数理解析研究所講究録 1426 巻 2005 年 101-105

The Reachability and Related Decision Problems for Semi-Constructor TRSs

Ichiro Mitsuhashi, Michio Oyamaguchi, Toshiyuki Yamada
(Faculty of Engineering, Mie University)
三橋一郎, 大山口通夫, 山田俊行 (三重大学工学部)

Abstract

This paper shows that reachability is undecidable for confluent monadic and semi-constructor TRSs, and joinability and confluence are undecidable for monadic and semi-constructor TRSs. Here, a TRS is monadic if the height of the right-hand side of each rewrite rule is at most 1, and semi-constructor if all defined symbols appearing in the right-hand side of each rewrite rule occur only in its ground subterms.

1 Introduction

In this paper, we consider the reachability problem for confluent monadic and semi-constructor TRSs posed by our previous paper [4]. Here, a TRS is monadic if the height of the right-hand side of each rewrite rule is at most 1, and semi-constructor if all defined symbols appearing in the right-hand side of each rewrite rule occur only in its ground subterms. We give a negative answer to this problem. This undecidability result is compared with the decidability results of joinability and unification for the same class [4, 3].

Moreover, we show that joinability and confluence are undecidable for monadic and semi-constructor TRSs.

2 Preliminaries

We assume that the reader is familiar with standard definitions of rewrite systems [1] and we just recall here the main notations used in this paper.

Let F be a finite set of operation symbols graded by an arity function $\mathsf{ar}\colon F \to \mathbf{N}(=\{0,1,2,\cdots\})$, $F_n = \{f \in F \mid \mathsf{ar}(f) = n\}$. We use x, y as variables, f as an operation symbol, r, s, t as terms. Let $\mathsf{V}(s)$ be the set of variables occurring in s. The *height* of a term is defined as follows: $\mathsf{height}(a) = 0$ if a is a variable or a constant and $\mathsf{height}(f(t_1,\ldots,t_n)) = 1 + \max\{\mathsf{height}(t_1),\ldots,\mathsf{height}(t_n)\}$ if $n > 0$. The *root symbol* of a term is defined as $\mathsf{root}(a) = a$ if a is a variable and $\mathsf{root}(f(t_1,\ldots,t_n)) = f$.

A position in a term is expressed by a sequence of positive integers, and positions are partially ordered by the prefix ordering $\leq$. Let $\mathcal{O}(s)$ be the set of positions of s. For a set of positions W, let $\mathsf{Min}(W)$ be the set of its minimal positions(w.r.t. $\leq$).

Let $s_{|p}$ be the subterm of s at position p. For a sequence $(p_1,\cdots,p_n)$ of pairwise parallel positions and terms $t_1,\cdots,t_n$, we use $s[t_1,\cdots,t_n]_{(p_1,\ldots,p_n)}$ to denote the term obtained from s by replacing each subterm $s_{|p_i}$ by $t_i (1 \leq i \leq n)$. For a set of function symbols F, let $\mathcal{O}_F(s) = \{p \in \mathcal{O}(s) \mid \mathsf{root}(s_{|p}) \in F\}$. For a string of unary function symbols $u = a_1 a_2 \cdots a_k$ and a term t, let $u(t)$ be an abbreviation for $a_1(a_2(\cdots a_k(t)))$.

A *rewrite rule* $\alpha \to \beta$ is a directed equation over terms. A *TRS* R is a set of rewrite rules. Let $\leftarrow$ be the inverse of $\to$, $\leftrightarrow = \to \cup \leftarrow$, and $\downarrow = \to^* \cdot \leftarrow^*$. t is *reachable* from s if $s \to^* t$. r is *confluent* on TRS R if for every $s \leftarrow^*_R r \to^*_R t$, $s \downarrow t$. A TRS R is *confluent* if every r is confluent on R. Let $\gamma\colon s_1 \overset{p_1}{\leftrightarrow} s_2 \cdots \overset{p_{n-1}}{\leftrightarrow} s_n$ be a *rewrite sequence*. This sequence is abbreviated to $\gamma\colon s_1 \leftrightarrow^* s_n$. Let $|\gamma|$ be the number of steps of γ. γ is called p-invariant if $q > p$ for any redex position q of γ, and we write $\gamma\colon s_1 \overset{p-\mathrm{inv}}{\leftrightarrow^*} s_n$.

The set D_R of *defined symbols* for a TRS R is defined as $D_R = \{\mathsf{root}(\alpha) \mid \alpha \to \beta \in R\}$. A term s is *semi-constructor* if for every subterm t of s, t has no variable or $\mathsf{root}(t)$ is not a defined symbol.

Definition 1 A rule $\alpha \to \beta$ is *monadic* if $\mathsf{height}(\beta) \leq 1$, *semi-constructor* if β is semi-constructor. A TRS R is *monadic* if every rule in R is monadic, *semi-constructor* if every rule in R is semi-constructor.

3 Undecidability of joinability for monadic and semi-constructor TRSs

We have shown that joinability is undecidable for linear semi-constructor TRSs [4]. In this section, we show that joinability for monadic and semi-constructor TRSs is undecidable by a reduction from the Post's Correspondence Problem (PCP). Let $P = \{\langle u_i, v_i\rangle \in \Sigma^* \times \Sigma^* \mid 1 \leq i \leq n\}$ be an instance of the PCP. The corresponding TRS R_P is constructed as follows. Let $F = F_0 \cup F_1 \cup F_2$ where $F_0 = \{0, \mathsf{c}, \mathsf{d}, \$\}$, $F_1 = \{\mathsf{e}_i \mid 1 \leq i \leq n\}(= E) \cup \Sigma$, $F_2 = \{\mathsf{f}, \mathsf{g}\}$.

$$\begin{aligned} R_P &= \{0 \to \mathsf{e}_i(0) \mid 1 \leq i \leq n\} \cup \{0 \to \mathsf{f}(\mathsf{c}, \mathsf{d})\} \\ &\cup \{b \to a(b),\ b \to a(\$) \mid b \in \{\mathsf{c}, \mathsf{d}\}, a \in \Sigma\} \\ &\cup \{\mathsf{f}(x, x) \to \mathsf{g}(x, x)\} \\ &\cup \{\mathsf{e}_i(\mathsf{g}(u_i(x), v_i(y))) \to \mathsf{g}(x, y) \mid 1 \leq i \leq n\} \end{aligned}$$

R_P is monadic. Here, $D_{R_P} = \{0, \mathsf{c}, \mathsf{d}, \mathsf{f}\} \cup E$, so R_P is semi-constructor.

Lemma 2 $0 \to^*_{R_P} \mathsf{g}(\$, \$)$ iff PCP P has a solution.

Proof. $0 \to^*_{R_P} \mathsf{g}(\$, \$)$ iff there exists $i_1 \cdots i_m \in \{1, \cdots, n\}^*$ such that $0 \to^{m+1} \mathsf{e}_{i_m} \cdots \mathsf{e}_{i_1}(\mathsf{f}(\mathsf{c}, \mathsf{d})) \to^+ \mathsf{e}_{i_m} \cdots \mathsf{e}_{i_1}(\mathsf{f}(u_{i_1} \cdots u_{i_m}(\$), u_{i_1} \cdots u_{i_m}(\$))) \to \mathsf{e}_{i_m} \cdots \mathsf{e}_{i_1}(\mathsf{g}(u_{i_1} \cdots u_{i_m}(\$), u_{i_1} \cdots u_{i_m}(\$))) \to^m \mathsf{g}(\$, \$)$ iff $u_{i_1} \cdots u_{i_m} = v_{i_1} \cdots v_{i_m}$. □

Since $\mathsf{g}(\$, \$)$ is a normal form, the following theorem holds.

Theorem 3 Both joinability and reachability for monadic and semi-constructor TRSs are undecidable.

4 Undecidability of reachability for confluent monadic and semi-constructor TRSs

We give a stronger result for reachability, that is, reachability for confluent monadic and semi-constructor TRSs is undecidable. Note that joinability is decidable for the same class [4, 3]. Let $\hat{F} = F \cup \{1\}$.

$$\begin{aligned} \hat{R}_P = R_P &\cup \{\$ \to 1\} \cup \{a(1) \to 1 \mid a \in \Sigma\} \\ &\cup \{\mathsf{e}_i(\mathsf{g}(1, v_i(y))) \to \mathsf{g}(1, y), \\ &\quad\ \mathsf{e}_i(\mathsf{g}(u_i(x), 1)) \to \mathsf{g}(x, 1), \\ &\quad\ \mathsf{e}_i(\mathsf{g}(1, 1)) \to \mathsf{g}(1, 1) \mid 1 \leq i \leq n\} \end{aligned}$$

$\hat{R}_P$ is monadic. Here, $D_{\hat{R}_P} = D_{R_P} \cup \{\$\} \cup \Sigma$, so $\hat{R}_P$ is semi-constructor. First, we show the confluence of $\hat{R}_P$.

4.1 Confluence of $\hat{R}_P$

To show the confluence of $\hat{R}_P$, we need some definitions and lemmata.

Definition 4 The set of Σ-*strings* is defined as follows.

- $1, \mathsf{c}, \mathsf{d}$ and $\$$ are Σ-strings.
- $a(t)$ is a Σ-string if t is a Σ-string and $a \in \Sigma$.

Lemma 5 For any Σ-string s, the following properties hold.

(1) For any $\gamma : s \leftrightarrow^* t$, t is a Σ-string.

(2) $s \to^* 1$.

Proof.

(1) By induction on $|\gamma|$.

(2) By induction on the structure of s. □

Corollary 6 Every Σ-string is confluent.

Lemma 7 Let $\gamma : u(s) \to^* t$ where $u \in \Sigma^+$. Then, if $\mathrm{root}(s) \notin \{1, \mathsf{c}, \mathsf{d}, \$\} \cup \Sigma$ and $u(s)_{|p1} = s$ then γ is p-invariant.

Proof. By induction on $|\gamma|$. □

Definition 8 The set of E-*strings* is defined as follows.

- $0, \mathsf{f}(t_1, t_2)$ and $\mathsf{g}(t_1, t_2)$ are E-strings if t_1, t_2 are Σ-strings.
- $\mathsf{e}_i(t)$ is an E-string if t is an E-string and $i \in \{1, \cdots, n\}$.

Lemma 9 For any E-string s, the following properties hold.

(1) For any $\gamma : s \leftrightarrow^* t$, t is an E-string.

(2) $s \to^* \mathsf{g}(1, 1)$.

Proof.

(1) By induction on $|\gamma|$.

(2) By induction on the structure of s. Basis : For any Σ-strings s_1, s_2, $\mathsf{f}(s_1, s_2) \to^* \mathsf{f}(1, 1) \to \mathsf{g}(1, 1)$ and $\mathsf{g}(s_1, s_2) \to^* \mathsf{g}(1, 1)$ by Lemma 5(2), and $0 \to \mathsf{f}(\mathsf{c}, \mathsf{d}) \to^* \mathsf{g}(1, 1)$. Thus, $s \to^* \mathsf{g}(1, 1)$ if $s = \mathsf{f}(s_1, s_2), \mathsf{g}(s_1, s_2)$ or 0. Induction step : Let $s = \mathsf{e}_i(s')$ for some $i \in \{1, \cdots, n\}$. By the induction hypothesis, $s' \to^* \mathsf{g}(1, 1)$. Thus, $\mathsf{e}_i(s') \to^* \mathsf{g}(1, 1)$. □

Corollary 10 Every E-string is confluent.

The following lemma is used as a component of the proof of Lemma 12.

Lemma 11 For any $i \in \{1,\cdots,n\}$ and terms r_1, r_2, the following properties hold.

(1) If $s \overset{\varepsilon}{\leftarrow} \mathsf{e}_i(\mathsf{g}(r_1,r_2)) \overset{\varepsilon\text{-inv}}{\to^*} t$ then there exist terms t_1, t_2 such that $t \to^* \mathsf{g}(t_1,t_2)$.

(2) If $\mathsf{g}(s_1,s_2) \leftarrow^* \mathsf{e}_i(\mathsf{g}(r_1,r_2)) \to^* \mathsf{g}(t_1,t_2)$ and $\mathsf{g}(r_1,r_2)$ is confluent then $\mathsf{g}(s_1,s_2) \downarrow \mathsf{g}(t_1,t_2)$.

Proof.

(1) Let $t = \mathsf{e}_i(\mathsf{g}(t'_1,t'_2))$. If r_1 is a Σ-string then $t'_1 \to^* 1$ by Lemma 5. Otherwise, $r_1 \neq 1$. Thus, $r_1 = u_i(r'_1)$ for some term r'_1 by $\mathsf{e}_i(\mathsf{g}(r_1,r_2)) \overset{\varepsilon}{\to} s$. By Lemma 7, $t'_1 = u_i(t''_1)$, where $r'_1 \to^* t''_1$. Similarly, $t'_2 \to^* 1$ or $t'_2 = v_i(t''_2)$ for some term t''_2. Thus, $t \to^* \mathsf{g}(t_1,t_2)$, where $t_1 \in \{1, t''_1\}$ and $t_2 \in \{1, t''_2\}$.

(2) By the definition of R_P, $\mathsf{e}_i(\mathsf{g}(r_1,r_2)) \overset{\varepsilon\text{-inv}}{\to^*} \mathsf{e}_i(\mathsf{g}(s'_1,s'_2)) \to \mathsf{g}(s''_1,s''_2) \overset{\varepsilon\text{-inv}}{\to^*} \mathsf{g}(s_1,s_2)$ and $\mathsf{e}_i(\mathsf{g}(r_1,r_2)) \overset{\varepsilon\text{-inv}}{\to^*} \mathsf{e}_i(\mathsf{g}(t'_1,t'_2)) \to \mathsf{g}(t''_1,t''_2) \overset{\varepsilon\text{-inv}}{\to^*} \mathsf{g}(t_1,t_2)$. Thus, $s'_1 \leftarrow^* r_1 \to^* t'_1$, $s''_1 \to^* s_1$ and $t''_1 \to^* t_1$. First, we show that $s_1 \downarrow t_1$.

Case of $s'_1 = t'_1 = 1$: Obviously, $s''_1 = s_1 = t''_1 = t_1 = 1$.

Case of $s'_1 = 1$ and $t'_1 = u_i(t''_1)$: Obviously, $s''_1 = s_1 = 1$. By Lemma 5, t_1 is a Σ-string and $t_1 \to^* 1$.

Case of $s'_1 = u_i(s''_1)$ and $t'_1 = 1$: Similar to the previous one.

Case of $s'_1 = u_i(s''_1)$ and $t'_1 = u_i(t''_1)$: By confluence of $\mathsf{g}(r_1,r_2)$, r_1 is confluent. Thus, $u_i(s_1) \downarrow u_i(t_1)$. If s_1 is a Σ-string then $s_1 \downarrow t_1$ by Corollary 6. Otherwise, $s_1 \downarrow t_1$ by Lemma 7.

Similarly, $s_2 \downarrow t_2$. Thus, $\mathsf{g}(s_1,s_2) \downarrow \mathsf{g}(t_1,t_2)$. □

Now, we show the confluence of $\hat{R}_P$.

Lemma 12 $\hat{R}_P$ is confluent.

Proof. We show that for any $\gamma : s \leftarrow^* r \to^* t$, $s \downarrow t$ by induction on $\mathsf{height}(r)$.

Basis : If $r \in \{\mathsf{c}, \mathsf{d}, 1\}$ then $s \downarrow t$ by Corollary 6, else if $r = 0$ then $s \downarrow t$ by Corollary 10. Otherwise, $s = r = t$ since r is a normal form.

Induction step : If γ is ε-invariant then $s \downarrow t$ by the induction hypothesis. So, we consider that γ has an ε-reduction. Let $\gamma_s : r \to^* s$ and $\gamma_t : r \to^* t$. Without lost of generality, we assume that γ_s has an ε-reduction and $\mathsf{root}(r) \in \Sigma \cup \{\mathsf{f}\} \cup E$.

Case of $\mathsf{root}(r) \in \Sigma$: $\gamma_s : r = a(r_1) \overset{\varepsilon\text{-inv}}{\to^*} a(1) \to 1 = s$ holds for some $a \in \Sigma$ and r_1. By Lemma 5, $t \to^* 1$.

Case of $\mathsf{root}(r) = \mathsf{f}$: $\gamma_s : r = \mathsf{f}(r_1,r_2) \overset{\varepsilon\text{-inv}}{\to^*} \mathsf{f}(r',r') \to \mathsf{g}(r',r') \overset{\varepsilon\text{-inv}}{\to^*} \mathsf{g}(s_1,s_2) = s$ holds for some terms r_1, r_2, r', s_1, s_2. If γ_t is ε-invariant then $t = \mathsf{f}(t_1,t_2)$ where $r_1 \to^* t_1$ and $r_2 \to^* t_2$. In this case, $s \to^* \mathsf{g}(r_0,r_0) \leftarrow^* t$ for some r_0 by Figure 1(i). If γ_t has an ε-reduction then $\gamma_t : r = \mathsf{f}(r_1,r_2) \overset{\varepsilon\text{-inv}}{\to^*} \mathsf{f}(r'',r'') \to \mathsf{g}(r'',r'') \overset{\varepsilon\text{-inv}}{\to^*} \mathsf{g}(t_1,t_2) = t$ holds for some terms r'', t_1, t_2. In this case, $s \to^* \mathsf{g}(r_0,r_0) \leftarrow^* t$ for some r_0 by Figure 1(ii).

Case of $\mathsf{root}(r) \in E$: $\gamma_s : r = \mathsf{e}_i(r_1) \overset{\varepsilon\text{-inv}}{\to^*} \mathsf{e}_i(\mathsf{g}(s'_1,s'_2)) \to \mathsf{g}(s''_1,s''_2) \overset{\varepsilon\text{-inv}}{\to^*} \mathsf{g}(s_1,s_2) = s$ holds for some terms $r_1, s'_1, s'_2, s''_1, s''_2, s_1, s_2$ and $i \in \{1,\cdots,n\}$. If γ_t is ε-invariant then $t = \mathsf{e}_i(t_1)$ where $r_1 \to^* t_1$. By the induction hypothesis, there exists a term t' such that $\mathsf{e}_i(\mathsf{g}(s'_1,s'_2)) \overset{\varepsilon\text{-inv}}{\to^*} t' \overset{\varepsilon\text{-inv}}{\leftarrow^*} t$. By Lemma 11(1), $t' \to^* \mathsf{g}(t'_1,t'_2)$ for some t'_1, t'_2. Here, $\mathsf{g}(s'_1,s'_2)$ is confluent by the induction hypothesis and $r_1 \to^* \mathsf{g}(s'_1,s'_2)$. Thus, $s \downarrow \mathsf{g}(t'_1,t'_2)$ by Lemma 11(2). (See Figure 1(iii).) If γ_t has an ε-reduction then $\gamma_t : r = \mathsf{e}_i(r_1) \overset{\varepsilon\text{-inv}}{\to^*} \mathsf{e}_i(\mathsf{g}(t'_1,t'_2)) \to \mathsf{g}(t''_1,t''_2) \overset{\varepsilon\text{-inv}}{\to^*} \mathsf{g}(t_1,t_2) = t$ holds for some terms $t'_1, t'_2, t''_1, t''_2, t_1, t_2$. There exists a term s' such that $s \to^* s' \leftarrow^* \mathsf{e}_i(\mathsf{g}(t'_1,t'_2))$ as shown in Figure 1(iii). Here, $\mathsf{root}(s') = \mathsf{g}$ by $\mathsf{root}(s) = \mathsf{g}$. By the induction hypothesis and $r_1 \to^* \mathsf{g}(t'_1,t'_2)$, $\mathsf{g}(t'_1,t'_2)$ is confluent. Thus, $s' \downarrow t$ by Lemma 11(ii). (See Figure 1(iv).) □

4.2 Reachability for confluent monadic and semi-constructor TRSs

Lemma 13 For any $\gamma : s \to^*_{\hat{R}_P} t$, if s has 1 as its subterm then so does t.

Proof. Since for any $\alpha \to \beta \in \hat{R}_P$, $\mathsf{V}(\alpha) = \mathsf{V}(\beta)$ and if α has 1 as its subterm then so does β. □

Lemma 14 $0 \to^*_{\hat{R}_P} \mathsf{g}(\$,\$)$ iff $0 \to^*_{R_P} \mathsf{g}(\$,\$)$.

Proof. Only if part: Let $\gamma : 0 \to^*_{\hat{R}_P} \mathsf{g}(\$,\$)$. We assume to the contrary that γ must have $\hat{R}_P \setminus R_P$ reduction, i.e., $\gamma : 0 \to^*_{\hat{R}_P} s \to_{\hat{R}_P \setminus R_P} t \to^*_{\hat{R}_P} \mathsf{g}(\$,\$)$ for some s, t. By the definition of $\hat{R}_P$, t has 1 as its subterm. By Lemma 13, $\mathsf{g}(\$,\$)$ has 1 as its subterm, a contradiction. If part : By $R_P \subseteq \hat{R}_P$. □

By Lemmata 2, 12, and 14, the following theorem holds.

Theorem 15 Reachability for confluent monadic and semi-constructor TRSs is undecidable.

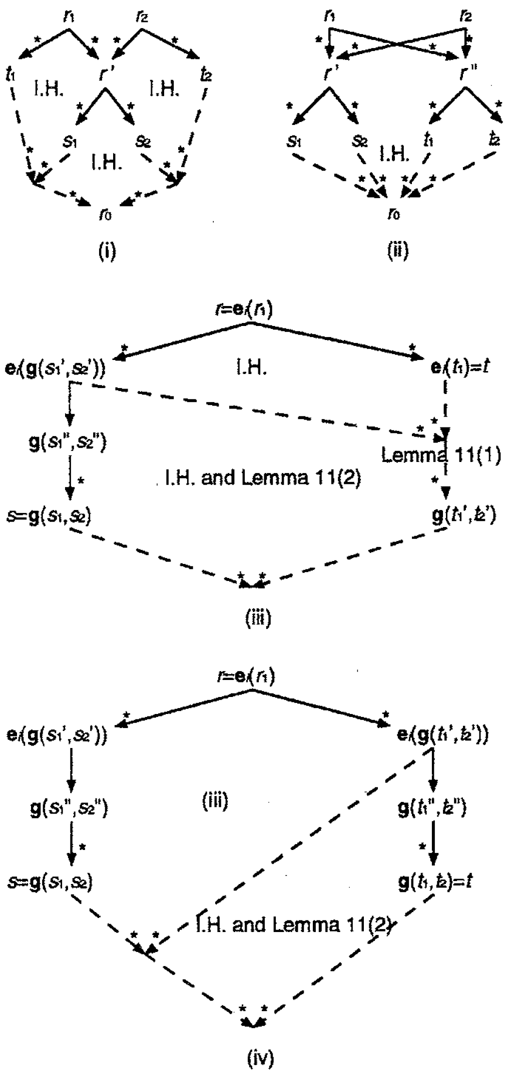

Figure 1:

5 Undecidability of confluence of monadic and semi-constructor TRSs

We show that confluence of monadic and semi-constructor TRSs is undecidable.

Let $F' = F'_0 \cup F'_1$ where $F'_0 = \{2\}$, $F'_1 = \{\mathsf{h}\}$.

$$R = \{\mathsf{h}(x) \to \mathsf{h}(0),\ \mathsf{h}(\mathsf{g}(\$,\$)) \to 2\}$$

$\hat{R}_P \cup R$ is monadic. Here, $D_R = \{\mathsf{h}\}$, so $\hat{R}_P \cup R$ is semi-constructor.

Lemma 16 For any s with $\mathrm{root}(s) \in F'$, the following properties hold.

(1) If $s \to_{\hat{R}_P \cup R} t$ then $\mathrm{root}(t) \in F'$.

(2) If $0 \to^*_{\hat{R}_P} \mathsf{g}(\$,\$)$ then $s \to^*_{\hat{R}_P \cup R} 2$.

The proof is straightforward, so omitted.

Lemma 17 Let $s \to_{\hat{R}_P \cup R} t$, $\mathrm{Min}(\mathcal{O}_{F'}(s)) = \{p_1, \cdots, p_m\}$, and $\mathrm{Min}(\mathcal{O}_{F'}(t)) = \{q_1, \cdots, q_n\}$. Then, $s[2, \cdots, 2]_{(p_1, \cdots, p_m)} \to_{\hat{R}_P} t[2, \cdots, 2]_{(q_1, \cdots, q_n)}$ or $s[2, \cdots, 2]_{(p_1, \cdots, p_m)} = t[2, \cdots, 2]_{(q_1, \cdots, q_n)}$.

Proof. Let $s \xrightarrow{p}_{\hat{R}_P \cup R} t$. If there exists $i \in \{1, \cdots, m\}$ such that $p_i \le p$ then $\{p_1, \cdots, p_m\} = \{q_1, \cdots, q_n\}$ by Lemma 16(1). Thus, $s[2, \cdots, 2]_{(p_1, \cdots, p_m)} = t[2, \cdots, 2]_{(q_1, \cdots, q_n)}$. Otherwise, obviously $s \to_{\hat{R}_P} t$. Since every function symbol in F' does not occur in $\hat{R}_P$, $s[2, \cdots, 2]_{(p_1, \cdots, p_m)} \to_{\hat{R}_P} t[2, \cdots, 2]_{(q_1, \cdots, q_n)}$. □

Lemma 18 $\hat{R}_P \cup R$ is confluent iff $0 \to^*_{\hat{R}_P} \mathsf{g}(\$,\$)$.

Proof. Only if part : By $\mathsf{h}(0) \leftarrow_{\hat{R}_P} \mathsf{h}(\mathsf{g}(\$,\$)) \to_{\hat{R}_P} 2$, confluence ensures that $\mathsf{h}(0) \downarrow_{\hat{R}_P \cup R} 2$. Since 2 is a normal form, $\mathsf{h}(0) \to^*_{\hat{R}_P \cup R} 2$. Thus, there exists a shortest sequence γ that satisfies $\gamma : \mathsf{h}(0) \to^*_{\hat{R}_P \cup R} \mathsf{h}(\mathsf{g}(\$,\$)) \to_R 2$. Since γ is shortest, $\mathsf{h}(0) \xrightarrow{\varepsilon\text{-inv}}{}^*_{\hat{R}_P \cup R} \mathsf{h}(\mathsf{g}(\$,\$))$. Thus, there exists $\gamma' : 0 \to^*_{\hat{R}_P \cup R} \mathsf{g}(\$,\$)$. Obviously, every function symbol occuring in γ' belongs to $\hat{F}$. Thus, $0 \to^*_{\hat{R}_P} \mathsf{g}(\$,\$)$. By Lemma 14, $0 \to^*_{R_P} \mathsf{g}(\$,\$)$.

If part: Let $s \leftarrow^*_{\hat{R}_P \cup R} r \to^*_{\hat{R}_P \cup R} t$. By Lemma 17, $s[2, \cdots, 2]_{(p_1, \cdots, p_m)} \leftarrow^*_{\hat{R}_P} r[2, \cdots, 2]_{(o_1, \cdots, o_l)} \to^*_{\hat{R}_P} t[2, \cdots, 2]_{(q_1, \cdots, q_n)}$, where $\mathrm{Min}(\mathcal{O}_{F'}(r)) = \{o_1, \cdots, o_l\}$, $\mathrm{Min}(\mathcal{O}_{F'}(s)) = \{p_1, \cdots, p_m\}$, and $\mathrm{Min}(\mathcal{O}_{F'}(t)) = \{q_1, \cdots, q_n\}$. Since $\hat{R}_P$ is confluent by Lemma 12, $s[2, \cdots, 2]_{(p_1, \cdots, p_m)} \downarrow_{\hat{R}_P} t[2, \cdots, 2]_{(q_1, \cdots, q_n)}$. By $0 \to^*_{\hat{R}_P} \mathsf{g}(\$,\$)$ and Lemma 16(2), $s \to^*_{\hat{R}_P \cup R} s[2, \cdots, 2]_{(p_1, \cdots, p_m)}$ and $t \to^*_{\hat{R}_P \cup R} t[2, \cdots, 2]_{(q_1, \cdots, q_n)}$. Thus, $s \downarrow_{\hat{R}_P \cup R} t$. □

By Lemmata 2 and 18, the following theorem holds.

Theorem 19 Confluence of monadic and semi-constructor TRSs is undecidable.

6 Confluence of flat TRSs

In [2], the undecidability of confluence of flat TRSs has been claimed, but we found that the proof is incorrect. In this section, we explain its flaw.

Definition 20 [2] A rule $\alpha \to \beta$ is *flat* if $\mathsf{height}(\alpha) \le 1$ and $\mathsf{height}(\beta) \le 1$.

In [2], first the undecidability of reachability has been obtained by showing that $0 \to^*_{R_1} 1$ iff there exists a solution for PCP for the following TRS R_1. |

$$R_1 = R_0 \cup \{0 \to \mathsf{f}(q_A^{(3)}, q_A^{(4)}, q_A^{(5)}, q_B^{(13)}, q_B^{(14)}, q_A^{(6)}, q_B^{(15)}, q_B^{(16)}), \mathsf{f}(x_1, x_2, x_1, y_{11}, y_{12}, x_2, y_{11}, y_{12}) \to \mathsf{g}(x_1, x_2, x_1, y_{11}, y_{12}, x_2, y_{11}, y_{12}), \mathsf{g}(x_0, x_0, y_{17}, y_{17}, y_{18}, y_{18}, y_{10}, y_{10}) \to 1\}$$

Here, R_0 has many rules, so omitted (see [[2],p.267]).

Next, the undecidability of confluence has been obtained by showing the claim that $R_1 \cup R_2$ is confluent iff $0 \to_{R_1}^* 1$ for the following TRS R_2.

$$\begin{aligned} R_2 &= \{2 \to 0,\ 2 \to 1\} \cup \{c \to 0 \mid c \in \Xi_0 \setminus \{0,1\}\} \\ &\cup \{d(x) \to 0,\ d(1) \to 1 \mid d \in \Xi_1\} \\ &\cup \{\mathsf{f}(z_1, \cdots, z_8) \to 1,\ \mathsf{g}(z_1, \cdots, z_8) \to 1 \mid \\ &\quad \text{one of the } z_i \text{ is } 1, \\ &\quad \text{the others are distinct variables}\} \end{aligned}$$

Here, $\Xi = \Xi_0 \cup \Xi_1 \cup \{\mathsf{f}, \mathsf{g}\}$, which is a set of function symbols occuring in R_1. Ξ_0, Ξ_1 have many symbols, so omitted (see [[2],p.267]). Note that Ξ_0 has $q_A^{(3)}, q_A^{(4)}, q_A^{(5)}, q_A^{(6)}, q_B^{(13)}, q_B^{(14)}, q_B^{(15)}, q_B^{(16)}$.

However, the proof of the only-if part of the claim is incorrect. The proof claims that if $0 \to_{R_1}^* 1$ does not hold then $R_1 \cup R_2$ is not confluent because of the peak $0 \leftarrow_{R_2} 2 \to_{R_2} 1$. But, the claim overlooks that $0 \to_{R_1} \mathsf{f}(q_A^{(3)}, q_A^{(4)}, q_A^{(5)}, q_B^{(13)}, q_B^{(14)}, q_A^{(6)}, q_B^{(15)}, q_B^{(16)}) \to_{R_2}^8 \mathsf{f}(0,0,0,0,0,0,0,0) \to_{R_1} \mathsf{g}(0,0,0,0,0,0,0,0) \to_{R_1} 1$. Thus, the undecidability of confluence of flat TRSs has not been shown. Now, Jacquemard claims that the proof can be corrected.

Acknowledgements

This work was supported in part by Grant-in-Aid for Scientific Research 15500009 from Japan Society for the Promotion of Science.

References

[1] F. Baader and T. Nipkow. *Term Rewriting and All That.* Cambridge University Press, 1998.

[2] F. Jacquemard. Reachability and confluence are undecidable for flat term rewriting systems. *Inf.Process.Lett.*, 87:265–270, 2003.

[3] I. Mitsuhashi, M. Oyamaguchi, Y. Ohta, and T. Yamada. The joinability and related decision problems for confluent semi-constructor TRSs. Submitted for publication.

[4] I. Mitsuhashi, M. Oyamaguchi, Y. Ohta, and T. Yamada. The joinability and unification problems for confluent semi-constructor TRSs. In V. van Oostrom, editor, *Proc.15th RTA*, pages 285–300. LNCS 3091, 2004.