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Cofinal types around $\mathcal{P}_\kappa\lambda$ and the tree property for directed sets

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Abstract

Generalizing a result of Todorčević, we prove the existence of directed sets D, E such that $D \not\geq \mathcal{P}_\kappa\lambda$ and $E \not\geq \mathcal{P}_\kappa\lambda$ but $D \times E \geq \mathcal{P}_\kappa\lambda$ in the Tukey ordering. As an application, we show that the tree property for directed sets introduced by Hinnion is not preserved under products. Most of the results appear in [14].

1 Introduction

Any notion of convergence, described in terms of sequences, nets or filters, involves directed sets, or at least a particular kind of them. In general, directed sets are considered to express the type of convergence. Tukey defined an ordering on the class of all directed sets [17]. This ordering, now called Tukey ordering, was studied by Schmidt [15], Isbell [11],[12], Todorčević [16] and others. In particular, the directed sets of the form $\mathcal{P}_\kappa\lambda$ are of interest, because they possess some nice properties. In section 4 we generalize the directed sets $D(S)$ introduced by Todorčević to $D_\kappa(S)$, where κ is an arbitrary infinite regular cardinal. With these directed sets, we show (Theorem 4.8) that there exist directed sets D, E such that $D \not\geq \mathcal{P}_\kappa\lambda$ and $E \not\geq \mathcal{P}_\kappa\lambda$ but $D \times E \geq \mathcal{P}_\kappa\lambda$ in the Tukey ordering.

The notion of tree property for infinite cardinals (the nonexistence of an Aronszajn tree) is well known, and is related to a large variety of set theoretic statements. The tree property for directed sets was invented by Hinnion [10], and studied by Esser and Hinnion [8],[9]. It is a generalization of the usual tree property for infinite cardinals and especially, for $\mathcal{P}_\kappa\lambda$, it is closely related with the mild ineffability if κ is strongly inaccessible (see Corollary 7.5). By an application of the result mentioned above, we show (Theorem 8.1) that there exist two directed sets D, E for which $\mathsf{add}(D) = \mathsf{add}(E)$ is weakly compact, and both D and E have the tree property but $D \times E$ does not. It was an open problem whether such D, E exist [8].

2 Directed sets and cofinal types

By classifying directed sets into isomorphism types, and further identifying a directed set with its cofinal subset, we arrive at the notion of cofinal type. On the other hand, the same equivalence relation is deduced from a quasi-ordering on the class of all directed sets. First we state the definitions.

Definition 2.1 Let $\langle D, \leq_D\rangle, \langle E, \leq_E\rangle$ be directed sets. A function $f\colon E \to D$ which satisfies

$$\forall d \in D \exists e \in E \forall e' \geq_E e\ [f(e') \geq_D d]$$

is called a *convergent function*. If such a function exists we write $D \leq E$ and say *E is cofinally finer than D*. $\leq$ is transitive and is called the *Tukey ordering* on the class of directed sets. A function $g\colon D \to E$ which satisfies

$$\forall e \in E \exists d \in D \forall d' \in D\ [g(d') \leq_E e \to d' \leq_D d]$$

is called a *Tukey function*.

If there exists a directed set C into which D and E can be embedded cofinally, we say *D is cofinally similar with E*. In this case we write $D \equiv E$. $\equiv$ is an equivalence relation, and the eqivalence classes with respect to $\equiv$ are the *cofinal types*.

The following propositions give the connection between the definitions. For the proofs, consult [16]

Proposition 2.2 *For directed sets D and E, the follwing are equivalent.*

(a) $D \leq E$.
(b) *There exists a Tukey function $g\colon D \to E$.*
(c) *There exist functions $g\colon D \to E$ and $f\colon E \to D$ such that* $\forall d \in D \forall e \in E\ [g(d) \leq_E e \to d \leq_D f(e)]$.

Proposition 2.3 *For directed sets D and E, the follwing are equivalent.*

(a) $D \equiv E$.
(b) $D \leq E$ *and* $E \leq D$.

So we can regard $\leq$ as an ordering on the class of all cofinal types.

One should always keep in mind the distinction between the unbounded and the cofinal subsets of a directed set.

Proposition 2.4 *For directed sets D and E,*

(i) $f\colon E \to D$ *is convergent* *iff* $\forall C \subseteq E$ *cofinal* $[f[C]$ *cofinal*$]$.
(ii) $g\colon D \to E$ *is Tukey* *iff* $\forall X \subseteq D$ *unbounded* $[g[X]$ *unbounded*$]$.

With two or more directed sets, we can form the product of these, to which we will always give the product ordering.

Proposition 2.5 *For directed sets D and E, $D \times E$ is the least upper bound of $\{D, E\}$ in the Tukey ordering.*

The next two cardinal functions are the most basic ones, being taken up in various contexts (mostly on a paticular kind of directed sets).

Definition 2.6 For a directed set D,

$$\begin{aligned} \mathsf{add}(D) &\stackrel{\text{def}}{=} \min\{|X| \mid X \subseteq D \text{ unbounded}\}, \\ \mathsf{cof}(D) &\stackrel{\text{def}}{=} \min\{|C| \mid C \subseteq D \text{ cofinal}\}. \end{aligned}$$

These are the *additivity* and the *cofinality* of a directed set. $\mathsf{add}(D)$ is only well-defined for D without maximum. In the sequel, any statement referring to $\mathsf{add}(D)$ presupposes that D has no maximum.

Proposition 2.7 *For a directed set D (without maximum),*

$$\aleph_0 \leq \mathsf{add}(D) \leq \mathsf{cof}(D) \leq |D|.$$

Furthermore, $\mathsf{add}(D)$ is regular and $\mathsf{add}(D) \leq \mathsf{cf}(\mathsf{cof}(D))$. Here cf is the cofinality of a cardinal, which is the same as the additivity of it.

Proposition 2.8 *For directed sets D and E, $D \leq E$ implies*

$$\mathsf{add}(D) \geq \mathsf{add}(E) \quad \textit{and} \quad \mathsf{cof}(D) \leq \mathsf{cof}(E).$$

From the above proposition we see that these cardinal functions are invariant under cofinal similarity.

Example 2.9 (see [1, chapter 2]) Let $\mathcal{M}, \mathcal{N}$ be respectively the meager ideal and the null ideal, each ordered by inclusion. $\langle {}^\omega\omega, \leq^* \rangle$ is the eventual dominance order on the reals. We have $\langle {}^\omega\omega, \leq^* \rangle \leq \mathcal{M} \leq \mathcal{N}$ in the Tukey ordering, and thus

$$\aleph_1 \leq \mathsf{add}(\mathcal{N}) \leq \mathsf{add}(\mathcal{M}) \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathsf{cof}(\mathcal{M}) \leq \mathsf{cof}(\mathcal{N}) \leq 2^{\aleph_0}.$$

Proposition 2.10 *For directed sets D and E,*

$$\begin{aligned} \mathsf{add}(D \times E) &= \min\{\mathsf{add}(D), \mathsf{add}(E)\}, \\ \mathsf{cof}(D \times E) &= \max\{\mathsf{cof}(D), \mathsf{cof}(E)\}. \end{aligned}$$

3 The width of a directed set

In the following, κ is always an infinite regular cardinal. If P is partially ordered set, we use the notation $X_{\leq a} = \{x \in X \mid x \leq a\}$ for X a subset of P and $a \in P$.

The cofinal type of $\mathcal{P}_\kappa\lambda$ is an interesting topic by itself (see [16]). As usual, $\mathcal{P}_\kappa\lambda = \{x \subseteq \lambda \mid |x| < \kappa\}$ is ordered by inclusion.

Lemma 3.1 $\mathsf{add}(\mathcal{P}_\kappa\lambda) = \kappa$, *and* $\lambda \leq \mathsf{cof}(\mathcal{P}_\kappa\lambda) \leq \lambda^{<\kappa}$. *In particular, if* κ *is strongly inaccessible, then* $\mathsf{cof}(\mathcal{P}_\kappa\lambda) = \lambda^{<\kappa}$.

Proof For the last statement, notice that in general for a cofinal $C \subseteq \mathcal{P}_\kappa\lambda$, $\mathcal{P}_\kappa\lambda = \bigcup_{x\in C}\mathcal{P}x$, and thus $\lambda^{<\kappa} \leq 2^{<\kappa} \cdot |C|$. □

Lemma 3.2 *For a directed set* D, *if* $\mathsf{add}(D) \geq \kappa$ *and* $\mathsf{cof}(D) \leq \lambda$, *then* $D \leq \mathcal{P}_\kappa\lambda$.

It turns out that the following cardinal function, which seems to be a natural one, gives a suitable formulation of Theorem 7.1.

Definition 3.3 The *width* of a directed set D is defined by

$$\mathsf{wid}(D) \stackrel{\text{def}}{=} \sup\{|X|^+ \mid X \text{ is a thin subset of } D\},$$

where 'a *thin* subset of D' means

$$\forall d \in D[|X_{\leq d}| < \mathsf{add}(D)].$$

Example 3.4 Let κ, λ, μ be regular with $\lambda^{<\kappa} = \lambda$ and $\lambda \leq \mu$. Then for the directed set $\mu \times \mathcal{P}_\kappa\lambda$ ordered by

$$\langle\alpha, x\rangle \leq \langle\beta, y\rangle \iff \alpha \leq \beta \wedge x \subseteq y$$

we have

$$\begin{aligned} \mathsf{add}(\mu \times \mathcal{P}_\kappa\lambda) &= \kappa, \\ \mathsf{wid}(\mu \times \mathcal{P}_\kappa\lambda) &= \lambda^+, \\ \mathsf{cof}(\mu \times \mathcal{P}_\kappa\lambda) &= \mu. \end{aligned}$$

The second equation can be verified using Proposition 4.1.

Fix D and put $\kappa := \mathsf{add}(D)$.

Lemma 3.5 *For any cardinal* $\lambda \geq \kappa$, *the follwing are equivalent.*

(a) *D has a thin subset of size λ.*
(b) $D \geq \mathcal{P}_\kappa\lambda$.
(c) *There exists an order-preserving function* $f\colon D \to \mathcal{P}_\kappa\lambda$ *with* $f[D]$ *cofinal in* $\mathcal{P}_\kappa\lambda$.

Proof (a) ⇒ (b) Let $X \subseteq D$ be a thin subset of size λ. Define

$$\begin{array}{rcl} f : D & \to & \mathcal{P}_\kappa X \\ \Downarrow & & \Downarrow \\ d & \mapsto & X_{\leq d} \end{array}$$

Then f is (order-preserving and) convergent.

(b) $\Rightarrow$ (c) If $f : D \to \mathcal{P}_\kappa\lambda$ is convergent, define

$$\begin{array}{rcl} g : D & \longrightarrow & \mathcal{P}_\kappa\lambda \\ \Downarrow & & \Downarrow \\ d & \mapsto & \bigcap_{d' \geq d} f(d') \end{array}$$

Then g is convergent and also order-preserving.

(c) $\Rightarrow$ (a) For such g as above, pick for each $\alpha \in \lambda$ a $d_\alpha \in D$ such that $g(d_\alpha) \ni \alpha$, and put $X := \{d_\alpha \mid \alpha \in \lambda\}$. It is readily seen that X is thin. Furthermore $|X| = \lambda$ since $\bigcup_{d \in X} g(d) = \lambda$. □

Corollary 3.6

$$\begin{aligned} \mathsf{wid}(D) &= \sup\{\lambda^+ \mid D \geq \mathcal{P}_\kappa\lambda\} \\ &= \sup\{\lambda^+ \mid \exists f\colon D \to \mathcal{P}_\kappa\lambda \textit{ order-preserving with } f[D] \textit{ cofinal in } \mathcal{P}_\kappa\lambda\}. \end{aligned}$$

The next inequality is checked easily.

Lemma 3.7

$$\mathsf{add}(D)^+ \leq \mathsf{wid}(D) \leq \mathsf{cof}(D)^+.$$

Lemma 3.8 $\mathsf{wid}(D)$ *is never singular.*

Proof Assume $\lambda := \mathsf{wid}(D) > \mathrm{cf}(\lambda)$ for a directed set D with $\mathsf{add}(D) = \kappa$. Fix a sequence of ordinals $\langle \theta_\alpha \mid \alpha < \mathrm{cf}(\lambda)\rangle$ converging up to λ. Then there are convergent order-preserving mappings $f_\alpha : D \to \mathcal{P}_\kappa\theta_\alpha$ for all $\alpha < \mathrm{cf}(\lambda)$. Fix also a convergent order-preserving $g : D \to \mathcal{P}_\kappa\mathrm{cf}(\lambda)$. Consider

$$\begin{array}{rcl} h : D & \longrightarrow & \mathcal{P}_\kappa\lambda \\ \Downarrow & & \Downarrow \\ d & \mapsto & \bigcap_{\alpha \in g(d)} f_\alpha(d). \end{array}$$

h is order-preserving and convergent. Hence we have a contradiction. □

However, the next proposition will show that $\mathsf{wid}(D)$ can be a limit cardinal. For example, that for any strongly inaccessible λ there is a directed set D such that $\mathsf{wid}(D) = \lambda$.

Proposition 3.9 *Let κ be regular and let λ be strongly κ^+-inaccesible (i.e. λ is regular and $\forall \mu < \lambda\ [\mu^\kappa < \lambda]$). Then there exists a directed set D such that $\mathsf{add}(D) = \kappa$ and $\mathsf{wid}(D) = \lambda$.*

Proof Consider

$$D = \prod^{(\kappa^+)}_{\kappa \leq \alpha < \lambda} \mathcal{P}_\kappa\alpha.$$

I.e. D is the set of functions f such that $\mathrm{dom}(f) \subseteq \lambda \setminus \kappa$, $|\mathrm{dom}(f)| \leq \kappa$, and for all $\alpha \in \mathrm{dom}(f)$, $f(\alpha) \in \mathcal{P}_\kappa\alpha$. The order is given by

$$f \leq_D g \iff \mathrm{dom}(f) \subseteq \mathrm{dom}(g) \wedge \forall\alpha \in \mathrm{dom}(f)\ [f(\alpha) \subseteq g(\alpha)].$$

Since $\mathsf{add}(D) = \kappa$ and $\mathcal{P}_\kappa\alpha \leq D$ for each $\alpha \in \lambda \setminus \kappa$, $\mathsf{wid}(D) \geq \lambda$. To show that equality holds, let $\langle f_\alpha \mid \alpha < \lambda \rangle$ be a sequence of distinct elements in D. By the Δ-system lemma there are $d \subseteq \lambda \setminus \kappa$ and $A \subseteq \lambda$ such that $|A| = \lambda$ and $\mathrm{dom}(f_\alpha) \cap \mathrm{dom}(f_\beta) = d$ for distinct $\alpha, \beta \in A$. Then by noting that $|\prod_{\alpha \in d}^{(\kappa^+)} \mathcal{P}_\kappa\alpha| < \lambda$, there is a $g \in D$ which bounds κ-many f_α's. □

4 The directed sets $D_\kappa(S)$

One notices at once that if $\mathsf{add}(D) = \mathsf{add}(E)$, then $\mathsf{wid}(D \times E) \geq \max\{\mathsf{wid}(D), \mathsf{wid}(E)\}$. But unlike add and cof, the width of finite products cannot be computed easily. In this section we show that there are directed sets D, E such that $\mathsf{add}(D) = \mathsf{add}(E)$ and $\mathsf{wid}(D \times E) > \max\{\mathsf{wid}(D), \mathsf{wid}(E)\}$.

Before that, we will take a look at the case $\mathsf{add}(D) \neq \mathsf{add}(E)$.

Proposition 4.1 *If* $\mathsf{add}(D) < \mathsf{add}(E)$, *then* $\mathsf{wid}(D \times E) = \mathsf{wid}(D)$.

This is proved by the next lemma.

Lemma 4.2 *Let* $\kappa := \mathsf{add}(D) < \mathsf{add}(E)$. *Then*

$$\mathcal{P}_\kappa\lambda \leq D \times E \iff \mathcal{P}_\kappa\lambda \leq D$$

for any cardinal $\lambda \geq \kappa$.

Proof ($\Leftarrow$) Let $X \subseteq D \times E$ be a thin subset of size λ, and let $p : D \times E \to D$ be the projection. Put $Y := p[X]$. Then Y is thin and $|Y| = \lambda$, since for each $d \in Y$, $|p^{-1}[Y_{\leq d}]| < \kappa$.

($\Rightarrow$) Trivial, using transitivity of $\leq$. □

Now we turn to our main results on cofinal types.

Definition 4.3 Let κ, λ be both regular with $\kappa < \lambda$. We define the following directed set, where the ordering is given by inclusion. For $S \subseteq E^\lambda_\kappa = \{\alpha \in \lambda \mid \mathrm{cf}\alpha = \kappa\}$,

$$D_\kappa(S) \stackrel{\text{def}}{=} \{x \subseteq S \mid |x| \leq \kappa \text{ and } \forall y \subseteq x\ [\mathrm{otp}\, y = \kappa \to \sup y \in x]\}.$$

Here, otp denotes the order type of a set of ordinals.

Todorčević [16] defined and studied these directed sets for $\kappa = \omega$. Note that by letting $S := \{\alpha \in E^\lambda_\kappa \mid \alpha \text{ is not a limit point of elements of } E^\lambda_\kappa\}$, we have $D_\kappa(S) = \mathcal{P}_\kappa S \cong \mathcal{P}_\kappa\lambda$.

The following statements mimic Lemmas 1,2,3 and Theorems 4,6 in [16], but because of the assumption on cardinal arithmetic, they are not full generalizations.

Lemma 4.4 *Let $\omega \leq \kappa < \lambda$, where κ is regular and λ is strongly κ^+-inaccessible, and let $S, S' \subseteq E^\lambda_\kappa$ with S unbounded in λ. Then*

$$D_\kappa(S) \leq D_\kappa(S') \quad \textit{implies} \quad S' \setminus S \textit{ is nonstationary in } \lambda.$$

Proof Let $f\colon D_\kappa(S) \to D_\kappa(S')$ be a Tukey function. Without loss of generality, f depends only on its values for singletons, i.e. $f(x) = \bigcup_{\alpha\in x} f(\{\alpha\})$ for all nonempty $x \in D_\kappa(S)$. By the Δ-system lemma we obtain an $A \subseteq S$ of size λ and a $d \subseteq S'$ such that

$$\begin{aligned} &\forall \alpha, \beta \in A\ [\alpha \neq \beta \to f(\{\alpha\}) \cap f(\{\beta\}) = d], \\ &\forall \alpha \in A\ [\min(f(\{\alpha\}) \setminus d) > \sup d], \\ \text{and}\quad &\forall \alpha, \beta \in A\ [\alpha < \beta \to \sup(f(\{\alpha\}) \setminus d) < \min(f(\{\beta\}) \setminus d)]. \end{aligned}$$

Next, put

$$C_0 = \Big\{\alpha \in \lambda \mid \text{there exists a strictly increasing sequence } \langle \alpha_\xi \mid \xi < \kappa\rangle \text{ such that } \alpha = \sup\{\alpha_\xi \mid \xi < \kappa\} = \sup \bigcup_{\xi<\kappa} f(\{\alpha_\xi\})\Big\}$$

and let C be the topological closure of C_0 in λ (with respect to the order topology). C_0 is closed for κ-sequences and also unbounded in λ, and thus C becomes a club. For our aim, we demonstrate that $C \cap (S' \setminus S) = \emptyset$. Suppose there were a $\gamma \in C \cap (S' \setminus S)$. Then $\gamma \in C_0$, so fix a sequence $\langle \alpha_\xi \mid \xi < \kappa\rangle$ witnessing it. But $\gamma \in S' \setminus S$ implies that $\{\alpha_\xi \mid \xi < \kappa\}$ is unbounded in $D_\kappa(S)$ and that $\{\gamma\} \cup \bigcup_{\xi<\kappa} f(\{\alpha_\xi\})$ is an upper bound of $\{f(\alpha_\xi) \mid \xi < \kappa\}$ in $D_\kappa(S')$. This contradicts the assumption that f is Tukey. □

Theorem 4.5 *Let $\omega \leq \kappa < \lambda$, where κ is regular and λ is strongly κ^+-inaccessible. Denote by $\mathcal{D}(\kappa, \lambda)$ the set of cofinal types with additivity κ and cofinality λ. Then there are 2^λ many pairwise incomparable elements of $\mathcal{D}(\kappa, \lambda)$.*

Proof For $i \in \lambda \times 2$ let $A_i \subseteq E^\lambda_\kappa$ be pairwise disjoint stationary sets. For each $f \in {}^\lambda 2$, put $S_f := \bigcup_{i\in f} A_i$. Now $\langle D_\kappa(S_f) \mid f \in {}^\lambda 2\rangle$ is a family of pairwise incomparable elements of $\mathcal{D}(\kappa, \lambda)$. □

Lemma 4.6 ([14]) *Let $\omega \leq \kappa < \lambda$, where κ is regular and λ is strongly κ^+-inaccessible, and let $S, S' \subseteq E^\lambda_\kappa$ be unbounded in λ. Then*

$$D_\kappa(S) \times D_\kappa(S') \geq \mathcal{P}_\kappa\lambda \quad \textit{iff} \quad S \cap S' \textit{ is nonstationary in } \lambda.$$

Proof ($\Rightarrow$) This is proved by a similar argument as in Lemma 4.4.

($\Leftarrow$) Suppose that $S \cap S'$ is nonstationary in λ. Pick a club $C \subseteq \lambda$ disjoint from $S \cap S'$. For $\xi < \lambda$ pick recursively $\alpha_\xi \in S$ and $\beta_\xi \in S'$ so that for all $\xi < \zeta < \lambda$ there is a $\gamma \in C$ such that

$$\alpha_\xi, \beta_\xi < \gamma < \alpha_\zeta, \beta_\zeta.$$

Consider

$$\begin{array}{rcl} f:\mathcal{P}_\kappa\lambda & \to & D_\kappa(S)\times D_\kappa(S') \\ \cup\!\!\!\!\!\mid & & \cup\!\!\!\!\!\mid \\ x & \mapsto & \langle\{\alpha_\xi \mid \xi\in x\},\{\beta_\xi \mid \xi\in x\}\rangle. \end{array}$$

We show that this function is Tukey. First note that $X\subseteq\mathcal{P}_\kappa\lambda$ is unbounded iff $|\bigcup X|\geq\kappa$. If X is such, then

$$f[X]=\{\langle\{\alpha_\xi \mid \xi\in x\},\{\beta_\xi \mid \xi\in x\}\rangle \mid x\in X\}$$

is also unbounded, since there exists a $\gamma\in C$ which is a limit of two strictly increasing κ-sequences consisting of α_ξ ($\xi\in\bigcup X$) and β_ξ ($\xi\in\bigcup X$) respectively. □

Corollary 4.7 ([14]) *Under the same notations and assumptions as above,*

$$D_\kappa(S)\geq\mathcal{P}_\kappa\lambda \quad \textit{iff} \quad S \textit{ is nonstationary in } \lambda.$$

Proof Just take $S=S'$ in Lemma 4.6. □

Theorem 4.8 ([14]) *Let κ,λ be infinite regular cardinals with $\kappa^+<\lambda$ and λ strongly κ^+-inaccessible. Then there exist directed sets D_1 and D_2 such that*

$$D_i\not\geq\mathcal{P}_\kappa\lambda \quad \textit{for } i=1,2$$

but

$$D_1\times D_2\equiv\mathcal{P}_\kappa\lambda.$$

Proof To prove the Theorem, let A be any unbounded nonstationary subset of E^λ_κ. Split $E^\lambda_\kappa\setminus A$ into two disjoint stationary sets S_1' and S_2'. Then apply Lemma 4.6 to $D_\kappa(S_1'\cup A)\times D_\kappa(S_2'\cup A)$. That $D_i\leq\mathcal{P}_\kappa\lambda$ ($i=1,2$) is clear from Lemma 3.2. □

We will call such a pair D_1,D_2 of directed sets a *Tukey decomposition* of $\mathcal{P}_\kappa\lambda$.

Remark 4.9 We note that, in view of Lemma 4.2, the above D_1 and D_2 must satisfy $\mathsf{add}(D_1)=\mathsf{add}(D_2)$. Besides, D_1 and D_2 must have different cofinal types, because $D\times D\equiv D$ for any directed set D. (This follows from Proposition 2.5, or from the fact that the diagonal $\{\langle d,d\rangle \mid d\in D\}$ is cofinal in $D\times D$.)

5 The tree property for directed sets

Definition 5.1 (κ-tree) ([8]) Let D denote a directed set. A triple $\langle T,\leq_T,s\rangle$ is said to be a *κ-tree* on D if the following holds.

1) $\langle T,\leq_T\rangle$ is a partially ordered set.
2) $s\colon T\to D$ is an order preserving surjection.
3) For all $t\in T$, $s\restriction T_{\leq t}\colon T_{\leq t}\overset{\sim}{\to} D_{\leq s(t)}$ (order isomorphism).

4) For all $d \in D$, $|s^{-1}\{d\}| < \kappa$. We call $s^{-1}\{d\}$ the *level* d of T.

Note that under conditions 1)2)4), condition 3) is equivalent to 3'):

3') (downwards uniqueness principle) $\forall t \in T \forall d' \leq_D s(t) \exists!\, t' \leq_T t\, [s(t') = d']$.

We write $t \downarrow d$ for this unique t'.

If a κ-tree $\langle T, \leq_T, s\rangle$ satisfies in addition

5) (upwards access principle) $\forall t \in T \forall d' \geq_D s(t) \exists t' \geq_T t\, [s(t') = d']$,

then it is called a *κ-arbor* on D.

If D is an infinite regular cardinal κ, a 'κ-tree on κ' coincides with the classical 'κ-tree'. Moreover, an 'arbor' is a generalization of a 'well pruned tree'.

Definition 5.2 (tree property) ([8]) Let $\langle D, \leq_D\rangle$ be a directed set and $\langle T, \leq_T, s\rangle$ a κ-tree on D. $f\colon D \to T$ is said to be a faithful embedding if f is an order embedding and satisfies $s \circ f = \mathrm{id}_D$. If for each κ-tree T on D there is a faithful embedding from D to T, we say that D has the *κ-tree property.* If D has the $\mathsf{add}(D)$-tree property, we say simply D has the *tree property.*

We note that in [8] the tree property in our definition is called 'weakly ramifiable', and a κ-arbor is called κ-arborescence.

Classically, κ has the tree property (as a cardinal) if κ carries no Aronszajn tree, which is, in our words, a κ-tree on κ into which there is no faithful embedding.

Proposition 5.3 ([8]) *Let D be directed set and let $\kappa = \mathsf{add}(D)$. D has the tree property iff for any κ-arbor on D there is a faithful embedding into it.*

In [8], Esser and Hinnion posed the question whether the tree property for directed sets with the same additivity is preserved under products. In fact, for the case $\mathsf{add}(D) \neq \mathsf{add}(E)$, a positive result was given.

Proposition 5.4 ([8]) *Let D, E be directed sets and $\mathsf{add}(D) < \mathsf{add}(E)$. If D has the tree property, then $D \times E$ also has the tree property.*

Proof Put $\kappa := \mathsf{add}(D \times E) = \mathsf{add}(D)$. Let $\langle T, \leq_T, s\rangle$ be an arbitrary κ-tree on $D \times E$. We have to find a faithful embedding $f\colon D \times E \to T$.

First, for each $d \in D$, $T_d := s^{-1}[\{d\} \times E]$ is a κ-tree on $\{d\} \times E$ ($\cong E$). Now we have $\kappa < \mathsf{add}(E)$ and hence there exists a faithful embedding into T_d, and moreover the number of faithful embeddings is less than κ (see [8]). Let F_d be the set of all faithful embeddings from $\{d\} \times E$ to T_d, and let $\overline{g} : D_{\leq d} \times E \to \bigcup_{d' \leq_D d} T'_d$ denote the faithful embedding which is generated by $g \in F_d$. Define

$$T_* := \bigcup_{d \in D} \{\overline{g} \mid g \in F_d\},$$
$$\overline{g} \leq_* \overline{g'} \iff \overline{g} \subseteq \overline{g'},$$
$${s_*}^{-1}\{d\} := \{\overline{g} \mid g \in F_d\}$$

so that $\langle T_*, \leq_*, s_*\rangle$ becomes a κ-tree on D. Since we are assuming that D has the tree property, we get a faithful embedding $f_*\colon D \to T_*$. Define $f(d, e)$ to be $(f_*(d))(e)$, and this completes the proof. □

So we may concentrate on the case $\mathsf{add}(D) = \mathsf{add}(E)$.

The following proposition gives the connetcion between our problem and the Tukey ordering. It is implicit in [10] but we give a direct proof. This has the advantage that the related statements in [10] can now be obtained as corollaries.

Proposition 5.5 *If E has the tree property, $D \le E$ in the Tukey ordering and* $\mathsf{add}(D) = \mathsf{add}(E)$*, then D also has the tree property.*

Proof Let $\kappa := \mathsf{add}(D) = \mathsf{add}(E)$, and let $\langle T, \le_T, s\rangle$ be an arbitrary κ-arbor on D. We have to construct a corresponding κ-arbor on E.

Fix a pair of functions $g\colon D \to E$ and $f\colon E \to D$ such that

$$\forall d \in D \forall e \in E\ [g(d) \le_E e \to d \le_D f(e)]$$

(see Proposition 2.2). Define a κ-arbor $\langle T', \le', s'\rangle$ on E so that

$$s'^{-1}\{e\} = \left\{\langle\, e,\ T_{\le t} \cap s^{-1}g^{-1}[E_{\le e}]\rangle \,\middle|\, t \in s^{-1}\{f(e)\}\right\} \quad \text{for } e \in E,$$

$$\text{and} \qquad \langle e_1, A\rangle \le' \langle e_2, B\rangle \iff e_1 \le_E e_2 \ \wedge\ A \subseteq B \qquad \text{for } \langle e_1, A\rangle, \langle e_2, B\rangle \in T'.$$

We check that $T' = \bigcup_{e\in E} s'^{-1}\{e\}$ is actually a κ-arbor on E. It is straightforward that $\le'$ is transitive, that s' is order preserving, and that each level has size less than κ. To prove the upwards access property, fix $e_0, e \in E$ with $e_0 \le_E e$ and $t_0 \in s^{-1}\{f(e_0)\}$ arbitrarily. Take some upper bound of $\{f(e_0), f(e)\}$ in D, say d^*. By the upwards access property of T, there is some $t^* \in s^{-1}\{d^*\}$ with $t^* \ge_T t_0$. Then by the downwards uniqueness property of T,

$$\langle\, e_0,\ T_{\le t_0} \cap s^{-1}g^{-1}[E_{\le e_0}]\rangle \le' \langle\, e,\ T_{\le t^*} \cap s^{-1}g^{-1}[E_{\le e}]\rangle \in s'^{-1}\{e\}.$$

To prove downwards uniqueness, fix $e_0 \le_E e$ and $t \in s^{-1}\{f(e)\}$ arbitrarily. Take an upper bound d^* of $\{f(e_0), f(e)\}$ in D. By the upwards access property of T, we have a $t^* \in s^{-1}\{d^*\}$ with $t^* \ge_T t$. Put $t_0 := t^* \downarrow f(e_0)$. Then

$$\begin{aligned} &\langle\, e_0,\ T_{\le t_0} \cap s^{-1}g^{-1}[E_{\le e_0}]\rangle &=& \langle\, e_0,\ T_{\le t^*} \cap s^{-1}g^{-1}[E_{\le e_0}]\rangle \\ =\ &\langle\, e_0,\ T_{\le t} \cap s^{-1}g^{-1}[E_{\le e_0}]\rangle &=& \langle\, e,\ T_{\le t} \cap s^{-1}g^{-1}[E_{\le e}]\rangle \downarrow e_0. \end{aligned}$$

By assumption, there exists a faithful embedding $\varphi\colon E \to T'$. From it we can deduce a faithful embedding from D into T, by choosing the image to be exactly $\bigcup\{A \mid \exists e \in E\ [\langle e, A\rangle = \varphi(e)]\}$. □

Thus the tree property is a property applying to the cofinal type of a directed set.

Remark 5.6 We note that this proposition does not hold if $\mathsf{add}(D) \ne \mathsf{add}(E)$. $D = \omega_1$ and $E = \mathcal{P}_\omega(\omega_1)$ is a counterexample.

Corollary 5.7 ([8]) *If D has the tree property, then* $\mathsf{add}(D)$ *has the tree property in the classical sense.*

By Hechler's theorem (see [4]), the eventual dominance order on the reals $\langle {}^{\omega}\omega, \leq^* \rangle$ can be consistently cofinally similar with any directed set which has $\mathsf{add}(D) \geq \aleph_1$. Hence to obtain the following result we apply Hechler's theorem by taking $\langle D, \leq_D \rangle = \langle \kappa, \in \rangle$. For (1), we let $\kappa = \omega_1$, and for (2), we let κ be weakly compact.

Theorem 5.8

(1) ZFC *and* $\mathsf{ZFC}+$ *"$\langle {}^{\omega}\omega, \leq^* \rangle$ does not have the tree property" are equiconsistent.*

(2) $\mathsf{ZFC} + \exists$*weakly compact and* $\mathsf{ZFC}+$ *"$\langle {}^{\omega}\omega, \leq^* \rangle$ has the tree property" are equiconsistent.*

Since Hechler's theorem holds with $\langle {}^{\omega}\omega, \leq^* \rangle$ replaced by $\mathcal{M}$ [2] or $\mathcal{N}$ [5], we have analogous results for $\mathcal{M}$ and $\mathcal{N}$.

6 Mild ineffability

Mild ineffability was introduced by DiPrisco and Zwicker, and studied by Carr [6] in detail. It can be viewed as a kind of tree property for $\mathcal{P}_\kappa\lambda$. We give the definition and an overview on the basic facts. In all the statements of section 6 and 7, the possibility of taking $\kappa = \omega$ is not excluded.

Definition 6.1 (mild ineffability) ([6]) $\mathcal{P}_\kappa\lambda$ is said to be *mildly ineffable* (or κ is *mildly λ-ineffable*) iff for any given $\langle A_x \mid x \in \mathcal{P}_\kappa\lambda \rangle$ with $A_x \subseteq x$ for all x, there exists some $A \subseteq \lambda$ such that

$$\forall x \in \mathcal{P}_\kappa\lambda \exists y \in \mathcal{P}_\kappa\lambda \; [x \subseteq y \wedge A_y \cap x = A \cap x].$$

Proposition 6.2 ([6]) *For a cardinal κ, the following are equivalent:*

(a) *κ is mildly κ-ineffable.*

(b) *κ is strongly inaccessible and has the tree property.*

(c) *κ is weakly compact.*

Proposition 6.3 ([6]) *If κ is mildly λ-ineffable and $\kappa \leq \lambda' \leq \lambda$, then κ is mildly λ'-ineffable.*

The relation between mild ineffability and strong compactness for pairs of cardinals κ, λ is as follows.

Proposition 6.4 ([6]) *For cardinals $\kappa \leq \lambda$,*

(1) *If κ is mildly $2^{\lambda^{<\kappa}}$-ineffable then κ is λ-strongly compact.*

(2) *If κ is λ-strongly compact then κ is mildly λ-ineffable.*

Proof (1) Let $\mathcal{P}(\mathcal{P}_\kappa\lambda) = \{X_\alpha \mid \alpha < 2^{\lambda^{<\kappa}}\}$. For each $x \in \mathcal{P}_\kappa(2^{\lambda^{<\kappa}})$, we put

$$A_x := \{\alpha \in x \mid x \cap \lambda \in X_\alpha\}.$$

By the mild $2^{\lambda^{<\kappa}}$-ineffability of κ, there exists an $A \subseteq 2^{\lambda^{<\kappa}}$ such that

$$\forall x \in \mathcal{P}_\kappa(2^{\lambda^{<\kappa}}) \exists y \in \mathcal{P}_\kappa(2^{\lambda^{<\kappa}})\ [x \subseteq y \wedge A_y \cap x = A \cap x].$$

If we let $\mathcal{U} := \{X_\alpha \mid \alpha \in A\}$, then one can check (by applying the above formula to suitable x's) that $\mathcal{U}$ is a κ-complete fine ultrafilter on $\mathcal{P}_\kappa\lambda$.
(2) Assume that there exists a κ-complete fine ultrafilter $\mathcal{U}$ on $\mathcal{P}_\kappa\lambda$. We are given $\langle A_x \mid x \in \mathcal{P}_\kappa\lambda\rangle$ such that $A_x \subseteq x$ for all x. For each $\alpha < \lambda$, put $X_\alpha := \{x \in \mathcal{P}_\kappa\lambda \mid \alpha \in A_x\}$. Let $A := \{\alpha < \lambda \mid X_\alpha \in \mathcal{U}\}$. We check that this is the required set. Let $x \in \mathcal{P}_\kappa\lambda$ be arbitrary. Then $X_\alpha \in \mathcal{U}$ for $\alpha \in x \cap A$, and $\mathcal{P}_\kappa\lambda \setminus X_\alpha \in \mathcal{U}$ for $\alpha \in x \setminus A$. Put

$$X := \bigcap\{X_\alpha \mid \alpha \in x \cap A\} \cap \bigcap\{\mathcal{P}_\kappa\lambda \setminus X_\alpha \mid \alpha \in x \setminus A\} \in \mathcal{U}.$$

X is cofinal in $\mathcal{P}_\kappa\lambda$ since $\mathcal{U}$ is fine, so we can pick $y \in X$ with $y \supseteq x$, and thus $A_y \cap x = A \cap x$. □

Corollary 6.5 (GCH) *Assume κ is not strongly compact. Let λ be the least cardinal such that κ is not λ-strongly compact, and let μ be the least cardinal such that κ is not mildly μ-ineffable. Assume that λ is regular. Then $\mu = \lambda$ or $\mu = \lambda^+$.*

Corollary 6.6 ([6]) *For a cardinal κ, κ is mildly λ-ineffable for all $\lambda \geq \kappa$ iff κ is strongly compact.*

7 Characterization of the tree property by mild ineffability

The next theorem is stated in [9, Theorem 3.3] with a different formulation. Using the cardinal width, we can state the theorem in a more convenient way.

Theorem 7.1 ([14], cf [9]) *Let D be a directed set and let $\kappa := \mathsf{add}(D)$ be strongly inaccessible. The following are equivalent:*

(a) *D has the tree property.*
(b) *For all $\lambda < \mathsf{wid}(D)$, $\mathcal{P}_\kappa\lambda$ has the tree property.*
(c) *For all $\lambda < \mathsf{wid}(D)$, $\mathcal{P}_\kappa\lambda$ is mildly ineffable.*

The proof we give here is a combination of the proofs in [14] and [7]. It enabled a good deal of simplification.

Definition 7.2 ([7]) Let $\langle T, \leq_T, s\rangle$ be an arbor on a directed set D. We define an equivalence relation on D. For $d_1, d_2 \in D$,

$$d_1 \sim d_2 \iff \forall d' \in D\ [d' \geq d_1, d_2 \rightarrow \forall t_1 \in s^{-1}\{d_1\}\exists! t_2 \in s^{-1}\{d_2\}\forall u \in s^{-1}\{d'\} [t_1 \leq_T u \longleftrightarrow t_2 \leq_T u]].$$

In the above formula, we say that the $t_1 \in s^{-1}\{d_1\}$ and the corresponding $t_2 \in s^{-1}\{d_2\}$ are *linked.* Equivalent levels give the same partial information on how to take the faithful embedding. Notice that $d_1 \sim d_2$ does not imply that they are comparable.

Lemma 7.3 *For the relation defined above,*

$$d_1 \sim d_2 \iff \exists d' \in D\ [d' \geq d_1, d_2 \wedge \forall t_1 \in s^{-1}\{d_1\}\exists! t_2 \in s^{-1}\{d_2\}\forall u \in s^{-1}\{d'\} [t_1 \leq_T u \longleftrightarrow t_2 \leq_T u]].$$

Thus $\sim$ is in fact an equivalence relation on D.

Proof ($\Rightarrow$) Trivial, since D is directed.
($\Leftarrow$) Use upwards access and downwards uniqueness. □

Lemma 7.4 *Assume that $\kappa := \mathsf{add}(D)$ is strongly inaccessible, and let $\langle T, \leq_T, s\rangle$ be a κ-arbor on D. If $F \subseteq D$ is a set of representatives with respect to $\sim$, then F is thin.*

Proof Fix an arbitrary $d_0 \in D$. For each element $d \in D_{\leq d_0}$,

$$P_d := \{\{u \in s^{-1}\{d_0\} \mid u \geq_T t\} \mid t \in s^{-1}\{d\}\}$$

provides a partiton of $s^{-1}\{d_0\}$. By Lemma 7.3, we see that $P_{d_1} = P_{d_2}$ iff $d_1 \sim d_2$ for $d_1, d_2 \in D_{\leq d_0}$. Since κ is strongly inaccessible, the number of partitions of $s^{-1}\{d_0\}$ is less than κ. □

Proof of 7.1 (a) $\Rightarrow$ (b) Let $\mathsf{add}(D) \leq \lambda < \mathsf{wid}(D)$. Then $\mathcal{P}_\kappa\lambda \leq D$, so by Proposition 5.5 $\mathcal{P}_\kappa\lambda$ has the tree property.

(b) $\Rightarrow$ (c) It suffices to show, for an arbitrary λ, that the tree property for $\mathcal{P}_\kappa\lambda$ implies its mild ineffability. Assume that $\mathcal{P}_\kappa\lambda$ has the tree property. Suppose we are given a family $\langle A_x \mid x \in \mathcal{P}_\kappa\lambda\rangle$ such that $A_x \in {}^x 2$ for $x \in \mathcal{P}_\kappa\lambda$. Then

$$\langle\{A_y \restriction x \mid y \supseteq x\} \mid x \in \mathcal{P}_\kappa\lambda\rangle$$

is a κ-tree on $\mathcal{P}_\kappa\lambda$ since κ is strongly inaccessible. Therefore we have a faithful embedding, which is the same as an $A \in {}^\lambda 2$ such that

$$\forall x \in \mathcal{P}_\kappa\lambda \exists y \in \mathcal{P}_\kappa\lambda\ [x \subseteq y \wedge A_y \restriction x = A \restriction x].$$

(c) $\Rightarrow$ (a) Let $\langle T, \leq_T, s\rangle$ be a κ-arbor on D. Our goal is to produce a faithful embedding $f\colon D \to T$. Fix a set of representatives $F \subseteq D$ with respect to the equivalence $\sim$ defined above.

Put $\lambda := |F|$. Then $T^* := s^{-1}[F]$ also has size λ. As we have $\lambda < \mathsf{wid}(D)$, the assumption (c) says κ is mildly λ-ineffable.

We define a family $\langle A_x \mid x \in \mathcal{P}_\kappa T^*\rangle$ to which we will apply the mild ineffability. For each $x \in \mathcal{P}_\kappa T^*$, pick an upper bound $d \in D$ of $s[x]$, and fix $t \in s^{-1}\{d\}$. For $v \in x$ we put $A_x(v) = 1$ if $v \leq_T t$, and $A_x(v) = 0$ otherwise. Then we get an $A \in {}^{T^*}2$ such that

$$\forall x \in \mathcal{P}_\kappa T^* \exists y \in \mathcal{P}_\kappa T^* \; [x \subseteq y \wedge A_y \restriction x = A \restriction x].$$

It remains to derive the faithful embedding f from A. For each $d \in F$, let v_d be the unique $v \in s^{-1}\{d\}$ such that $A(v) = 1$. Then $d \mapsto v_d$ is an embedding from F to T^*. To extend this map to all of D, let $d \in D$ be arbitrary and let $d \sim d^* \in F$ be the corresponding representative. Now v_{d^*} is defined, and we can put $f(d)$ to be the unique $u \in s^{-1}\{d\}$ such that u and v_{d^*} are linked. One can verify that $f\colon D \to T$ is a faithful embedding. □

Corollary 7.5 *Let κ be strongly inaccessible and $\lambda \geq \kappa$. Then*

$\mathcal{P}_\kappa\lambda$ has the tree property iff κ is mildly $\lambda^{<\kappa}$-ineffable.

8 Application of the Tukey decomposition

Theorem 8.1 *Assume that κ is weakly compact but not strongly compact, and that $\lambda > \kappa^+$ is the least cardinal such that κ is not mildly λ-ineffable. Assume further that λ is strongly κ^+-inaccessible. Then there exist directed sets D_1 and D_2 with $\mathsf{add}(D_1) = \mathsf{add}(D_2) = \kappa$ such that*

D_1 and D_2 have the tree property

but

$D_1 \times D_2$ does not have the tree property.

Proof By the Theorem 4.8, we have directed sets D_1 and D_2 such that $D_i \not\geq \mathcal{P}_\kappa\lambda$ for $i = 1, 2$ but $D_1 \times D_2 \equiv \mathcal{P}_\kappa\lambda$. Recalling how D_1 and D_2 were defined (or by Remark 4.9), we see that $\mathsf{add}(D_1) = \mathsf{add}(D_2) = \kappa$. By Theorem 7.1, D_1 and D_2 have the tree property but $D_1 \times D_2$ does not have the tree property. □

At last, we discuss the consistency of the assumption in the above theorem.

We quote the following theorem.

Theorem 8.2 ([13]) *If λ is regular and κ is mildly λ-ineffable, then for each regular $\eta < \kappa$, any stationary set $S \subseteq E^\lambda_\eta$ is reflecting.*

Here we call $S \subseteq E^\lambda_\eta$ reflecting iff there is a limit ordinal $\gamma < \lambda$ such that $S \cap \gamma$ is stationary in γ. Otherwise S is called nonreflecting.

Assuming a strongly compact cardinal κ, we perform a forcing which destroies the mild λ^+-ineffability of κ and which at the same time preserves the mild λ-ineffability. By Theorem 8.2 the standard forcing which adds a nonreflecing stationary subset (see [3, Definition 4.14]) serves our purpose. To be precise, define P to be the forcing which consists of conditions $p \in {}^{<\lambda^+}2$ (i.e. p is a characteristic function for a subset of an ordinal $< \lambda^+$) such that if we let $S_p := p^{-1}\{1\}$, then $S_p \subseteq E^{\lambda^+}_\omega$ and for all limit ordinals $\gamma < \lambda^+$, $S_p \cap \gamma$ is nonstationary in γ. For $p, q \in P$, p extends q iff $p \supseteq q$. It is known [3] that P preserves cardinals, cofinalities, and GCH, and that P is λ-strategically closed.

This completes the proof.

Theorem 8.3 *If we assume the consistency of* ZFC$+\exists$*strongly compact, then* ZFC$+$*"the tree property for directed sets is not always preserved under products" is consistent.*

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