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Cayley Graphs in Laminations

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1 Introduction

As an analogy to hyperbolic 3-orbifolds associated with Kleinian groups, Lyubich and Minsky [7] have constructed hyperbolic orbifold 3-laminations associated with rational maps. Their construction involved in their first step the construction of natural extensions and regular leaf spaces.

However, the global structures of the regular leaf spaces of rational maps are not precisely known except only for a few examples: For $f_c(z) = z^2 + c$ with c in the main cardioid of the Mandelbrot set, all regular leaf spaces of f_c are topologically similar to that of $f_0(z) = z^2$, which is 2-dimensional extension of 2-adic solenoid[9, Example 2][7, §11]. It is also known that for $f_{1/4}(z) = z^2 + 1/4$, the regular leaf space of $f_{1/4}$ is obtained by applying pinching semiconjugacy on the regular leaf space of $f_c(z) = z^2 + c$ with c in the main cardioid of the Mandelbrot set [6]. Cases for other parameters, even hyperbolic parameters, are not well understood yet, since the Julia set in both dynamical plane and the regular leaf space is not "simple" anymore.

In this paper, as a first step toward understanding the regular leaf space of hyperbolic polynomials, we will describe the topological structure of the Julia set on the regular leaf space on $z^2 - 1$. The structure of this paper is as follows. In §3, we construct the *Cayley graph* in the regular leaf space of $z^2 - 1$ and state the main theorem (Theorem 3.1). In §4, we describe the monodromy group action on the fiber of a single point and show that the Cayley graph is actually the planer realization of this action in each leaf. In §5, we prove the main theorem 1, which is the topological classification of the Julia set in the regular leaf space in detail. In §6, we show the main theorem 2, which is the Hausdorff convergence of the Cayley graph to the Julia set by the iteration of $\hat{f}$. (Theorem 3.3). In §7, we list future problems and in the Appendix there are basic definitions and concepts in the theory of laminations, including several new definitions we suggest.

2 Preliminaries

2.1 The Julia set

We first recall some basic concepts in the dynamics of rational functions. We assume the reader be quite familiar with these concepts.

- For a rational map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, the *Fatou set* $F = F(f)$ is defined as the collection of points $z \in \overline{\mathbb{C}}$ around which the family of functions $\{f^n\}_{n=1}^{\infty}$ is normal.
- The *Julia set* $J = J(f)$ is a complement of the Fatou set in $\overline{\mathbb{C}}$.
- *Postcritical Set* $P = P(f)$ is defined as the closure of the forward orbit of all critical points.

2.2 Natural extension

Next we follow [7, §3]. For a rational map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, the *natural extension* $\mathcal{N}_f$ is the collection of backward orbits under f:

$$\mathcal{N}_f := \left\{\hat{z} = (z_0, z_{-1}, \ldots) : \ z_0 \in \overline{\mathbb{C}}, \ f(z_{-n-1}) = z_{-n}\right\}.$$

The lift of f and a natural projection are defined by

$$\begin{aligned} \hat{f}(\hat{z}) &:= (f(z_0), z_0, z_{-1}, \ldots) \ \text{ and} \\ \pi_{-n}(\hat{z}) &:= z_{-n}. \end{aligned}$$

We sometimes denote π_0 by π. This set $\mathcal{N}_f$ is equipped with a topology from $\overline{\mathbb{C}} \times \overline{\mathbb{C}} \times \cdots$. It is clear that $\hat{f}$ is a homeomorphism, and satisfies $\pi_{-n} \circ \hat{f} = f \circ \pi_{-n}$. For notational conveniences, for a periodic orbit $a_0 \mapsto a_1 \mapsto \ldots \mapsto a_n = a_0$, we will denote $(a_0, a_{n-1}, \ldots, a_1, a_0, a_{n-1}, \ldots) \in \mathcal{N}_f$

by $(\widehat{a_0, \dots a_{n-1}})$. Given a (forward) invariant set $X \subset \overline{\mathbb{C}}$, let $\hat{X} \in \mathcal{N}_f$ denote its *invariant lift* to $\mathcal{N}_f$, that is, the collection of orbits $\{z_n\} \subset X$. This is nothing but the natural extension of $f|X$. Note that it differs from $\pi^{-1}(X)$, unless X is completely invariant (that is, $f^{-1}(X) = X$).

2.3 The Regular leaf space

The *regular leaf space* $\mathcal{R}_f \in \mathcal{N}_f$ is the collection of points of $\mathcal{N}_f$ around which there is no branching point of infinite degree under π, *namely*,

$\mathcal{R}_f := \{\hat{z} = (z_0, z_{-1}, \dots) \in \mathcal{N}_f :$ There exists a neighborhood U_0 of z_0 such that its pull-back U_{-n} along the backward orbit $\hat{z}$ is eventually univalent$\}$.

A *leaf* of $\mathcal{R}_f$ is a path connected component of $\mathcal{R}_f$. We denote the leaf containing $\hat{z}$ by $L(\hat{z})$. By [7, Lemma 3.1], leaves of $\mathcal{R}_f$ are Riemann surfaces. Moreover,

Lemma 2.1 *Leaves of $\mathcal{R}_f$ have following properties:*

- *Each leaf L possess an intrinsic topology and a complex structure such that $\pi_{-n} : L \to \overline{\mathbb{C}}$ is an analytic branched covering for any n.*
- *$\pi_{-n} : L \to \overline{\mathbb{C}}$ branches at $\hat{z} = (z_0, z_{-1}, \dots) \in L$ if and only if $\hat{z}$ contains a critical point in $\{z_{-m}\}_{m>n}$.*
- *$\hat{f}$ maps $L(\hat{z})$ to $L(\hat{f}(\hat{z}))$ biholomorphically.*

Furthermore, if f is hyperbolic, then

- *Each leaf is ismorphic to the conformal plane $\mathbb{C}$.*
- *$\mathcal{R}_f$ is an* affine lamination, *namely, each transition function is an affine conformal mapping.*

Notes.

- For a general theory of laminations, see [3]. See also Appendix in this paper for basic terminologies. In this paper, we will not use any special terminologies from the theory of foliations and laminations beyond the Appendix.

- Local charts are actually given by π_{-n} for large enough n at every point in $\mathcal{R}_f$. Transition functions are given by f^m for some $m \in \mathbb{Z}$. From this, it immediately follows that $\mathcal{R}_f$ is a *Riemann surface lamination.*

Any set $\mathcal{X} \subset \mathcal{R}_f$ can be decomposed into parts sitting in each leaf, namely, $\mathcal{X} = \cup_{\hat{z} \in \mathcal{R}_f}(\mathcal{X} \cap L(\hat{z}))$. We will denote $\mathcal{X} \cap L(\hat{z})$ by $\mathcal{X}(\hat{z})$. This notation has a little ambiguity because if two different points $\hat{z} \in \mathcal{R}_f$ and $\hat{w} \in \mathcal{R}_f$ lie on the same leaf, then $\mathcal{X}(\hat{z}) = \mathcal{X}(\hat{w})$. However, this notation is convenient because it inherits the notation from $L(\hat{z})$.

2.4 The Julia set and Fatou sets in the natural extension

Since the Julia set and the Fatou set of f is completely invariant under f, we can define *the Julia set* $\mathcal{J} := \hat{J}$ and the *Fatou set* $\mathcal{F} := \hat{F}$ in the natural extension. Note that $\mathcal{J}$ is not necessarily path-connected (or locally connected) even if the Julia set $J \subset \overline{\mathbb{C}}$ is path-connected (or locally connected). Actually, $\mathcal{J}$ can be decomposed into $\mathcal{J}(\hat{z}) := \mathcal{J} \cap L(\hat{z})$, where $\hat{z}$ moves over different leaves of $\mathcal{N}_f$.

2.5 The dynamics of $z^2 - 1$

For the rest of this paper, we will restrict ourselves to the case $f(z) = z^2 - 1$. However, most theories can be generalized to any hyperbolic quadratic maps. This subsection recalls the basic dynamical properties of the map $f(z) = z^2 - 1$, and some related facts about the regular leaf space directly following from them. See [10] for details. This map f is *postcritically finite* with the postcritical set $P = \{0, -1, \infty\}$. The *immediate basin of attraction* $\mathcal{A}(\{0, -1\})$ for $\{0, -1\}$ consists of two connected components. We denote the component containing 0 by U_0, and the other component containing -1 by U_{-1}. By f, U_0 is mapped 2-1 onto U_{-1}, and U_{-1} is mapped univalently onto U_0. The basin of attraction for ∞ will be denoted by U_∞. This set U_∞ is completely invariant under f, and is mapped 2-1 onto itself. There exists a unique conformal map $\phi : \overline{\mathbb{C}} \setminus \overline{\mathbb{D}} \to U_\infty$ such that

- $f(\phi(z)) = \phi(z^2)$, and
- $\phi(z)/z \to 1$ as $z \to \infty$.

Moreover, since f is hyperbolic, this map ϕ continuously extends to $\overline{\phi}$: $\overline{\mathbb{C}} \setminus \mathbb{D} \to \overline{U_\infty}$. For each angle $t \in \mathbb{R}/\mathbb{Z}$, the *external ray* $R_t \subset \mathbb{C}$ is defined by

$$R_t = \{\phi(r \exp(2\pi i t)) : 1 < r < \infty\},$$

and for each radius $r \in (1, \infty)$, the *equipotential curve* Ω_r is defined by

$$\Omega_r = \{\phi(r \exp(2\pi i t)) : t \in \mathbb{R}/\mathbb{Z}\}.$$

To express subarcs of external rays and equipotential curves, we will often use the notation

$$R_t(r_0) := \left\{\bar{\phi}(r \exp(2\pi i t)) : 1 \le r \le r_0\right\} \text{ and}$$
$$\Omega_r(\theta_0, \theta_1) := \{\phi(r \exp(2\pi i t)) : \theta_0 \le \theta \le \theta_1, \text{ where we direct } \mathbb{R}/\mathbb{Z} \text{ clockwise}\}.$$

There are two repelling fixed points, both with real multipliers. Let β be the landing point of the invariant external ray R_0 and α be the other point. α is the landing point of $R_{1/3}$ and $R_{2/3}$, and they are transposed by the action of f. For further details on ray combinatorics, see [8].

In this special case for $z^2 - 1$, we have the following properties for the regular leaf space of f.

- $\mathcal{R}_f = \mathcal{N}_f \setminus \left\{\widehat{(0,-1)}, \widehat{(-1,0)}, \widehat{\infty}\right\}$.
- $\hat{z} = (z_0, z_{-1}, \dots) \in L(\hat{z})$ is a branching point of degree 2^k under $\pi_{-n} : L \to \overline{\mathbb{C}}$, where k is the number of -1's in $\{z_{-m}\}_{m \ge n}$.

3 Cayley graphs and Topology of Julia sets

This section will state the first main theorem, and will define the object needed to prove the theorem, the Cayley graph, and then state the second main theorem.

3.1 Main theorem 1

Let $\hat{\gamma} = (\gamma_0, \gamma_{-1}, \dots) \in \mathcal{R}_f$ be the backward orbit of $\gamma_0 := -\alpha$ where the backward orbit is taken so that all points stay on the boundary of $\mathcal{A}(\{0, -1\})$. There are two choices for taking such a backward orbit every other times we take the backward orbit from γ_{-2n+1} to γ_{-2n}, but either way will work for the statement below.

Theorem 3.1 *All Julia sets $\mathcal{J}(\hat{z})$ are connected. Every Julia set in each leaf $\mathcal{J}(\hat{z})$ is homeomorphic to one of the following:*

- $\mathcal{J}(\hat{\alpha})$. *Other than $\mathcal{J}(\hat{\alpha})$ itself, there is no $\mathcal{J}(\hat{z})$ which is homeomorphic to $\mathcal{J}(\hat{\alpha})$. The number of unbounded components in $L(\hat{\alpha}) \setminus \mathcal{J}(\hat{\alpha})$ is four.*
- $\mathcal{J}(\hat{\beta})$. *The number of unbounded components in $L(\hat{\beta}) \setminus \mathcal{J}(\hat{\beta})$ is one.*
- $\mathcal{J}(\hat{\gamma})$. *The number of unbounded components in $L(\hat{\gamma}) \setminus \mathcal{J}(\hat{\gamma})$ is two.*

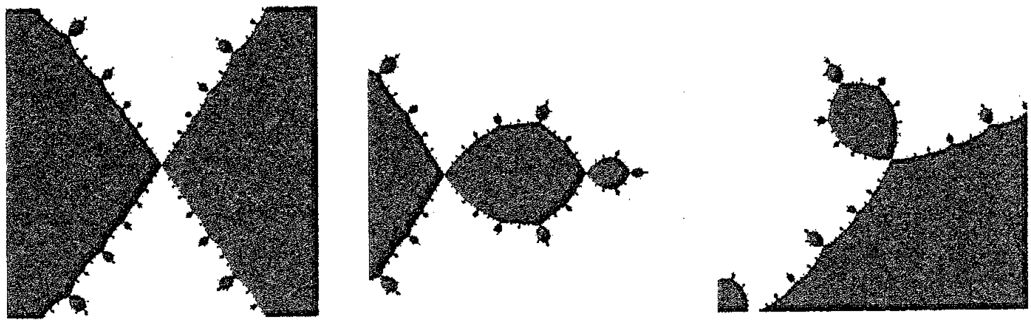

Figure 1: Filled Julia sets $\mathcal{K} = \widehat{K(f)}$ in $L(\hat{\alpha})$, $L(\hat{\beta})$, and $L(\hat{\gamma})$, where $K(f) = U_\infty^c$.

The proof will be completed at the end of §5. The main idea is to use the Cayley graph to "noose around" the Julia set.

3.2 Cayley graphs

Let $[a]$ and $[b]$ be two generators of the fundamental group $\pi_1(\overline{\mathbb{C}} \setminus P, \alpha)$, where $a, b : [0,1] \to \overline{\mathbb{C}} \setminus P$, $a(0) = a(1) = b(0) = b(1) = \alpha$. By abusing notation, we will often denote $[a]$ and $[b]$, or $a([0,1]) \subset \overline{\mathbb{C}} \setminus P$ and $b([0,1]) \subset \overline{\mathbb{C}} \setminus P$ simply by a and b. By taking $R_{1/3}(2) * \Omega_2(1/3, 2/3) * R_{2/3}(2)$ and $R_{2/3}(2) * \Omega_2(2/3, 1/3) * R_{1/3}(2)$ for a and b for example, we may take representatives a, b so that they don't intersect the Julia set (hence stay in U_∞) except they touch $\alpha \in J$ at their endpoints.

Definition. The *Cayley graph* $\mathcal{G} \subset \mathcal{R}_f$ is defined by $\pi^{-1}(a \cup b)$.

This is called a graph because it satisfies the following property:

Lemma 3.2 *For any $\hat{z} \in \mathcal{R}_f$, $\mathcal{G}(\hat{z}) \subset L(\hat{z})$ can be regarded as a locally finite planer graph, with vertices being $\pi^{-1}(\alpha) \cap L(\hat{z})$, and each edge being a connected component of $\pi^{-1}[a((0,1)) \cup b((0,1))] \cap L(\hat{z})$.*

Proof. Since $\pi : L(\hat{z}) \setminus \pi^{-1}(P) \to \overline{\mathbb{C}} \setminus P$ is a *non*-branched covering, π maps each connected component of $\pi^{-1}[a((0,1)) \cup b((0,1))] \cap L(\hat{z})$ univalently onto either $a((0,1)) \in \overline{\mathbb{C}} \setminus P$ or $b((0,1)) \in \overline{\mathbb{C}} \setminus P$. Local finiteness follows from the fact that $\pi|L(\hat{z})$ is a branched covering onto $\overline{\mathbb{C}}$. □

Remarks.

- In spite of the lemma above, $\mathcal{G}$ as a whole is *not* a graph in an ordinary sense: $\mathcal{G}$ has uncountably many path-connected components. However, $\mathcal{G}$ can be considered and should be understood as a *laminated graph*. See appendix for details.
- Actually Cayley graph in each leaf $\mathcal{G}(\hat{z})$ consists of a single path-connected component. This will be proved in §5.

3.3 Main theorem 2

The Cayley graph is a locally finite object on each leaf, so we may think that this object doesn't carry all combinatorial information about $\mathcal{J}$. However, if $\hat{f} : \mathcal{R}_f \mapsto \mathcal{R}_f$ is also given, then the following theorem shows that $\mathcal{G}$ carries *all* combinatorial information about $\mathcal{J}$.

Theorem 3.3 *For any compact set $K \subset L(\hat{z}), \hat{f}^{-n}G(\hat{f}^n(\hat{z})) \to \mathcal{J}(\hat{z})$ with respect to the Hausdorff topology on the collection of compact subsets of K.*

The proof of this theorem will be given in §6.

4 Monodromy action on the fiber $\pi^{-1}(\alpha)$

This section will explain the action of the fundamental group of $\overline{\mathbb{C}} \setminus P$ on the fiber $\pi^{-1}(\alpha)$ and its relationship with the Cayley graph.

The element of the fundamental group $\pi_1(\overline{\mathbb{C}} \setminus P, \alpha)$ acts on $\pi^{-1}(\alpha)$ in the following way: Take any element $g \in \pi_1(\overline{\mathbb{C}} \setminus P, \alpha)$. Since $\pi : L \setminus \pi^{-1}(P) \to \overline{\mathbb{C}} \setminus P$ is a covering, we can lift this path g to paths $\tilde{g}(L)$

in L for any L. The collection of these paths $\{g(L)\}$ defines the action of g on $\pi^{-1}(\alpha)$.

These actions generate a group action of $\pi_1(\overline{\mathbb{C}} \setminus P, \alpha)$ on $\pi^{-1}(\alpha)$.

Remark. The lift of g is actually *laminated* in $\mathcal{R}_f$, except at the lift of endpoints α. We suggest the terminology *laminated path* for the structure of this set. See Appendix for details.

The description of this group action is given in [1, §5]:

Theorem 4.1 *We have an identification $\pi^{-1}(\alpha) \cong \{0,1\}$ such that the action of $\hat{f}$ is conjugate to a Bernoulli shift. The group action of $\pi_1(\overline{\mathbb{C}} \setminus P, \alpha)$ on $\pi^{-1}(\alpha) \cong \{0,1\}$ is expressed by the following recursive formulae:*

$$a(0\bar{\theta}) = 1\bar{\theta}, a(1\bar{\theta}) = 0b(\bar{\theta}),$$
$$b(0\bar{\theta}) = 0b(\bar{\theta}), b(1\bar{\theta}) = 1\bar{\theta}.$$

where $\bar{\theta} = (\theta_0, \theta_1, \dots) \in \{0,1\}$ and a and b are two generators of $\pi_1(\overline{\mathbb{C}} \setminus P, \alpha)$.

Remark. This encoding $\pi^{-1}(\alpha) \cong \{0,1\}$ is not canonical as we will see in the proof below, but the expression *is* uniquely determined modulo conjugacy by $\pi(\overline{\mathbb{C}} \setminus P, \alpha)$. This statement easily follows from the proof below.

Proof Outline.

Step1. Coding tree. We first associate a digit $\bar{\theta} \in \{0,1\}$ for any point in $\pi^{-1}(\alpha)$. This procedure is generally known as *coding tree*, and can be described as follows. Label α by ϕ, an empty set, for notational conveniences. Connect α with each point of $f^{-1}(\alpha)$ by two paths in $\overline{\mathbb{C}} \setminus P$ (one of them happens to be a loop since $f^{-1}(\alpha) = \{\alpha, -\alpha\}$). Label them as l_x and l_y. Label two endpoints of l_x and l_y by x and y, respectively. Now we proceed by induction to label all points in $f^{-n}(\alpha)$ for $n > 0$. Suppose we have the n-character label for all points in $f^{-n}(\alpha)$. By taking the n-th iterated pullbacks of l_x and l_y, we have 2^n paths connecting each point in $f^{-n}(\alpha)$ to each point in $f^{-(n+1)}(\alpha)$. When a point q in $f^{-(n+1)}(\alpha)$

is connected from a point p in $f^{-n}(\alpha)$ by the iterated pullbacks of l_x (or l_y respectively), we add *to the left* a new character "x" (or "y" respectively) to the label of $p \in f^{-n}(\alpha)$ and use this to label $q \in f^{-(n+1)}(\alpha)$. Now we have assigned for an each point in $(z_0, z_{-1}, \dots) \in \pi^{-1}(\alpha)$ an encoding $(e_0, e_1, \dots)$, where $e_i \in \{x, y\}^n$ and $\sigma^{right}(e_{i+1}) = e_i$, where σ^{right} is the truncation of the rightmost character. By taking the limit of e_i and by identifying $\{x, y\}$ to $\{0, 1\}$, we obtain the encoding $\pi^{-1}(\alpha) \cong \{0, 1\}$. By construction, it is clear that the action of $\hat{f}$ is conjugate to a Bernoulli shift.

Remark. When we select l_x and l_y, we have the freedom of choice by the right action of $\pi_1(\overline{\mathbb{C}} \setminus P, \alpha)$ on the set of paths from α to $f^{-1}(\alpha)$. To fix our ideas, we take the trivial loop in $\pi_1(\overline{\mathbb{C}} \setminus P, \alpha)$ as l_x, and $R_{2/3}(2) * \Omega_2(2/3, 5/6) * R_{5/6}(2)$ as l_y in the following discussion.

Step2. Describing monodromy action. We will keep using the notation $\{x, y\}$ instead of $\{0, 1\}$ because the former notation is less confusing for the discussion below. Let us call $T = \cup_{n=0}^{\infty} f^{-n}(l_x \cup l_y)$ a *coding tree*. Notice that a and $b \in \pi_1(\overline{\mathbb{C}} \setminus P, \alpha)$ acts on T: The action on the n-the level points $f^{-n}(\alpha) \subset T$ is given by the n-th iterated pullbacks of a and b by f starting from each point of $f^{-n}(\alpha)$, and since the pullback of T is itself a part of T, the edges are mapped to edges by these actions. Since a circles around the critical value -1, the action of a on the first level $f^{-1}(\alpha) \cong \{x, y\}$ is a transposition. Let us call two preimages of a by p and q, where p starts from x and q starts from y. To determine the action on $f^{-(n+1)}(\alpha) \cong \{x, y\}^{n+1}$, notice that the path determining the action on $f^{-(n+1)}(\alpha)$ are pullbacks of p and q by f^n. To be more precise, the action on the point $x\theta \in \{x, y\}^{n+1}, \theta \in \{x, y\}^n$ (or $y\theta$, respectively) is determined by the pullback of p (or q, respectively). To locate these pullbacks, we consider about pulling back the whole loop $l_x * p * l_y^{-1}$ (or $l_y * q * l_x^{-1}$, respectively) based on α. Since $l_x * p * l_y^{-1}$ is homotopic to identity and $l_y * q * l_x^{-1}$ is homotopic to b in $\overline{\mathbb{C}} \setminus P$, we obtain

$$a(x\theta) = (y, l_x * p * l_y^{-1}(\theta)) = (y, \mathrm{id}(\theta)),$$
$$a(y\theta) = (x, l_y * q * l_x^{-1}(\theta)) = (x, b(\theta)).$$

Similarly we get the recursive expression for b as well. By passing to the limit, we get the expression in the theorem. $\square$

By the theorem above and by the definition of Cayley graph $\mathcal{G}$, the following proposition follows immediately. This is why we use the terminology *Cayley* graph.

Proposition 4.2 *Cayley graph $\mathcal{G}$ is the realization of this group action in $\mathcal{R}_f$. Namely, two points $\hat{z}, \hat{w} \in \pi^{-1}(\alpha)$ is connected by a single edge if and only if $g(\hat{z}) = \hat{w}$, where $g \in \{a, b, a^{-1}, b^{-1}\} \subset \pi_1(\overline{\mathbb{C}} \setminus P, \alpha)$.*

5 Proof of main theorem 1

Throughout the proof, we will often specify a point in $\pi^{-1}(\alpha)$ by its encoding $\{0,1\}$. We will use the notation $\hat{z}, \hat{w}$, etc., when we directly mention points in $\pi^{-1}(\alpha)$, and $\bar{\theta}, \bar{\eta}$, etc., when we mention points in $\pi^{-1}(\alpha)$ via $\{0,1\}$.

5.1 Combinatorics of $\mathcal{G}(\hat{z})$ and its relation to $\mathcal{J}(\hat{z})$

We first start with a lemma about the action of a and b on the fiber $\pi^{-1}(\alpha)$. Before stating this lemma, we need one definition. Let $\bar{\theta} = (\theta_0, \theta_1, \dots) \in \{0,1\}$. The *adding machine* add : $\{0,1\} \to \{0,1\}$ is defined by the following recursive formula: $\mathrm{add}(0\bar{\theta}) = 1\bar{\theta}$, $\mathrm{add}(1\bar{\theta}) = 0\mathrm{add}(\bar{\theta})$.

Lemma 5.1 *Let $\bar{\theta} = (\theta_0, \theta_1, \dots) \in \{0,1\}$.*

1. *If $\theta_{2k} = 0$ for $k < n$ and $\theta_{2n} = 1$, then $\langle b \rangle$ acts cyclically with order 2^n. This action restricted on $\{\bar{\eta} : \eta_i = \theta_i$ except for $i = 2k-1, 1 \leq k \leq n\}$ is conjugate to $+1 : \mathbb{Z}/2^n\mathbb{Z} \to \mathbb{Z}/2^n\mathbb{Z}$ via the identification $\bar{\eta} \mapsto \Sigma_{k=1}^{n} \eta_{2k-1} 2^{k-1}$.*

2. *If $\theta_{2k-1} = 0$ for $k < n$ and $\theta_{2n-1} = 1$, then $\langle a \rangle$ acts cyclically with order 2^n. This action restricted on $\{\bar{\eta} : \eta_i = \theta_i$ except for $i = 2k, 0 \leq k \leq n-1\}$ is conjugate to $+1 : \mathbb{Z}/2^n\mathbb{Z} \to \mathbb{Z}/2^n\mathbb{Z}$ via the identification $\bar{\eta} \mapsto \Sigma_{k=0}^{n-1} \eta_{2k} 2^k$.*

3. *If $\theta_{2k} = 0$ for all k, then $\langle b \rangle$ acts freely. This action restricted on $\{\bar{\eta} : \eta_{2i} = 0$ for all $i\}$ is conjugate to* add : $\{0,1\} \to \{0,1\}$ *via the identification $\bar{\eta} \mapsto \{\eta_{2k-1}\}_{k \geq 1}$.*

4. *If $\theta_{2k-1} = 0$ for all k, then $\langle a \rangle$ acts freely. This action restricted on $\{\bar{\eta} : \eta_{2i-1} = 0$ for all $i\}$ is conjugate to* add : $\{0,1\} \to \{0,1\}$ *via the identification $\bar{\eta} \mapsto \{\eta_{2k}\}_{k \geq 0}$.*

5. *If $\bar{\theta} = (\theta_0, \theta_1, \dots), \bar{\eta} = (\eta_0, \eta_1, \dots)$ satisfies $g(\bar{\theta}) = \bar{\eta}$ for some element $g \in \pi_1(\overline{\mathbb{C}} \setminus P, \alpha)$, then either there exists an $N > 0$ such that $\theta_n = \eta_n$ for all $n > N$ or $\mathcal{G}(\bar{\theta}) = \mathcal{G}(\bar{\eta}) = \mathcal{G}(\bar{0})$.*

Proof. The proof is straightforward by the recursive definition of a and b. $\square$

Next we show the main lemma which relates $\mathcal{G}$ to $\mathcal{J}$ by homotopy.

Lemma 5.2 *$\mathcal{G}(\hat{z})$ is homotopic to $\mathcal{J}(\hat{z})$ in $L(\hat{z}) \setminus \pi^{-1}(P)$ rel $\pi^{-1}(\alpha) \cap L(\hat{z})$. Therefore, the number of unbounded components of $L(\hat{z}) \setminus \mathcal{G}(\hat{z})$ is equal to the number of unbounded components of $L(\hat{z}) \setminus \mathcal{J}(\hat{z})$.*

Proof. Since we selected a and b so that they lie inside U_∞ except that endpoints are $\alpha \in \partial U_\infty$, we can shrink $a \cup b$ onto $J(f)$ by homotopy in $\overline{\mathbb{C}} \setminus P$ rel α. We can lift this homotopy by π into $L(\hat{z}) \setminus \pi^{-1}(P)$ to obtain the homotopy in the first statement. The second statement immediately follows from the first statement. $\square$

5.2 Connectivity of $\mathcal{J}(\hat{z})$

To complete the statement of the theorem about unbounded components in the compliment, we have to investigate whether we have more than two separate path-connected components of $\mathcal{J}(\hat{z})$ in one leaf. The following lemma claims that this is impossible.

Lemma 5.3 *$\mathcal{J}(\hat{z})$ (or equivalently, $\mathcal{G}(\hat{z})$) is path connected.*

Proof. Take two points $\hat{z} = (z_0, z_{-1}, \dots)$ and $\hat{w} = (w_0, w_{-1}, \dots)$ in $\mathcal{J}(\hat{z})$.

Step1. Connecting $\hat{z}$ to a point in $\pi^{-1}(\alpha)$. Since f is hyperbolic, all point in $J \subset \overline{\mathbb{C}}$ is a landing point of some external ray. Take one ray R_t landing on z_0. Let us connect $z_0 \in \overline{\mathbb{C}}$ to α by a path $l = R_t(2) * \Omega_2(t, 1/3) * R_{1/3}(2)$. Since $l \subset \overline{\mathbb{C}} \setminus P$, we can lift this path by π and obtain a path $\tilde{l}$ connecting $\hat{z}$ to some point in $\pi^{-1}(\alpha)$. Since we can shrink l by homotopy rel $\{z_0, \alpha\}$ in $\overline{\mathbb{C}} \setminus P$ onto a path-connected subset of J by following external rays, we can shrink $\tilde{l}$ by homotopy rel $\pi^{-1}(\alpha) \cup \{\hat{z}\}$ in $L(\hat{z}) \setminus \pi^{-1}(P)$ onto a path-connected subset of $\mathcal{J}(\hat{z})$. This gives a path in $\mathcal{J}(\hat{z})$connecting $\hat{z}$ to some point in $\pi^{-1}(\alpha)$.

Step2. Connecting two points in $\pi^{-1}(\alpha)$. By *Step1*, we may assume $\hat{z}, \hat{w} \in \pi^{-1}(\alpha)$. Let us connect $\hat{z}$ to $\hat{w}$ by a path $\tilde{l}$ in $L(\hat{z}) \setminus \pi^{-1}(P)$. This is possible because $\pi^{-1}(P) \cap L(\hat{z})$ is a locally finite set. Let us now project $\tilde{l}$ by π in $\overline{\mathbb{C}} \setminus P$ and call it l. Since l is a loop based on α, l is homotopic rel α in $\overline{\mathbb{C}} \setminus P$ to some loop l' generated by $\{a, b, a^{-1}, b^{-1}\} \subset \pi_1(\overline{\mathbb{C}} \setminus P, \alpha)$. By taking a further homotopy following external rays onto J, l' is homotopic in $\overline{\mathbb{C}} \setminus P$ to a path-connected subset of J. We can lift these two homotopies by π, and obtain a path-connected subset of $\mathcal{J}(\hat{z})$ which connects $\hat{z}$ to $\hat{w}$. □

By the above three lemmas, we have the following.

Proposition 5.4 *Let $\bar{\theta} = (\theta_0, \theta_1, \dots) \in \{0,1\}$ be the encoding for $\pi^{-1}(\alpha)$.*

1. *$\mathcal{J}(\bar{0})$ is the unique Julia set which has 4 unbounded components in $L(\bar{0}) \setminus \mathcal{J}(\bar{0})$.*
2. *If $\bar{\theta}$ satisfies either ($\theta_{2k} = 0$ for k large enough) or ($\theta_{2k-1} = 0$ for k large enough), but not both, then $\mathcal{J}(\bar{\theta})$ has 2 unbounded components in $L(\bar{\theta}) \setminus \mathcal{J}(\bar{\theta})$.*
3. *If neither ($\theta_{2k} = 0$ for k large enough) nor ($\theta_{2k-1} = 0$ for k large enough), then $\mathcal{J}(\bar{\theta})$ has 1 unbounded component.*

5.3 Construction of homeomorphisms

Now we are at the stage of constructing the continuous map between $\mathcal{J}(\hat{z})$ of the same type in 5.4. We only need consider cases 2 and 3. First we introduce some notations to decompose $\mathcal{J}(\hat{z})$ into small pieces. When we are in case 2, we can find $\bar{\theta} \in \pi^{-1}(\alpha)$ in $\mathcal{J}(\hat{z})$ which satisfies [either ($\theta_{2k} = 0$ for all k) or ($\theta_{2k-1} = 0$ for all k), but not both]. On this $\bar{\theta}$, either $\langle a \rangle$ or $\langle b \rangle$, but not both, acts freely. Let us denote this infinite orbit of points $\{a^n(\bar{\theta})\}_{n\in}$ or $\{b^n(\bar{\theta})\}_{n\in}$ by $\{\bar{\theta}^n\}_{n\in}$. by 5.2, if we remove one such point $\bar{\theta}^n$ from $\mathcal{J}(\hat{z})$, we will obtain two unbounded components and one bounded component. Let us denote this bounded component by $\mathcal{J}_b(\bar{\theta}^n)$ (where "b" stands for the "bubble"). Now if we remove two points $\bar{\theta}^n$ and $\bar{\theta}^{n-1}$, then we have two unbounded components, two bounded components of the form $\mathcal{J}_b(\bar{\theta}^n)$ and $\mathcal{J}_b(\bar{\theta}^{n-1})$, and one more bounded component in between $\bar{\theta}^n$ and $\bar{\theta}^{n-1}$. Let us denote this component by $\mathcal{J}_a(\bar{\theta}^n, \bar{\theta}^{n-1})$ (where "a" stands for the "arc"). Now $\mathcal{J}(\hat{z})$ can be decomposed as follows:

$$\mathcal{J}(\hat{z}) \setminus \{\bar{\theta}^n\}_{n\in} = \bigcup_{n\in} (\mathcal{J}_b(\bar{\theta}^n) \cup \mathcal{J}_a(\bar{\theta}^{n-1}, \bar{\theta}^n)). \tag{1}$$

When we are in case 3 in 5.4, neither $\langle a \rangle$ nor $\langle b \rangle$ acts free on any point in $\pi^{-1}(\alpha) \cap \mathcal{J}(\hat{z})$, but we have the following lemma.

Lemma 5.5 *There exists a sequence $\{\bar{\theta}^n\}_{n=0}^{\infty}$ which satisfies the following:*

- $\bar{\theta}^{2n} \in \langle b \rangle(\bar{\theta}^{2n-1})$ *and* $\bar{\theta}^{2n+1} \in \langle a \rangle(\bar{\theta}^{2n})$ *for all* n.
- $\bar{\theta}^n \to \bar{0}$ *as* $n \to \infty$,

where the metric on $\{0,1\}$ *in the second statement is given by a natural* cylinder metric, *defined by*

$$d(\bar{\theta}, \bar{\eta}) = \sum_{n=0}^{\infty} \frac{|\theta_n - \eta_n|}{2^n}, \ \textit{for } \bar{\theta} = (\theta_0, \theta_1, \dots) \textit{ and } \bar{\eta} = (\eta_0, \eta_1, \dots).$$

Proof. We start by selecting any $\bar{\theta} \in \pi^{-1}(\alpha)$. If $\theta_i = 0$ for $0 \le i \le 2k-1$, and $\theta_{2k} \ne 0$, then let $\bar{\theta}^0 = \bar{\theta}$. If $\theta_i = 0$ for $0 \le i \le 2k$, and $\theta_{2k+1} \ne 0$, then there exists an m and k' such that $\bar{\eta} = b^m(\bar{\theta})$ satisfies $\eta_i = 0$ for $0 \le i \le 2k'-1$, and $\eta_{2k'} \ne 0$. We will use this $\bar{\eta}$ as $\bar{\theta}^0$. Now by induction suppose we have obtained $\{\bar{\theta}^i\}_{0\le i\le 2n}$ (or $\{\bar{\theta}^i\}_{0\le i\le 2n-1}$, respectively). Since we are now in case 3 of 5.4, $F = \langle a \rangle(\bar{\theta}^{2n})$ (or $\langle b \rangle(\bar{\theta}^{2n-1})$, respectively) is a finite set. By 5.1, there is a *unique element* $\bar{\eta} \in F$ such that the number of consecutive zeros from the first digit becomes maximum. We now define $\bar{\theta}^{2n+1}$ (or $\bar{\theta}^{2n}$, respectively) by this $\bar{\eta}$. This sequence $\{\bar{\theta}^n\}_{n=0}^{\infty}$ obviously satisfies properties in the lemma. $\square$

Example. Consider $L(\overline{001})$ (actually this is equal to $L(\{\widehat{p_0, p_1, p_2}\})$, where $\{p_0, p_1, p_2\}$ is a repelling periodic orbit of period 3, and p_0 is a landing point of $R_{3/7}$). In this case the sequence $\{\bar{\theta}^n\}_{n=0}^{\infty}$ becomes $\bar{\theta}^n = (0)^{3n}\overline{001}$, where $(0)^k$ means k consecutive zeros.

Now by 5.2, if we remove a point $\bar{\theta}^0$ from $\mathcal{J}(\hat{z})$, then we obtain one unbounded component and one other bounded component. Let us denote this bounded component by $\mathcal{J}_b(\bar{\theta}^0)$. If we remove two points $\bar{\theta}^{n-1}$ and $\bar{\theta}^n$ from $\mathcal{J}(\hat{z})$, then we have one unbounded component, one bounded component which contains $\mathcal{J}_b(\bar{\theta}^0)$, and two other bounded components. To differentiate these two other components consistently, let us put the orientation on each leaf by lifting the orientation of $\mathbb{C}$ by π. We will denote these components by $\mathcal{J}_r(\bar{\theta}^n, \bar{\theta}^{n-1})$ or $\mathcal{J}_l(\bar{\theta}^{n-1}, \bar{\theta}^n)$ according as they exist on the *r*ight side or on the *l*eft side of $\bar{\theta}^n$.

We have the following decomposition in this case:

$$\mathcal{J}(\hat{z}) \setminus \{\bar{\theta}^n\}_{n=1}^{\infty} = \mathcal{J}_b(\bar{\theta}^0) \cup \bigcup_{n=0}^{\infty} (\mathcal{J}_r(\bar{\theta}^n, \bar{\theta}^{n-1}) \cup \mathcal{J}_l(\bar{\theta}^{n-1}, \bar{\theta}^n)). \qquad (2)$$

By following the same argument as above, we can decompose $\mathcal{G}(\hat{z}) \setminus \{\bar{\theta}^n\}$ into connected components. We will use the same notation $\mathcal{G}_b(\bar{\theta}^n)$, $\mathcal{G}_a(\bar{\theta}^{n-1}, \bar{\theta}^n))$, $\mathcal{G}_r(\bar{\theta}^n, \bar{\theta}^{n-1})$, and $\mathcal{G}_l(\bar{\theta}^{n-1}, \bar{\theta}^n)$ for each corresponding part in $\mathcal{G}(\hat{z})$.

We also need notations for some subsets of the Julia set $J \subset \overline{\mathbb{C}}$. Let us take two distinct points p and q in $(\cup_{n=0}^{\infty} f^{-n}(\alpha)) \cap \partial U_0$. If we remove these two points from J, we obtain four connected components: Two components which share a single point p or q with ∂U_0, and the other two components which share arcs with endpoints p and q with ∂U_0. Let us denote two components of the first type by $J(p)$ and $J(q)$ according as they share the point p or q, and the components of the second type by $J(p,q)$ or $J(q,p)$ according as they exist on the right side of p or on the left side of p. We have the following lemma about $J(q,p)$.

Lemma 5.6 *$J(p,q)$ are all homeomorphic to each other for any $p \neq q$.*

Proof. Suppose $p \in f^{-n}(\alpha)$ and $q \in f^{-m}(\alpha)$ with $m \geq n$. Let us denote $J(p,q) \cap f^{-m}(\alpha)$ by $\{a_0, a_1, \dots, a_k\}$ where we name these points clockwise in ∂U_0. By removing $a_0, a_1, \dots, a_k$, we cut $J(p,q)$ further into small pieces $J(a_i, a_{i+1})$ and $J(a_i)$. It is clear that these pieces $J(a_i, a_{i+1})$ can be mapped into one another for any i, by some branch of $f^{-m} \circ f^m$. We can also map $J(a_i, a_{i+1})$, onto $J(a_i, a_{i+2})$ by some branch of $f^{-(m-2)} \circ f^m$, combined with some branch of $f^{-m} \circ f^m$, if necessary. Finally, all $J(a_i)$ can be mapped homeomorphically into one another by the some branch of $f^{-l} \circ f^k$, for some $l, k \leq m$.

To show any two sets $J(p,q)$ and $J(p',q')$ are mutually homeomorphic with each other, we cut both sets into small pieces by $f^{-n}(\alpha)$ for n sufficiently large and adjust number of pieces by three homeomorphisms explained above, and map each piece homeomorphically onto each piece. This completes the proof. □

The following lemma about $J(p)$ can be proved in almost the same way, so we will leave out the proof.

Lemma 5.7 *$J(p)$ are all homeomorphic to each other for any p.*

The following proposition together with two lemmas above and two decompositions (1) and (2) completes the proof of the theorem.

Proposition 5.8 *Let $\bar\theta \in \pi^{-1}(\alpha)$ and $\{\bar\theta^n\} \subset \pi^{-1}(\alpha)$ as above.*

- *$\mathcal{J}_b(\bar\theta^n)$ is homeomorphic to $J(-\alpha)$ for any $\bar\theta$ and n.*
- *$\mathcal{J}_r(\bar\theta^n, \bar\theta^{n-1})$, $\mathcal{J}_l(\bar\theta^{n-1}, \bar\theta^n)$, and $\mathcal{J}_a(\bar\theta^{n-1}, \bar\theta^n)$ are all homeomorphic to $J(-\alpha,\alpha)$ for any $\bar\theta$ and n.*

Proof. The idea is to project each part of $\mathcal{J}(\bar\theta) \setminus \{\bar\theta^n\}$ not by π, but by π^{-n} for n sufficiently large. The following lemma precisely describes how large n should be.

Lemma 5.9 *Let $\bar\eta = (\eta_0, \eta_1, \dots) \in \{\bar\theta^n\}$ as above.*

- *If $\bar\eta$ satisfies $\eta_i = 0$ for $0 \le i \le k-1$ and $\eta_k \ne 0$, then π_{-k} maps $\mathcal{J}_b(\bar\eta)$ homeomorphically onto $J(-\alpha)$.*
- *Either π_{-1} or π_{-2} maps $\mathcal{J}_a(\bar\theta^{n-1}, \bar\theta^n)$ homeomorphically onto $J(-\alpha,\alpha)$ or $J(\alpha,-\alpha)$ for any n.*
- *If $\bar\theta^{n-1} = (\theta_0^{n-1}, \theta_1^{n-1}, \dots)$ and $\bar\theta^n = (\theta_0^n, \theta_1^n, \dots)$ satisfy*

$$\theta_i^{n-1} = 0 \text{ for } 0 \le i \le l-1 \text{ and } \theta_l^{n-1} \ne 0, \text{ and}$$
$$\theta_i^n = 0 \text{ for } 0 \le i \le k-1 \text{ and } \theta_k^n \ne 0$$

  for $l < k$, then π_{-k} maps $\mathcal{J}_r(\bar\theta^n, \bar\theta^{n-1})$ onto $J(\alpha, \alpha_{l-k})$ and $\mathcal{J}_l(\bar\theta^{n-1}, \bar\theta^n)$ onto $J_l(\alpha_{l-k}, \alpha)$, where α_{l-k} is some point in $f^{l-k}(\alpha) \cap \partial U_0$.

Proof of Lemma. Recall that in the proof of Theorem 4.1 we associated each point of $f^{-n}(\alpha)$ with a digit in $\{0,1\}^n$. It follows by construction that if $\hat{z} = (z_0, z_{-1}, \dots) \in \pi^{-1}(\alpha)$ corresponds to the digit $\bar{\theta} = (\theta_0, \theta_{-1}, \dots) \in \{0,1\}$, then $z_{-n} = \pi_{-n}(\hat{z}) \in f^{-n}(\alpha)$ corresponds to the digit $(\theta_0, \theta_{-1}, \dots, \theta_{n-1})$. Now if $\bar{\eta}$ satisfies $\eta_i = 0$ for $0 \leq i \leq k-1$ and $\eta_k \neq 0$, then π_{-k} maps $\bar{\eta}$ to $(\eta_0, \eta_1, \dots, \eta_{k-1}) = (0)^k \cong \alpha$. By using lemma 5.1 several times, it is easy to check that by π_{-k}, $\mathcal{G}_b(\bar{\eta})$ is mapped univalently onto the connected component of $f^{-k}(a \cup b) \setminus \alpha$ containing $-\alpha$. We can shrink $\mathcal{G}_b(\bar{\eta})$ by homotopy onto $\mathcal{J}_b(\bar{\eta})$ to obtain the first statement. Next, in the second case, by the definition of $\{\bar{\theta}^n\}$, it is easy to check that either π_{-1} or π_{-2} maps $\{\bar{\theta}^{n-1}, \bar{\theta}^n\}$ into $\{\alpha, -\alpha\}$. Hence the statement follows by the similar argument. Finally, in the last case, by the definition of $\{\bar{\theta}^n\}$ again, it is easy to check π_{-k} maps $\bar{\theta}^n$ onto α and $\bar{\theta}^{n-1}$ onto some point of $f^{l-k}(\alpha) \cap \partial U_0$. By following the similar argument as in the first case again, we have the last statement.

□

The lemma above and lemma 5.6 implies the proposition. □

6 Hausdorff convergence of Cayley graphs: Proof of main theorem 2

This section proves theorem 3.3.

Proof. First recall the following lemma. This immediately follows from the fact that f is hyperbolic.

Lemma 6.1 *Let $\{a, b : [0,1] \to \mathbb{C} \setminus P\}$ be generators of $\pi_1(\mathbb{C} \setminus P)$. Then $f^{-n}(a([0,1]) \cup b([0,1])) \to J(f)$ with respect to the Hausdorff topology on compact subsets of $\mathbb{C}$.*

Now for the proof of the theorem, it is enough to show

$$\pi : \hat{f}^{-n} G(\hat{f}^n(\hat{z})) \to f^{-n}(a([0,1]) \cup b([0,1]))$$

is a non-branched covering. To see this, it is enough to show

$$\hat{f}^{-n} G(\hat{f}^n(\hat{z})) = \pi^{-1}(f^{-n}(a([0,1]) \cup b([0,1]))) \cap L(\hat{z}),$$

because π is already a branched covering and $\hat{f}^{-n}G(\hat{f}^n(\hat{z}))$ never passes through $\pi^{-1}(P)$. Take $\hat{\zeta} \in \hat{f}^{-n}G(\hat{f}^n(\hat{z}))$. We have $f^n \circ \pi(\hat{\zeta}) = \pi(\hat{f}^n(\hat{\zeta})) \in \pi(G(\hat{f}^n(\hat{z})) \subset a([0,1]) \cup b([0,1])$. On the other hand, if we take $\hat{\zeta} \in \pi^{-1}(f^{-n}(a([0,1]) \cup b([0,1]))) \cap L(\hat{z})$, then $\hat{f}^n(\hat{\zeta}) \in L(\hat{f}^n(\hat{z}))$ and $\pi \circ \hat{f}^n(\hat{\zeta}) = f^n \circ \pi(\hat{\zeta}) \in a([0,1]) \cup b([0,1])$. Therefore $\hat{f}^n(\hat{\zeta}) \in G \cap L(\hat{f}^n(\hat{z})) = G(\hat{f}^n(\hat{z}))$.

$\square$

7 Future problems

Since we are still at the stage of looking at the regular leaf space of independent quadratic parameters, there are various possible generalizations and future directions.

1. *Other hyperbolic parameters.* Describe the structure of $\mathcal{R}_f$ for all quadratic hyperbolic parameters. Can any two of them homeomorphic to each other? The description of Julia sets in each leaf can be performed along the line of this paper with a little modification and generalization. The description of $\mathcal{R}_f$ is harder and now still in progress. However, we started to realize that the Cayley graph plays a crucial role.

2. *Other parameters.* Describe the structure of $\mathcal{R}_f$ when f is Parabolic, Misiurewicz, Feigenbaum, Siegel, Cremer, etc. The case for Parabolics was partly solved in [6]. Misiurewicz case and Feigenbaum case are now in progress.

3. *Increase dimensions.* It is known that $(\mathcal{R}_f \setminus \mathcal{J})/\langle\hat{f}\rangle$ becomes a Riemann surface lamination and is called *2-lamination.* It is also possible to hyperbolize each leaf of $\mathcal{R}_f$ consistently enough to get a metrizable space $\mathcal{H}_f$ and even to take the quotient by the extension of $\hat{f}$ to hyperbolized leaves. These objects obtained by these processes are called *hyperbolic 3-laminations.* These three objects are sometimes easier to deal with and considered as more important than $\mathcal{R}_f$ because there are no more any irregular points around which the structure is highly twisted. Questions: Describe the structure of 2- and 3- laminations for above parameters in 1 and 2. Are they non-holomorphic when parameters are combinatorially different?

8 Appendix: General definition around laminations

The first subsection will describe basic concepts in the theory of laminations. The following section will be a brief suggestion of how to make lamination a "workable" object in algebraic topology.

8.1 Lamination: General concepts

In this paper, a *lamination* will be a Hausdorff topological space $\mathcal{X}$ equipped with a covering $\{\cup U_i\}$ and coordinate charts $\phi_i : U_i \to T_i \times D_i$, where D_i is homeomorphic to a domain in $\mathbb{R}^n$ and T_i is a topological space. The *transition maps* $\phi_{ij} = \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ are required to be homeomorphisms that take leaves to leaves (see [3]).

Subsets of the form $\phi_i^{-1}(\{t\} \times D)$ are called local leaves. The requirement on the transition maps implies that the local leaves piece together to form *global leaves*, which are n-manifolds immersed injectively in $\mathcal{X}$.

As usual we may restrict the class of transition maps to obtain finer structures on $\mathcal{X}$. If D_i are taken to lie in $\mathbb{C}$ and ϕ_{ij} are conformal maps, we call $\mathcal{X}$ a *Riemann surface lamination* and note that the global leaves have the structure of Riemann surfaces. If ϕ_{ij} are further restricted to be complex affine maps $z \mapsto az + b$, then we call $\mathcal{X}$ a *(complex) affine lamination*, and the global leaves have a (complex) affine structure. If the leaves of an affine lamination are isomorphic to the complex plane, we also call it a $\mathbb{C}$-lamination. One can similarly consider real affine laminations, but as they will not play a role in this paper we shall assume from now on that "affine" means "complex affine".

8.2 Laminated graphs

As in the theory of Manifolds, by Zorn's lemma, there exsits a unique atlas which is *maximal*. We will use this maximal atlas in the following definitions.

A *laminated point* is a set $\mathcal{P} \subset \mathcal{X}$ which satisfies $\phi_i(\mathcal{P}) = T_i \times \{q\}$ for some i, where q is a single point in D_i. Similarly, a *laminated path* is a set $\mathcal{P} \subset \mathcal{X}$ which satisfies $\phi_i(\mathcal{P}) = T_i \times \gamma$ for some i, where γ is a closed path in D_i. *Endpoints of a laminated path* is defined by $\phi_i^{-1}(T_i \times (\text{endpoints of } \gamma))$.

Definition. A union of a finite collection of laminated paths (or *laminated edges) and a finite collection of laminated points* $\{\mathcal{P}_i\}$ *(or* laminated vertices) is called a *laminated graph* $\mathcal{G}$ if endpoints of all laminated paths is a subset of $\cup\mathcal{P}_i$.

The following is a natural result of the definition.

Proposition 8.1 *Let* $\mathcal{G}$ *be a laminated graph in a lamination* $\mathcal{X}$.

- *In each leaf a laminated graph is a locally finite graph.*
- *Around any point* p *in the laminated graph* $\mathcal{G}$, *there exists a neighborhood* U_i *and a chart* ϕ_i *such that* $\phi_i(U_i \cap \mathcal{G}) = T_i \times G$, *where* G *is a graph in* D_i *(We allow half-open edge to happen for* G*).*

References

[1] L. Bartholdi, R. Grigorchuk, V. Nekrashevych. From Fractal Groups to Fractal Sets. *http://arxiv.org/abs/math.GR/0202001*

[2] W. Brandt, Über eine Verallgemeinerung des Gruppengriffes. *Math. Ann.* **96** (1926) 360-366.

[3] A. Candel and L. Conlon. *Foliations I.* American Mathematical Society, 2000.

[4] C. Cabrera and Y. Tanaka. On the regular leaf space of $z^2 - 1$, in preparation.

[5] A. Hatcher. *Algebraic Topology.* Cambridge University Press, 2002.

[6] T. Kawahira. On the regular leaf space of the cauliflower. *Kodai Math Journal* **26** (1997) 17–94.

[7] M. Lyubich and Y. Minsky. Laminations in holomorphic dynamics. *J. Diff. Geom.* **47, No.2** (2003) 167-178.

[8] J. Milnor. Periodic Orbits, External Rays and the Mandelbrot Set: An Expository Account. *Asterisque*261 (2000) 277-333.

[9] D. Sullivan. Linking the universalities of Milnor-Thurston, Feigenbaum and Ahlfors-Bers. *Topological Methods in Modern Mathematics*, L. Goldberg and Phillips, editor, Publish or Perish, 1993

[10] J. Milnor. *Dynamics in one complex variable: Introductory lectures.* vieweg, 1999.