Title	FINDING FAMILIES OF UNITS OF CYCLIC FIELDS (Algebraic number theory and related topics)
Author(s)	Thaine, F.
Citation	数理解析研究所講究録 (2005), 1451: 207-215
Issue Date	2005-10
URL	http://hdl.handle.net/2433/47738
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録 1451 巻 2005 年 207-215

FINDING FAMILIES OF UNITS OF CYCLIC FIELDS

F. THAINE

Department of Mathematics and Statistics - CICMA, Concordia University,
1455, de Maisonneuve Blvd. W., Montreal, Quebec, H3G 1M8, Canada

ABSTRACT. Let $m > 2$. Let K be the $m \times m$ matrix $[\delta_{i+1,j}]_{i,j}$, where $\delta_{i,j} = 1$ if $i \equiv j \bmod m$ and $\delta_{i,j} = 0$ otherwise. Let $1 \leq k \leq \varphi(m)$, $t_1, \ldots, t_k$ indeterminates, $\alpha = t_1 + t_2\zeta_m + \cdots + t_k\zeta_m^{k-1}$ and $q = N(\alpha) \in \mathbb{Z}[t_1, \ldots, t_k]$, where N is the norm from $\mathbb{Q}(t_1, \ldots, t_k, \zeta_m)$ to $\mathbb{Q}(t_1, \ldots, t_k)$. Suppose q is irreducible. We show a method to construct an $m \times m$ matrix C, with entries in $\mathbb{Q}[t_1, \ldots, t_k]$, which has the following properties: The characteristic polynomial Φ of C is cyclic over $\mathbb{Q}(t_1, \ldots, t_k)$. The field $\mathbb{Q}(t_1, \ldots, t_k)[C]$ is a splitting field for Φ and the roots of Φ in this field are the conjugates $C, K^{-1}CK, \ldots, K^{-(m-1)}CK^{m-1}$ of C. If we give $t_1, \ldots, t_k$ integer values, such that q is a prime number (so $q \equiv 1 \bmod m$), then C is (basically) the matrix of cyclotomic numbers of order m corresponding to q, the $K^{-i}CK^i$ can be identified with the Gaussian periods of degree m in $\mathbb{Q}(\zeta_q)$ and the units of the cyclic subfield of degree m of $\mathbb{Q}(\zeta_q)$ can be identified with the linear combinations, over $\mathbb{Z}$, of the $K^{-i}CK^i$ which have determinant ± 1. This gives us the following method to find families of units of cyclic fields of degree m at the parameter t. Regard the t_i as elements of $\mathbb{Z}[t]$. Find linear combinations over $\mathbb{Z}[t]$ of the matrices $K^{-i}CK^i$, $i = 0, \ldots, m-1$, which have determinant ± 1. The characteristic polynomials of these matrices, when they are irreducible, are cyclic, and their roots are units. We show several examples of such families of units. When m is prime and a matrix C as above, with entries in $\mathbb{Q}[t]$, is given there is a procedure, found by René Schoof and to be published elsewhere, to decide whether or not this method will produce a family of units.

1) CYCLIC POLYNOMIALS

I thank the organizer Professor Masanori Morishita for the invitation. I also thank Professor Ki-ichiro Hashimoto for his support and for some useful comments, and Professor Masanari Kida for showing me some material that I used in preparing this talk. This work originated in a question posed to me by René Schoof about generalizing a family of cyclic polynomials found by Emma Lehmer.

Let $D = \mathbb{Z}[t_1, \ldots, t_k]$, with $t_1, \ldots, t_k$ indeterminates. Let $\mathbb{K}$ be the field of fractions of D and $\overline{\mathbb{K}}$ an algebraic closure of $\mathbb{K}$. Let $m \geq 2$ and $P = x^m + c_{m-1}x^{m-1} + \cdots + c_0 \in D[x]$ a cyclic polynomial; that is P is irreducible over $\mathbb{K}$ with cyclic Galois group. We can regard P as a family of cyclic polynomials at the parameters $t_1, \ldots, t_k$. We are interested in finding such families P whose constant term c_0 is 1 or -1. Those are the polynomials P as above whose roots are algebraic units when we give integer values to the parameters $t_1, \ldots, t_k$. Those families are, in general, difficult to construct for arbitrary m. To show a few examples:

For $m = 2$, $P = x^2 + tx \pm 1$.
For $m = 3$, $P = x^3 - tx^2 - (t+3)x - 1$, studied by D. Shanks (1974).
For $m = 5$, $P = x^5 + t^2x^4 - (2t^3 + 6t^2 + 10t + 10)x^3 + (t^4 + 5t^3 + 11t^2 + 15t + 5)x^2 + (t^3 + 4t^2 + 10t + 10)x + 1$, Emma Lehmer, 1988 (reference [2]). If for some integer t the number $q = t^4 + 5t^3 + 15t^2 + 25t + 25$ is prime, then P is cyclic and any set of four of its roots is a fundamental system of units of the rink of integers of its decomposition field. This family was used by Schoof and Washington (1988) to construct a real p-cyclotomic field with class number divisible by a large prime (reference [3]).
Families for $m = 4$ and $m = 6$ were found by M-N. Gras (1977, 1987).
For $m = 8$ a family was found by Y.Y. Shen (1988).
More examples and references can be found in [6].
For $m = 7$ no such families are known, but Professors Hashimoto and Hoshi, using their geometric method to construct families of cyclic polynomials, have found (reference [1]) a family $x^7 - (t^3 + t^2 + 5t + 6)x^6 + \cdots + t^7$, with constant term t^7.
In this talk we show a method to search for such families, of cyclic polynomials with constant term ± 1, for arbitrary $m \geq 2$ and give several examples.

Let $\theta_0, \theta_1, \ldots, \theta_{m-1}$ be the roots of P in $\overline{\mathbb{K}}$, $L = \mathbb{K}[\theta_0] = \mathbb{K}[\theta_0, \ldots, \theta_{m-1}]$ and τ a generator of $\mathrm{Gal}(L/K)$. Suppose that $\theta_0, \theta_1, \ldots, \theta_{m-1}$ are linearly independent over $\mathbb{K}$ and that they are labeled so that $\tau(\theta_i) = \theta_{i+1}$ (indices modulo m). For $0 \leq i, j \leq m-1$ define the elements $a_{i,j} \in \mathbb{K}$ by

$$\theta_0 \theta_i = \sum_{j=0}^{m-1} a_{i,j} \theta_j. \tag{1}$$

Let $A = [a_{i,j}]_{0 \leq i,j < m}$. We call A the multiplication matrix of the θ_i.

By applying powers of τ to (1), we get, for all i, j,

$$\theta_i \theta_j = \sum_{k=0}^{m-1} a_{i-j,k-j} \theta_k \tag{2}$$

(indices modulo m).

We use the following version of Kronecker's delta: for $i, j \in \mathbb{Z}$,

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i \equiv j \bmod m \\ 0 & \text{if } i \not\equiv j \bmod m. \end{cases}$$

Let K be the $m \times m$ matrix $[\delta_{i+1,j}]_{i,j}$; that is $K = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \end{bmatrix}$. We have $K^m = I$, the identity matrix. It follows from formula (2) and a little linear algebra, that P is the characteristic polynomial of A, that $\mathbb{K}[A]$ is a splitting field for P and that the conjugates $K^{-i}AK^i$ of A $(i = 0, \ldots, m-1)$ belong to $\mathbb{K}[A]$. It follows that P splits in $\mathbb{K}[A][x]$ as $P = (x - A)(x - K^{-1}AK) \ldots (x - K^{-(m-1)}AK^{m-1})$. Hence we can identify the conjugates $K^{-i}AK^i$ with the roots θ_i of P.

We will show how to construct some multiplication matrices as above, whose characteristic polynomials are cyclic with constant term ± 1. By knowing the multiplication matrices we get more information than by just knowing the irreducible polynomials. For example, suppose m and n are relatively prime integers ≥ 2. Suppose A and B are multiplication matrices of orders m and n respectively of the roots cyclic polynomials over D with constant terms ± 1. Then we can construct a cyclic polynomial over D, with constant term ± 1 of degree mn by calculating first the multiplication matrix of its roots. So, for example, since we can construct such polynomials for m =2, 4, 8, 3 and 5, we can construct several families of cyclic polynomials at two parameters of degrees 10, 12, 15, 20, 30, etc., even families at three parameters of degree 120. In fact, let $A = [a_{i,j}]$ and $B = [b_{i,j}]$. Denote by θ_i the roots of $\det(xI - A)$ and by η_i the roots of $\det(xI - B)$. They are units. Since $\gcd(m, n) = 1$ we have that the $\theta_i \eta_j$, $0 \leq i \leq m-1$, $0 \leq j \leq n-1$, are linearly independent (over $\mathbb{K}$) units in the composite field $\mathbb{K}[\theta_0, \eta_0]$. If we arrange these elements as $\theta_0\eta_0, \theta_1\eta_1, \ldots, \theta_{mn-1}\eta_{mn-1}$ (indices of θ modulo m and indices of η modulo n), then their multiplication matrix is the $mn \times mn$ matrix $E = [e_{i,j}]$ with $e_{i,j} = a_{|i|_m,|j|_m} b_{|i|_n,|j|_n}$, where $|i|_m$ is the integer such that $0 \leq |i|_m < m-1$ and $|i|_m \equiv i \bmod m$. (This fact had also been noticed by Professor Hashimoto who pointed out that E is the tensor product of the matrices A and B.) As we said before, $P = \det(xI - E)$ is a cyclic polynomial of degree mn with constant term equal to ± 1. We will see later examples of this composition.

2) Generalizations of Gaussian periods

We are going to construct the multiplication matrices of some generalizations of Gaussian periods. Since these generalizations behave in a very similar manner than the actual Gaussian periods, we recall some properties of these numbers. Let $m > 2$. Let $q \equiv 1 \bmod m$ be a prime number, $f = (q-1)/m$, ζ_q a primitive q-th root of 1 and s a primitive root modulo q. For $0 \leq i \leq m-1$, define

$$\eta_i = \sum_{j=0}^{f-1} \zeta_q^{s^{i+mj}}.$$

These are the Gaussian periods of degree m in $\mathbb{Q}[\zeta_q]$. They are real numbers if f is even and complex nonreal if f is odd. The Gaussian periods $\eta_0, \eta_1, \ldots, \eta_{m-1}$ form a normal integral basis of the only subfield ($\mathbb{Q}[\eta_0]$) of degree m of $\mathbb{Q}[\zeta_q]$. In particular all units of the ring of integers of this subfield are linear combination over $\mathbb{Z}$ of the η_i. The irreducible polynomial of these periods is $\Phi = (x-\eta_0)(x-\eta_1)\ldots(x-\eta_{m-1}) \in \mathbb{Z}[x]$; it is a cyclic polynomial. Its multiplication matrix is basically the matrix of cyclotomic numbers of order m. Call it $C = [c_{i,j}]$. We can express the numbers $c_{i,j}$ in terms of some Jacobi sums $J_{a,b}$ and we can construct those Jacobi sums using Stickelberger Theorem and some roots of 1. More precisely, let ζ_m be a primitive m-th root of 1. The Jacobi sums $J_{a,b}$, $0 \leq a, b \leq m-1$, are elements of $\mathbb{Q}[\zeta_m]$ defined by

$$J_{a,b} = -\sum_{k=2}^{q-1} \zeta_m^{a \operatorname{ind}_s(k) + b \operatorname{ind}_s(1-k)}$$

where $\mathrm{ind}_s(k)$ is the least nonnegative integer such that $s^{\mathrm{ind}_s(k)} \equiv k \bmod q$. We have

$$c_{i,j} = -f\delta_{(q-1)/2,i} - \frac{1}{m^2}\sum_{a=0}^{m-1}\sum_{b=0}^{m-1}(-1)^{fa}\zeta_m^{-ia-jb}J_{a,b}.$$

Let Q be the prime ideal of $\mathbb{Z}[\zeta_m]$ over q such that $s^f \equiv \zeta_m \bmod Q$, and suppose that Q is a principal ideal: $Q = (\alpha)$. Then by Stickelberger Theorem we have that, for all a, b,

$$(J_{a,b}) = \Big(\prod_{\substack{1\le c<m\\(c,m)=1}} \sigma_c^{-1}(\overline{\alpha})^{\left[\frac{(a+b)c}{m}\right]-\left[\frac{ac}{m}\right]-\left[\frac{bc}{m}\right]}\Big)$$

(an equality of ideals), where the over bar is complex conjugation and $[\rho]$ is the integral part of the real number ρ. To take out the brackets we have to multiply by some roots of unity, this can be tricky when m is not a prime number. We have that $q = N_{\mathbb{Q}[\zeta_m]/\mathbb{Q}}(\alpha)$. This is the only prime that ramifies in $\mathbb{Q}[\eta_0]/\mathbb{Q}$, and if $(q) = \mathcal{P}^m$, then the ideal $\mathcal{P}$ divides all elements $\mu = d_0\eta_0 + d_1\eta_1 + \cdots + d_{m-1}\eta_{m-1}$ with $d_i \in \mathbb{Z}$ and $d_0 + d_1 + \cdots + d_{m-1} = 0$. So q divides the norm (from $\mathbb{Q}[\eta_0]$ to $\mathbb{Q}$) of such an element μ.

As we said before, we can apply this method to a much more general situation, working, for example, over D instead of $\mathbb{Z}$. We start with $m > 2$ and $\alpha = t_1 + t_2\zeta_m + \cdots + t_k\zeta_m^{k-1}$, $t_1, \ldots, t_k$ indeterminates, $1 \le k \le \varphi(m)$, such that $q = N_{\mathbb{K}[\zeta_m]/\mathbb{K}}(\alpha)$ is irreducible (or at least squarefree). By the above method we get matrices C that behave very much like matrices of cyclotomic numbers. In particular their characteristic polynomials Φ are cyclic of degree m over $\mathbb{K}$ and the matrices $C, K^{-1}CK, \ldots, K^{-(m-1)}CK^{m-1}$ can be identified with the roots of Φ, that is with the generalized Gaussian periods. (For a more detailed account of cyclic polynomials over characteristic zero domains see [5].) We have a simple MAPLE program to construct such $m \times m$ matrices C (see the last section of [4] for C generalizing real Gaussian periods, and [5] for C generalizing complex Gaussian periods), and once we have such C our problem of finding families of units of cyclic fields becomes one of elementary algebra; indeed, we are searching for matrices M of the form $M = b_0C + b_1K^{-1}CK + \cdots + b_{m-1}K^{-(m-1)}CK^{m-1}$, with $b_i \in D$ such that $\det(M) = \pm 1$ (determinants correspond to norms) and such that $g(x) = \det(xI - M)$ is irreducible, then automatically $g(x)$ is cyclic. In searching for such matrices M, a very useful trick is the following. The matrices $N = d_0C + d_1K^{-1}CK + \cdots + d_{m-1}K^{-(m-1)}CK^{m-1}$ such that $d_0 + d_1 + \cdots + d_{m-1} = 0$ are (as in the case of linear combinations, over $\mathbb{Z}$, of Gaussian periods) such that q divides their norms, that is $q|\det(N)$. It is easier to find matrices N with $\det(N) = \pm q$ than matrices M with $\det(M) = \pm 1$. Once we have such a matrix N, then any of the matrices $K^{-i}NK^iN^{-1}$, $i = 2, \ldots, m-1$, has determinant ± 1 and characteristic polynomial in $D[x]$. This is basically our method. Now we show some examples.

3) Examples

For $m = 3$, with $\alpha = n + t\zeta_3$, we get $q = n^2 - nt + t^2$ and

$$C = \frac{1}{9}\begin{bmatrix} -a-b-9 & a-3(q-1) & b-3(q-1) \\ a & b & c \\ b & c & a \end{bmatrix},$$

where $a = q+n-2t-2$, $b = q+n+t-2$ and $c = q-2n+t+1$.

Let $W = C - K^{-1}CK$. We have that $\det(W) = tq/3$. For $A = K^{-1}WKW^{-1}$, we have that $P = \det(xI - A) = x^3 + 3\frac{n}{t}x^2 + 3\frac{n-t}{t}x - 1$.

Taking $t = 1$, we get $q = n^2 - n + 1$ and $P = x^3 + 3nx^2 + (3n-3)x - 1$.

Taking $t = 3$, we get $q = n^2 - 3n + 9$ and $P = x^3 + nx^2 + (n-3)x - 1$, Shanks' polynomial.

Also we have, for example, $\det(W - K^{-1}WK) = (2n-t)q$, $\det(W - 2K^{-1}WK - K^{-2}WK^2) = (6n+t/3)q$, $\det(W - 2K^{-1}WK + 2K^{-2}WK^2) = (-12n-17t/3)q$ and $\det(W - 3K^{-1}WK + 2K^{-2}WK^2) = (-20n - 17t)q$. By finding values of n and t, depending on one parameter, such that those determinants equal $\pm q$ we can obtain more families of cyclic polynomials; for example:

For $t = 2n+1$, we have $q = 3n^2+3n+1$, and for $U = W - K^{-1}WK$ and $A = K^{-1}UKU^{-1}$, we get $P = \det(xI - A) = x^3 + (9n+6)x^2 + (9n+3)x - 1$.

In this and in the following examples, observe that P has the form $x^3 + ux^2 + (u-3)x - 1$ as before, but the conductor is distinct than that corresponding to the Shanks' polynomial, in particular, when n runs through the integers, the sets of prime values taken by the conductors q are different.

For $t = -18n + 3$, we have $q = 343n^2 - 111n + 9$, and for $U = W - 2K^{-1}WK - K^{-2}WK^2$ and $A = K^{-1}UKU^{-1}$, we get $P = \det(xI - A) = x^3 + (343n - 54)x^2 + (343n - 57)x - 1$.

For $n = -17u + 7$ and $t = 36u - 15$, we have $q = 2197u^2 - 1825u + 379$, and for $U = W - 2K^{-1}WK + 2K^{-2}WK^2$ and $A = K^{-1}UKU^{-1}$, we get
$P = \det(xI - A) = x^3 + (2197u - 911)x^2 + (2197u - 914)x - 1$.

For $n = -17u - 6$ and $t = 20u + 7$, we have $q = 1029u^2 + 723u + 127$, and for $U = W - 3K^{-1}WK + 2K^{-2}WK^2$ and $A = K^{-1}UKU^{-1}$, we get
$P = \det(xI - A) = x^3 + (3087u + 1086)x^2 + (3087u + 1083)x - 1$.

For $m = 4$, with $\alpha = n + t\zeta_4$, we get $q = n^2 + t^2$ and

$$C = \frac{1}{16}\begin{bmatrix} -a-b-c-16 & a-4(q-1) & b-4(q-1) & c-4(q-1) \\ a & c & d & d \\ b & d & b & d \\ c & d & d & a \end{bmatrix},$$

where $a = q + 2n - 4t - 3$, $b = q + 2n - 3$, $c = q + 2n + 4t - 3$ and $d = q - 2n + 1$.

Let $W = C - K^{-1}CK$. We have that $\det(W) = -t^2q/16$. For $A = K^{-1}WKW^{-1}$, we have that $P = \det(xI - A) = x^4 + 4\frac{n}{t}x^3 - 6x^2 - 4\frac{n}{t}x + 1$.

Taking $t = 1$ and $t = 2$, we get some families.

Taking $t = 4$, we get $q = n^2 + 16$ and $P = x^4 + nx^3 - 6x^2 - nx + 1$, first found by M-N. Gras.

For $A = K^{-2}WK^2W^{-1}$, we have that $P = x^4 + 4x^3 - \frac{16n^2+10t^2}{t^2}x^2 + 4x + 1$.

Taking $t = 1$, $t = 2$ and $t = 4$, we get more families.

Let $U = W - K^{-1}WK$. We have that $\det(U) = t^2q/4$. For $A = K^{-1}UKU^{-1}$, we have that $P = \det(xI - A) = x^4 + \frac{8n^2+8t^2+4nt}{t^2}x^3 + \frac{20n^2+14t^2}{t^2}x^2 + \frac{8n^2+8t^2-4nt}{t^2}x + 1$.

Taking $t = 1$ and $t = 2$, we get more families.

For $A = K^{-2}UK^2U^{-1}$, we have that
$P = \det(xI - A) = x^4 + \frac{-16n^2-12t^2}{t^2}x^3 + \frac{32n^2+38t^2}{t^2}x^2 + \frac{-16n^2-12t^2}{t^2}x + 1$.

Taking $t = 1$, $t = 2$ and $t = 4$, we get more families.

Also we have, for example, $\det(W - 3K^{-2}WK^2 + 3K^{-3}WK^3) = -(24n + 7t)^2q/16$. If we take $n = -7u - 8$ and $t = 24u + 28$, then $q = 625u^2 + 1456u + 848$ and, for $U = W - 3K^{-2}WK^2 + 3K^{-3}WK^3$ and $A = K^{-1}UKU^{-1}$, we get
$P = \det(xI - A) = x^4 - (625u + 728)x^3 - 6x^2 + (625u + 728)x + 1$

For $m = 5$, with $\alpha = n + t\zeta_5 + u\zeta_5^2 + v\zeta_5^3$, we get

$q = 2v^2ut + 2vt^2n + 2vu^2n - 3v^2tn + 2v^2un + v^4 + 2u^2tn + v^2n^2 + 2vtn^2 - u^3n - v^3t - vutn - u^3t + u^2n^2 - tn^3 - un^3 - ut^3 + u^2t^2 - vn^3 - vt^3 - vu^3 + v^2t^2 + v^2u^2 - v^3n - 3vun^2 - 3ut^2n + n^4 + 2vut^2 - 3vu^2t + 2utn^2 + t^2n^2 + u^4 + t^4 - t^3n - v^3u$ and

$$C = \frac{1}{25}\begin{bmatrix} -a-b-c-d-25 & a-5(q-1) & b-5(q-1) & c-5(q-1) & d-5(q-1) \\ a & d & e & f & e \\ b & e & c & f & f \\ c & f & f & b & e \\ d & e & f & e & a \end{bmatrix},$$

where
$a = q - 4 + 3n^2 + 3v^2 - 9tn - 2u^2 + 3t^2 + 6tu - 4nu - 4vt + vn - 4vu$,
$b = q - 4 + 3n^2 - 7v^2 + tn + 3u^2 + 3t^2 - 4tu - 9nu + vt + 6vn + vu$,
$c = q - 4 + 3n^2 + 3v^2 - 4tn - 7u^2 - 2t^2 + 6tu + 6nu + 6vt - 9vn + vu$,
$d = q - 4 + 3n^2 - 2v^2 + 6tn + 3u^2 - 7t^2 + tu + nu + 6vt - 4vn - 4vu$,
$e = q + 1 - 2n^2 - 2v^2 + tn - 2u^2 + 3t^2 - 4tu + nu + vt + vn + 6vu$ and
$f = q + 1 - 2n^2 + 3v^2 + tn + 3u^2 - 2t^2 + tu + nu - 4vt + vn - 4vu$.

With $\alpha = n+2+\zeta_5+2\zeta_5^2$, we get $q = n^4+5n^3+15n^2+25n+25$. Let $W = C-K^{-1}CK$. We have that $\det(W) = -q$.

For $A = -K^{-2}WK^2W^{-1}$, we have that $P = \det(xI - A) =$
$x^5+n^2x^4-(2n^3+6n^2+10n+10)x^3+(n^4+5n^3+11n^2+15n+5)x^2+(n^3+4n^2+10n+10)x+1$,
Lehmer's polynomial.

For $A = K^{-1}WKW^{-1}$, we have that $P = \det(xI - A) =$
$x^5 + (2n^2 + 5n + 10)x^4 + (n^4 + 5n^3 + 17n^2 + 25n + 25)x^3 + (n^4 + 3n^3 + 7n^2 + 5n + 5)x^2 - (n^3 + 3n^2 + 5n + 5)x - 1$.

For $m = 6$, with $\alpha = n + t\zeta_6$, we get $q = n^2 + nt + t^2$ and

$$C = \frac{1}{36}\begin{bmatrix} -a-b-c-d-e-36 & a-6(q-1) & b-6(q-1) & c-6(q-1) & d-6(q-1) & e-6(q-1) \\ a & e & f & f & f & f \\ b & f & d & f & f & f \\ c & f & f & c & f & f \\ d & f & f & f & b & f \\ e & f & f & f & f & a \end{bmatrix},$$

where $a = q + 4n - 7t - 5$, $b = q + 4n - t - 5$, $c = q + 4n + 2t - 5$, $d = q + 4n + 5t - 5$, $e = q + 4n + 11t - 5$ and $f = q - 2n - t + 1$. Let $W = C - K^{-1}CK$. We have that $\det(W) = -t^4 q/5184$.
For $A = K^{-1}WKW^{-1}$, we have that $P = \det(xI - A) =$
$x^6 + 6\frac{n}{t}x^5 - 15\frac{n+t}{t}x^4 + 20x^3 + 15\frac{n}{t}x^2 - 6\frac{n+t}{t}x + 1$.
Taking $t = 1$, we get $q = n^2 + n + 1$ and $P = x^6 + 6nx^5 - (15n + 15)x^4 + 20x^3 + 15nx^2 - (6n + 6)x + 1$.
Taking $t = 3$, we get $q = n^2 + 3n + 9$ and $P = x^6 + 2nx^5 - (5n + 15)x^4 + 20x^3 + 5nx^2 - (2n + 6)x + 1$, first found by M-N. Gras.

For $A = K^{-2}WK^2W^{-1}$, we have that $P = \det(xI - A) =$
$x^6 + 6\frac{n+t}{t}x^5 - 3\frac{24n^2+24t^2+19nt}{t^2}x^4 + 20\frac{9n^2+8t^2+9nt}{t^2}x^3 - 3\frac{24n^2+29t^2+29nt}{t^2}x^2 - 6\frac{n}{t}x + 1$.

Taking $t = 1$ and $t = 3$, we get more families.

For $A = K^{-3}WK^3W^{-1}$, we have that $P = \det(xI - A) =$
$x^6 + 6x^5 - 3\frac{48n^2+43t^2+48nt}{t^2}x^4 + 4\frac{72n^2+77t^2+72nt}{t^2}x^3 - 3\frac{48n^2+43t^2+48nt}{t^2}x^2 + 6x + 1$.

Taking $t = 1$, $t = 2$, $t = 3$, $t = 4$, $t = 6$ and $t = 12$, we get more families.

For $m = 7$ we could not find a family of cyclic polynomials in $\mathbb{Z}[x]$ whose roots are units, but the following family of cyclic polynomials, with constant term n^7, was found by Professors Hashimoto and Hoshi using a different method (see [1]).

$P = x^7 - (n^3 + n^2 + 5n + 6)x^6 + (9n^3 + 9n^2 + 24n + 12)x^5 + (n^7 + n^6 + 9n^5 - 5n^4 - 15n^3 - 22n^2 - 36n - 8)x^4 - (n^8 + 5n^7 + 12n^6 + 24n^5 - 6n^4 + 2n^3 - 20n^2 - 16n)x^3 + (2n^8 + 7n^7 + 19n^6 + 14n^5 + 2n^4 + 8n^3 - 8n^2)x^2 - (n^8 + 4n^7 + 8n^6 + 4n^4)x + n^7$.

This can be obtained using our method as follows: Take $\alpha = n - 2 + (1 - \zeta_7)(1 - \zeta_7^3)(1 + \zeta_7 + \zeta_7^3)$ and $U = C - K^{-2}CK^2$. We have that $\det(U) = -n^4 q$. With $A = -nK^{-2}UK^2U^{-1}$, we have that $P = \det(xI - A)$. Here $q = N(\alpha) = n^6 + 2n^5 + 11n^4 + n^3 + 16n^2 + 4n + 8$. If we take $A_1 = nK^{-6}UK^6U^{-1}$, we get another family $P_1 = \det(xI - A_1) =$

$x^7 - (n^5 + 2n^4 + 11n^3 + 2n^2 + 10n + 2)x^6 + (n^9 + 4n^8 + 16n^7 + 26n^6 + 31n^5 + 45n^4 + 21n^3 + 12n^2 + 12n - 4)x^5 + (n^{11} + 2n^{10} + 12n^9 + 7n^8 + 39n^7 + 58n^6 + 74n^5 + 68n^4 + 74n^3 + 24n^2 + 16n + 8)x^4 - (n^{12} + 4n^{11} + 16n^{10} + 27n^9 + 33n^8 + 54n^7 + 19n^6 + 2n^5 - 9n^4 - 14n^3 - 20n^2)x^3 - (2n^{11} + 7n^{10} + 30n^9 + 40n^8 + 56n^7 + 66n^6 + 46n^5 + 30n^4 + 20n^3 + 8n^2)x^2 - (n^{10} + 3n^9 + 12n^8 + 10n^7 + 12n^6 + 10n^5 + 4n^4)x - n^7$.

Also, for $\alpha = n - 1 + (1 - \zeta_7)(1 - \zeta_7^3)$ and $U = C - K^{-2}CK^2$, we have that $\det(U) = n^4 q$. If we take $A_2 = -nK^{-1}UKU^{-1}$, we get $P_2 = \det(xI - A_2) =$

$x^7+(n^3+4n^2+3n+6)x^6+(3n^5+6n^4+15n^3+15n^2+12n+12)x^5+(3n^7+4n^6+20n^5+11n^4+27n^3+6n^2+12n+8)x^4+(n^9+n^8+8n^7+n^6+7n^5-17n^4-2n^3-20n^2)x^3-(n^8+7n^7+11n^6+19n^5+16n^4+8n^3+8n^2)x^2-(n^9+5n^7-4n^6+2n^5-4n^4)x+n^7.$

If we take $A_3=nK^{-5}UK^5U^{-1}$, we get $P_3=\det(xI-A_3)=$

$x^7+(2n^3+n^2+6n-2)x^6+(n^6+n^5+5n^4+n^3-2n^2-16n-4)x^5-(n^7+12n^5+13n^4+36n^3+16n^2-8)x^4-(n^8+4n^7+6n^6+10n^5-26n^3-28n^2-16n)x^3+(n^9+2n^8+9n^7+18n^6+38n^5+36n^4+28n^3+8n^2)x^2+(n^9+2n^8+5n^7+5n^6+4n^5+4n^4)x-n^7.$

Here $q=N(\alpha)=n^6+n^5+8n^4+n^3+22n^2+8n+8$.

Observe that if $q_1(n)=n^6+2n^5+11n^4+n^3+16n^2+4n+8$ and $q_2(n)=n^6+n^5+8n^4+n^3+22n^2+8n+8$, then $q_2(n)=n^6q_1(2n^{-1})/8$ and $q_1(n)=n^6q_2(2n^{-1})/8$.

For $m=8$, take for example $\alpha=n+t\zeta_8$, we get $q=n^4+t^4$. Let $U=C-K^{-2}CK^2$. We have that $\det(U)=-t^{12}q/4096$.

Take $t=1$. For $A=K^{-1}UKU^{-1}$, we have $P=\det(xI-A)=x^8+8nx^7+(-16n^4+28n^2-16)x^6+(-64n^5-16n^4+56n^3-64n-16)x^5+(-64n^6+96n^4-64n^2+26)x^4+(64n^6+128n^5+16n^4+64n^2+72n+16)x^3+(-32n^5-16n^4-28n^2-32n-16)x^2-8n^3x+1$.

For $A=K^{-2}UK^2U^{-1}$, we have $P=\det(xI-A)=x^8+(16n^4-8n^2+16)x^7+(80n^4+52)x^6+(-64n^6+64n^4-8n^2+64)x^5+(-64n^6-48n^4-64n^2+22)x^4+(-16n^4-56n^2-16)x^3+(16n^4-12)x^2+8n^2x+1$.

For $A=K^{-4}UK^4U^{-1}$, we have $P=\det(xI-A)=x^8+(-16n^4-8)x^7+(64n^6-32n^4+64n^2-4)x^6+(16n^4+72)x^5+(-128n^6-192n^4-128n^2-122)x^4+(16n^4+72)x^3+(64n^6-32n^4+64n^2-4)x^2+(-16n^4-8)x+1$.

Take $t=2$. For $A=K^{-1}UKU^{-1}$, we have $P=\det(xI-A)=x^8+4nx^7+(-n^4+7n^2-16)x^6+(-2n^5-n^4+7n^3-32n-16)x^5+(-n^6+6n^4-16n^2+26)x^4+(n^6+4n^5+n^4+16n^2+36n+16)x^3+(-n^5-n^4-7n^2-16n-16)x^2-n^3x+1$.

For $A=K^{-2}UK^2U^{-1}$, we have $P=\det(xI-A)=x^8+(n^4-2n^2+16)x^7+(5n^4+52)x^6+(-n^6+4n^4-2n^2+64)x^5+(-n^6-3n^4-16n^2+22)x^4+(-n^4-14n^2-16)x^3+(n^4-12)x^2+2n^2x+1$.

For $A=K^{-4}UK^4U^{-1}$, we have $P=\det(xI-A)=x^8+(-n^4-8)x^7+(n^6-2n^4+16n^2-4)x^6+(n^4+72)x^5+(-2n^6-12n^4-32n^2-122)x^4+(n^4+72)x^3+(n^6-2n^4+16n^2-4)x^2+(-n^4-8)x+1$.

Now we construct families at two parameters of degrees 10 and 12. For example, for $m=2$, the matrix $A=\begin{bmatrix} n+1/n & 1/n \\ -1/n & -1/n \end{bmatrix}$ is the multiplication matrix of the roots of the polynomial $P_1=\det(xI-A)=x^2-nx-1$ (see [6]). For $m=5$, $\alpha=t-1-\zeta_5-2\zeta_5^2$ and $W=C-K^{-1}CK$, we have that $q=t^4-t^3+6t^2-6t+11$,

$$B=K^{-2}WK^2W^{-1}=\frac{1}{25}\begin{bmatrix} a & a & c & a-25 & a-25 \\ a & a & c-25 & a & a-25 \\ b & b-25 & d & f & f \\ a-25 & a & e & g & a \\ a-25 & a-25 & e & a & g+25 \end{bmatrix},$$

where $a = -t^4 + t^3 - 4t^2 + 5t$, $b = 4t^4 - 4t^3 + 21t^2 - 10t + 50$, $c = -t^4 + t^3 - 9t^2 + 20t$, $d = 4t^4 - 4t^3 + 41t^2 - 20t + 50$, $e = -t^4 + t^3 - 9t^2 - 5t$, $f = 4t^4 - 4t^3 + 21t^2 - 35t + 50$ and $g = -t^4 + t^3 - 4t^2 + 30t - 25$, and $P_2 = \det(xI - B) = x^5 + (-t^2 - 2t - 1)x^4 + (2t^3 + 4t - 4)x^3 + (-t^4 + t^3 - 2t^2 + 4t + 3)x^2 + (-t^3 + t^2 - 5t + 3)x - 1$. Let E be the composite of A and B as defined in Section 1. Then we have the following family at two parameters of degree 10.

$P = \det(xI - E) = x^{10} - n(t^2 + 2t + 1)x^9 + (2n^2t^3 + 4n^2t - 4n^2 - t^4 - 6t^2 + 4t - 9)x^8 + n(-n^2t^4 + n^2t^3 - 2n^2t^2 + 4n^2t + 3n^2 + 2t^5 + t^4 + 9t^3 - 2t^2 + 8t + 5)x^7 + (-n^4t^3 + n^4t^2 - 5n^4t - n^2t^6 - n^2t^5 - n^2t^4 - 3n^2t^3 + 13n^2t^2 - 10n^2t + 3n^4 + 15n^2 + 2t^6 - 2t^5 + 14t^4 - 16t^3 + 36t^2 - 22t + 28)x^6 - n(n^2t^5 + n^2t^4 + 4n^2t^3 + 6n^2t^2 - n^2t + n^4 + 2n^2 + 2t^7 - 2t^6 + 11t^5 - 13t^4 + 18t^3 - 6t^2 + t + 8)x^5 + (-n^4t^2 - 2n^4t - 2n^2t^6 + 2n^2t^5 - 14n^2t^4 + 14n^2t^3 - 28n^2t^2 + 24n^2t - n^4 - 16n^2 - t^8 + 2t^7 - 9t^6 + 16t^5 - 34t^4 + 38t^3 - 54t^2 + 36t - 35)x^4 - n(2n^2t^3 + 4n^2t - 4n^2 + t^7 - 2t^6 + 8t^5 - 14t^4 + 20t^3 - 23t^2 + 9t - 3)x^3 + (-n^2t^4 + n^2t^3 - 2n^2t^2 + 4n^2t + 3n^2 + t^6 - 2t^5 + 9t^4 - 14t^3 + 27t^2 - 22t + 15)x^2 + n(t^3 - t^2 + 5t - 3)x - 1.$

For $m = 3$, $\alpha = n + 3\zeta_3$ and $W = C - K^{-1}CK$, we have that $q = n^2 - 3n + 9$,

$$A = K^{-1}WKW^{-1} = \frac{1}{9}\begin{bmatrix} a & a & c \\ a & a-9 & c+9 \\ b & b+9 & d \end{bmatrix},$$

where $a = n^2 - 5n + 13$, $b = -2n^2 + 7n - 23$, $c = n^2 - 2n + 1$ and $d = -2n^2 + n - 17$, and $P_1 = \det(xI - A) = x^3 + nx^2 + (n - 3)x - 1$.

For $m = 4$, $\alpha = t + 2\zeta_4$ and $W = C - K^{-1}CK$, we have that $q = t^2 + 4$,

$$B = K^{-1}WKW^{-1} = \frac{1}{8}\begin{bmatrix} a & a & c & a \\ a & a-8 & c+8 & a \\ b & b+8 & d & b-8 \\ a & a & c-8 & a+8 \end{bmatrix},$$

where $a = t^2 - 2t + 5$, $b = -3t^2 + 2t - 11$, $c = t^2 + 2t + 1$ and $d = -3t^2 - 10t - 15$, and $P_2 = \det(xI - B) = x^4 + 2tx^3 - 6x^2 - 2tx + 1$. Let E be the composite of A and B as defined in Section 1. Then we have the following family at two parameters of degree 12.

$P = \det(xI - E) = x^{12} - 2ntx^{11} + (4nt^2 - 6n^2 + 12n - 12t^2 - 36)x^{10} + (2n^3t + 6n^2t - 18nt + 8t^3 + 30t)x^9 + (-4n^3t^2 + 20n^2t^2 - 28nt^2 + n^4 - 4n^3 + 50n^2 - 160n + 72t^2 + 342)x^8 + (-2n^4t - 8n^2t^3 + 10n^2t + 16nt^3 + 16nt - 48t^3 - 210t)x^7 + (-36n^2t^2 + 108nt^2 - 6n^4 + 36n^3 - 138n^2 + 252n - 156t^2 - 522)x^6 + (2n^4t - 24n^3t + 8n^2t^3 + 98n^2t - 32nt^3 - 140nt + 72t^3 + 234t)x^5 + (4n^3t^2 - 16n^2t^2 + 16nt^2 + n^4 - 8n^3 + 68n^2 - 140n + 60t^2 + 285)x^4 + (2n^3t - 24n^2t + 72nt - 8t^3 - 84t)x^3 + (-4nt^2 - 6n^2 + 24n - 54)x^2 + (-2nt + 6t)x + 1.$

References

1. K. Hashimoto and A. Hoshi, *Families of cyclic polynomials obtained from geometric generalization of Gaussian period relations.*, Mathematics of Computation, to appear.
2. E. Lehmer, *Connection between Gaussian periods and cyclic units*, Math. Comp. **50** (1988), 535–541.
3. R. Schoof and L. Washington, *Quintic polynomials and real cyclotomic fields with large class numbers*, Math. Comp. **50** (1988), 543–556.
4. F. Thaine, *Jacobi sums and new families of irreducible polynomials of Gaussian periods*, Math. Comp. **70, no. 236** (2001), 1617–1640.
5. F. Thaine, *Cyclic polynomials and the multiplication matrices of their roots*, J. Pure Appl. Algebra **188, no. 1-3** (2004), 247–286.
6. L. Washington, *Abelian number fields of small degree*, Proc. KAIST Math. Workshop, 5, Korea Adv. Sci. Tech., Taejon (1990), 63–78.