \dots, m_2 - 1)$$

with

$$\Gamma^J = \{(M, [\lambda, \mu]) \mid M \in \Gamma, \lambda, \mu \in \mathbb{Z}\}$$

and its subgroup

$$\Gamma_\infty^J = \{\gamma \in \Gamma^J \mid 1|_{k, m} \gamma = 1\} = \{(\pm \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}, [0, \mu]) \mid n, \mu \in \mathbb{Z}\}$$

where 1 denotes the constant function.

Using (1a) and (1b) this is rewritten explicitly

$$(8a) \quad \begin{aligned} E_{k, m, s, \chi_p}^\infty(\tau, z) &= \frac{1}{2} \sum_{\substack{c, d \in \mathbb{Z} \\ (c, d) = 1 \\ c \equiv 0 \pmod{p}}} \frac{\chi_p(d)}{(c\tau + d)^k} \sum_{\substack{\lambda \in \mathbb{Z} \\ u \equiv 2m_1 m_2 s \\ \pmod{2m}}} e^m \left( \left( \lambda + \frac{u}{2m} \right)^2 M\tau + 2 \left( \lambda + \frac{u}{2m} \right) \frac{z}{c\tau + d} - \frac{cz^2}{c\tau + d} \right) \\ &= \frac{1}{2} \sum_{\substack{u \in \mathbb{Z} \\ u \equiv 2m_1 m_2 s \\ \pmod{2m}}} q^{u^2/4m} (\zeta^u + (-1)^k \zeta^{-u}) + \underbrace{\dots}_{c \neq 0 \text{ part}}, \quad (s = 0, 1, \dots, m_2 - 1). \end{aligned}$$

For the cusp 0 we define  $E_{k,m,s,\chi_p}^0(\tau, z)$  by

$$(8b) \quad E_{k,m,s,\chi_p}^0(\tau, z) := \tau^{-k} e^{m \left( \frac{-z^2}{\tau} \right)} E_{k,m,s,\chi_p}^\infty \left( -\frac{1}{\tau}, \frac{z}{\tau} \right).$$

**b) Eisenstein series  $E_{k,m,t,\chi_p}^{\kappa,(\chi)}$ :** For the greatest integer whose square divides  $mp$ , i.e. for  $Q(mp) = \prod_{p^\lambda \parallel mp} p^{\lfloor \lambda/2 \rfloor}$ , we set

$$(9) \quad E_{k,m,t,\chi_p}^{\kappa,(\chi)}(\tau, z) := \sum_{s \pmod{Q(mp)/t}} \chi(s) E_{k,m,ts,\chi_p}^\kappa(\tau, z)$$

where  $t$  are divisors of  $Q(mp)$ , and  $\chi$  is a primitive Dirichlet character modulo  $F$  with  $F \mid \frac{Q(mp)}{t}$  and  $\chi(-1) = (-1)^k$ .

Then we can show

$$(9a) \quad \begin{aligned} E_{k,m,t,\chi_p}^{\infty,(\chi)} | T_l &= \sigma_{2k-3,\chi_p^2}^{(\chi)}(l) E_{k,m,t,\chi_p}^{\infty,(\chi)} & \text{if } (l, mp) = 1 \\ E_{k,m,t,\chi_p}^{\infty,(\chi)} | W_n &= \chi(\lambda_n) E_{k,m,t,\chi_p}^{\infty,(\chi)} & \text{if } n \parallel mp \end{aligned}$$

where  $\sigma_{k-1,\chi_p^2}^{(\chi)}(l) := \sum_{0 < d \mid l} d^{k-1} \chi_p^2(l/d) \overline{\chi(d)} \chi(l/d)$ , and  $\lambda_n$  denotes any integer as above.

**c) Eisenstein series  $E_{k,\chi_p^2}^{\kappa,(\chi)}(\tau)$ :** (cf. [Miy]). We define Eisenstein series in the space of  $M_k(mp, \chi_p^2)$  by

$$(10) \quad E_{k,\chi_p^2}^{\infty,(\chi)}(\tau) = \sum_{l \geq 0} \sigma_{k-1,\chi_p^2}^{(\chi)}(l) q^l \text{ with } \sigma_{k-1,\chi_p^2}^{(\chi)}(l) := \sum_{0 < d \mid l} d^{k-1} \chi_p^2(l/d) \overline{\chi(d)} \chi(l/d) \quad (l \neq 0)$$

$$\text{and } \sigma_{k-1,\chi_p^2}^{(\chi)}(0) := \begin{cases} 0 & (m > 1) \\ \frac{1}{2} (1 - p^{k-1}) \cdot \zeta(1-k) = \frac{1}{2} L(1-k, \chi_p^2) & (m = 1), \end{cases}$$

$$\text{and } E_{k,\chi_p^2}^{0,(\chi)}(\tau) := \tau^{-k} E_{k,\chi_p^2}^{\infty,(\chi)} \left( \frac{-1}{p\tau} \right).$$

**d) The space of cuspforms, Eisenstein series and new forms:** We define  $J_{k,m,p}^{\text{Eis,new}} := J_{k,m}^{\text{Eis,new}}(\Gamma_0(p), \chi_p)$  as the span of the functions  $E_{k,m,1,\chi_p}^{\kappa,(\chi)}$  ( $\kappa = 0, \infty$ ) if  $mp = F^2$  ( $F \in \mathbb{N}$ ) and 0 otherwise, i.e.

$$J_{k,m,p}^{\text{Eis,new}} = \begin{cases} \langle E_{k,m,1,\chi_p}^{\kappa,(\chi)} \mid \chi \pmod{F} \rangle \text{ if } mp = F^2, \\ \{0\} \text{ otherwise.} \end{cases}$$

Analogous to [EZ] we can estimate  $E_{k,m,1,\chi_p}^{\kappa,(\chi)}$  from  $E_{k,1,1,\chi_p}^{\kappa,(\chi)}$  by  $U_l$  and  $V_l$  operators:

$$E_{k,m,1,\chi_p}^{\kappa,(\chi)} = m^{-k+1} \prod_{q|m} (1+q^{-k+1})^{-1} \sum_{d^2|m} \mu(d) E_{k,1,1,\chi_p}^{\kappa,(\chi)} |U_d V_{m/d^2}$$

Moreover we can show:

$$(11) \quad J_{k,m,p}^{\text{Eis}} = \bigoplus_{\substack{l,l' \\ l^2 l' | mp \\ l_1 l_2 = l^2 l'}} J_{k, \frac{m}{l_1}, \frac{p}{l_2}}^{\text{Eis,new}} |U_l V_{l'}.$$

We define the space of Eisenstein series of modular forms.  $M_k^{\text{Eis,new}}(mp, \chi_p^2)$  is defined to be zero if  $mp$  is not a square, while if  $mp$  is a square then  $M_k^{\text{Eis,new}}(mp, \chi_p^2)$  is defined to be the span of the series  $E_{k,\chi_p^2}^{0,(\chi)}(\tau)$  and  $E_{k,\chi_p^2}^{\infty,(\chi)}(\tau)$ . For  $d \geq 1$  we define an operator  $B_d$  by

$$(f|B_d)(\tau) := f(d\tau).$$

Using this operator we obtain

$$M_k^{\text{Eis}}(mp, \chi_p^2) = \bigoplus_{rd|mp} M_k^{\text{Eis,new}}(r, \chi_p^2) |B_d.$$

**3. Main Theorem:** In this paragraph we construct the mapping between Eisenstein series, and gain the main theorem of this paper:

**Definition.**  $M_{2k-2}^-(mp, \chi_p^2)$  denote the space of all forms  $f \in M_{2k-2}(mp, \chi_p^2)$  satisfying the functional equation

$$f\left(\frac{-1}{mp\tau}\right) = (-1)^k (mp)^{k-1} \tau^{2k-2} f(\tau).$$

(Here the minus-sign means that the  $L$ -series of such an  $f$  satisfies a functional equation under  $s \rightarrow 2k - 2 - s$  with root number  $-1$  and, in particular, vanishes at  $s = k - 1$ ). Let be

$$M_{2k-2}^{\text{Eis},-}(mp, \chi_p^2) := M_{2k-2}^{\text{Eis}}(mp, \chi_p^2) \cap M_{2k-2}^-(mp, \chi_p^2).$$

Now we have followings:

**Theorem.** Let  $k, m$  be integers with  $k \geq 4$  even and  $m > 0$ . Let  $\chi, \chi_p$  be as above. For any fixed fundamental discriminant  $d_0 < 0$  and any fixed integer  $r_S$  with  $d_0 \equiv r_S^2 \pmod{4m}$  there is a map

$$\mathcal{S}_{d_0, r_S} : J_{k, m}^{\text{Eis}}(\Gamma_0(p), \chi_p) \xrightarrow{\text{into}} M_{2k-2}^{\text{Eis}, -}(mp, \chi_p^2)$$

given by

$$\sum_{\substack{D \leq 0 \\ r \in \mathbb{Z} \\ D \equiv r^2 \pmod{4m}}} e_{k, m}^{\kappa, (\chi)}(D, r) q^{\frac{r^2 - D}{4m}} \zeta^r \mapsto \sum_{l \geq 0} \left\{ \sum_{a|l} a^{k-2} \chi_p(a) \varepsilon_{d_0}(a) e_{k, m}^{\kappa, (\chi)}\left(\frac{l^2}{a^2} d_0, \frac{l}{a} r_S\right) \right\} q^l$$

with the convention

$$\sum_{a|0} a^{k-2} \chi_p(a) \varepsilon_{d_0}(a) e_{k, m}^{\kappa, (\chi)}(0, 0) := \frac{1}{2} e_{k, m}^{\kappa, (\chi)}(0, 0) \cdot L(2 - k, \chi_p \cdot \varepsilon_{d_0}).$$

The maps  $\mathcal{S}_{d_0, r_S}$  commute with all Hecke operators  $T_l$  ( $(l, mp) = 1$ ) and Atkin-Lehner involutions  $W_n$  ( $n \parallel mp$ ), and map Eisenstein series to Eisenstein series.

**Remark:** 1) For integers  $l, n > 0$  with  $(l, mp) = 1$  and  $n \parallel mp$  we denote by  $T_l$  and  $W_n$  the  $l$ -th Hecke operator and the  $n$ -th Atkin-Lehner involution on  $M_k(mp, \chi_p^2)$ , respectively. Thus, for any  $f \in M_k(mp, \chi_p^2)$  one has

$$f|T_l = \sum_{r \geq 0} \sum_{d|(l, r)} \chi_p^2(d) d^{k-1} a_f\left(\frac{lr}{d^2}\right) q^r$$

$$f|W_n(\tau) = n^{k/2} (cmp\tau + nd)^{-k} f\left(\frac{an\tau + b}{cmp\tau + dn}\right).$$

Here  $a_f(\tau)$  denote the  $r$ -th Fourier coefficients of  $f$  and  $a, b, c, d$  are any integers satisfying  $adn^2 - bcmp = n$ .

Let be

$$M_{2k-2}^{\text{new}, -}(mp, \chi_p^2) := M_{2k-2}^{\text{new}}(mp, \chi_p^2) \cap M_{2k-2}^-(mp, \chi_p^2).$$

Comparing 9a) with the description of Eisenstein series  $E_{2k-2, \chi_p^2}^{\infty, (\chi)} \in M_{2k-2}^{\text{Eis}, \text{new}}(mp, \chi_p^2)$  with  $mp = F^2$ , we gain

$$E_{2k-2, \chi_p^2}^{\kappa, (\chi)}|T_l = \sigma_{2k-3, \chi_p^2}^{\kappa, (\chi)}(l) E_{2k-2, \chi_p^2}^{\kappa, (\chi)}$$

$$E_{2k-2, \chi_p^2}^{\kappa, (\chi)}|W_n = \chi(\lambda_n) E_{2k-2, \chi_p^2}^{\kappa, (\chi)}$$



where  $(l, F^2) = 1, n \parallel F^2$  and  $\lambda_n \equiv -1 \pmod{2n}$  and  $\lambda_n \equiv +1 \pmod{2mp/n}$ .

2) For the proof we need the explicit form of Fourier coefficients of Jacobi-Eisenstein series  $E_{k,1,\chi_p}^\infty$ , which are calculated in [Hay2] analogous to [EZ] p17, where

$$\begin{aligned} E_{k,m,\chi_p}^\infty(\tau, z) &:= \sum_{\gamma \in \Gamma_\infty^J \setminus \Gamma^J} (1|_{k,m}\gamma)(\tau, z) \\ &= \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1 \\ c \equiv 0 \pmod{N}}} \frac{\chi_p(d)}{(c\tau + d)^k} \sum_{\lambda \in \mathbb{Z}} e^m \left( \lambda^2 \frac{a\tau + b}{c\tau + d} + 2\lambda \frac{z}{c\tau + d} - \frac{cz^2}{c\tau + d} \right). \end{aligned}$$

For the Fourier coefficients we have

**Proposition.** Assume  $\chi_p(-1) = (-1)^k$ . Then the series  $E_{k,m,\chi_p}^\infty$  ( $k \geq 4$ : even) for the cusp  $\kappa = \infty$  converges and defines a non-zero element of  $J_{k,m,p}$ . For

$$E_{k,m,\chi_p}^\infty(\tau, z) = \sum_{\substack{\tau, D \in \mathbb{Z} \\ D \leq 0 \\ D \equiv r^2 \pmod{4m}}} e_{k,m}^\infty(D, r) q^{\frac{r^2 - D}{4m}} \zeta^r.$$

we have the constant terms  $e_{k,m}^\infty(0, r)$  equals 1 if  $r \equiv 0 \pmod{2m}$  and 0 otherwise.

For  $D < 0$  we have

$$e_{k,1,\chi_p}^\infty(D, r) = \lambda_{k,p,\chi_p} \cdot \frac{L_D(2-k, \chi_p)}{L(3-2k, \chi_p^2)} \cdot \frac{c_p(2k-3, D)}{1-p^{2-2k}} \text{ with } \lambda_{k,p,\chi_p} = \frac{\chi_p(-1)^{1/2}}{p^{k-1/2}}$$

and  $c_p(2k-3, D)$  elementary representation with  $p^{2k-3}$  and  $D$ .

Using  $V_l, U_l$  operators we can determine the Fourier coefficients with  $m > 1$  (see;2).

The  $L$ -functions  $L(s, \chi_p^2)$  and  $L(s, \varepsilon_{D_0} \cdot \chi_p)$  are given by  $L(s, \chi_p^2) = \sum_{n>0} \chi_p^2(n) n^{-s}$  and convolution  $L(s, \varepsilon_{D_0} \cdot \chi_p) = \sum \varepsilon_{D_0}(n) \chi_p(n) n^{-s}$ , respectively. For  $D = D_0 f^2$  with  $f \in \mathbb{Z}$  and fundamental discriminant  $D_0$  we define

$$L_D(s, \chi_p) = \begin{cases} 0 & \text{if } D \not\equiv 0, 1 \pmod{4}, \\ L(2s-1, \chi_p) & \text{if } D = 0, \\ L(s, \varepsilon_{D_0} \cdot \chi_p) \cdot \gamma_{D_0, \chi_p}^s(f) & \text{if } D \equiv 0, 1 \pmod{4}, D \neq 0 \end{cases}$$

where  $\gamma_{D_0, \chi_p}^s(f) = \sum_{d|f} \mu(d) \varepsilon_{D_0}(d) \chi_p(d) d^{-s} \sigma_{1-2s, \chi_p^2}(f/d)$

with  $\sigma_{s, \chi_p^2}(d) = \sum_{d'|d} \chi_p^2(d') d'^s$ .

Proof: [Hay2].

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