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Path description of conserved quantities of generalized periodic box-ball systems

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Abstract

The present article is a review on the conserved quantities of periodic box-ball systems (PBBS) with arbitrary kinds of balls and box capacity greater than or equal to one. By introducing the notion of nonintersecting paths on the two dimensional array of boxes, we give a combinatorial formula for the conserved quantities of the generalized PBBS using these paths.

1 Introduction

The box-ball system (BBS) is a reinterpretation of a soliton cellular automaton proposed by Takahashi-Satsuma [1] as a dynamical system of balls in a one dimensional array of boxes [2]. Hence, the BBS shows both a feature of cellular automata (CAs) and that of solitons.

The periodic box-ball system (PBBS) is the BBS in which the updating rule is extended to be compatible with a periodic boundary condition [8]. Let us consider a one-dimensional array of N boxes. A periodic boundary condition is imposed by assuming that the Nth box is adjacent to the first one. (We may imagine that the boxes are arranged in a circle.) In the generalized PBBS (GPBBS), the capacity of the nth $(1 \leq n \leq N)$ box is denoted by a positive integer θ_n, and we suppose that there are M kinds of balls distinguished by an integer index j $(1 \leq j \leq M)$. When ${}^{\forall}n\ \theta_n = 1$ and $M = 1$, the GPBBS coincides with the PBBS. Then, the rule for the time evolution of the GPBBS from time step t to $t+1$ is given as follows:

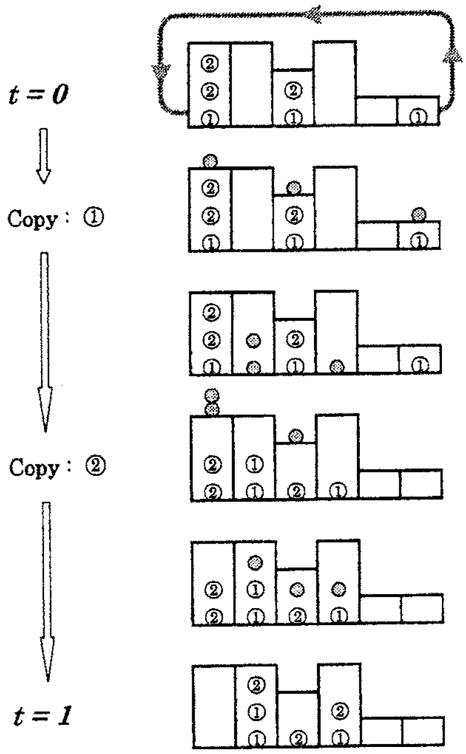
1. At each box, create the same number of copies of the balls with index 1.

2. Choose one of the copies arbitrarily and move it to the nearest box with an available space to the right of it.

3. Choose one of the remaining copies and move it to the nearest available box on the right of it.

4. Repeat the above procedure until all the copies have been moved.
5. Delete all the original balls with index 1.
6. Perform the same procedure for the balls with index 2.
7. Repeat this procedure successively until all of the balls are moved.

An example of the time evolution of the GPBBS according to this rule is shown in Fig. 1.



$$N = 6,\ M = 2,\ \theta_1 = \theta_2 = \theta_4 = 3,\ \theta_3 = 2,\ \theta_5 = \theta_6 = 1.$$

Figure 1: Time evolution rule for PBBS.

In this article, we investigate the conserved quantities of the GPBBS for arbitrary M. In section 2, we derive the path description of characteristic polynomial of particular matrices. In section 3, we briefly summarize the results of Ref. [9], which we will use in the subsequent sections. In section 4, we treat the ndKP equation which corresponds to the GPBBS. We shall obtain an explicit expression for the conserved quantities of the ndKP equation. Using the results in section 4, we construct the conserved quantities of the GPBBS in section 5. In section 6, we discuss algebraic aspects of the GPBBS with respect to the Affine Weyl group and the crystals of quantum affine algebra. Section 7 is devoted to the concluding remarks.

2 Path description of a particular determinant

For a particular matrix B which contains a parameter μ in upper half elements, we give a combinatorial description for coefficients of the characteristic polynomial $\det(\lambda I - B)$ in λ and μ in terms of nonintersecting paths (theorem 2.1).

Let

$$B := (D_0 - \Upsilon)(D_1 - \Upsilon)\cdots(D_M - \Upsilon), \tag{2.1}$$

where D_i $(i = 0, 1, \ldots, M)$ are diagonal matrices,

$$D_i := \begin{bmatrix} x_{1,i} & & & & \\ & x_{2,i} & & & \\ & & x_{3,i} & & \\ & & & \ddots & \\ & & & & x_{N,i} \end{bmatrix},$$

and

$$\Upsilon := \begin{bmatrix} & & & \mu \\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

If we put $D_i^{(0)} := D_i$ and

$$D_i^{(r)} := \begin{bmatrix} x_{N-r+1,i} & & & & & & \\ & x_{N-r+2,i} & & & & & \\ & & \ddots & & & & \\ & & & x_{N,i} & & & \\ & & & & x_{1,i} & & \\ & & & & & \ddots & \\ & & & & & & x_{N-r,i} \end{bmatrix}$$

for $0 < r < N$, we have

$$D_i^{(r+1)}\Upsilon = \Upsilon D_i^{(r)}.$$

Hereafter, for $i = 0$, we define

$$\sum_{c_1 < c_2 < \cdots < c_i} \cdots := 1.$$

Then, using the notation $D_i^{(r)}$, we obtain

$$B = \sum_{\ell=0}^{M+1} (-1)^\ell \left(\sum_{\substack{0\le h_1<h_2<\cdots \\ \cdots<h_{M-\ell+1}\le M}} D_{h_1}^{(h_1)} D_{h_2}^{(h_2-1)} \cdots D_{h_{M-\ell+1}}^{(h_{M-\ell+1}-M+\ell)} \right) \Upsilon^\ell \tag{2.2}$$

and

$$\begin{aligned} &\det(\lambda I - B) \\ &= \sum_{k=0}^{N} (-1)^{N-k} \lambda^k \sum_{j=0}^{N-k} \mu^j \sum_{\substack{X\subset\{1,2,\dots,N\} \\ \sharp X=N-k}} \sum_{\substack{J\subset X \\ \sharp J=j}} \\ &\quad \sum_{\sigma\in S_X^J} \operatorname{sgn}(\sigma) \left(\prod_{n\in J} \left(\sum_{\substack{0\le h_1<h_2<\cdots \\ \cdots<h_{M-N-n+\sigma(n)+1}\le M}} \prod_{i=1}^{M-N-n+\sigma(n)+1} x_{n-h_i+i-1,h_i} \right) \right. \\ &\quad \left. \times \left(\prod_{n\in X-J} \left(\sum_{\substack{0\le h_1<h_2<\cdots \\ \cdots<h_{M-n+\sigma(n)+1}\le M}} \prod_{i=1}^{M-n+\sigma(n)+1} x_{n-h_i+i-1,h_i} \right) \right) \right) \end{aligned} \tag{2.3}$$

A combinatorial description of the coefficients is possible.

By $C_{N,M+1}$ we denote the $N\times(M+1)$ boxes in Fig. 2 and by (n,m)-box the box at the nth column in the $(m+1)$th row. We assume that the Nth column is adjacent to the first one.

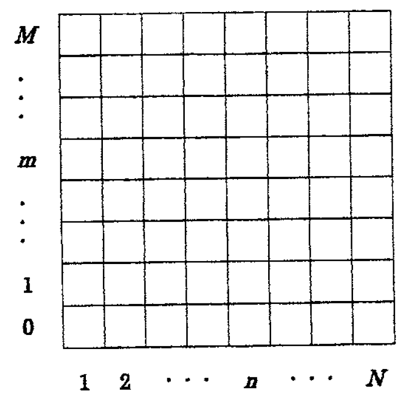


Figure 2: $N\times(M+1)$ boxes.

Let a and b be column indices $(a,b=1,2,\dots,N)$. A *path* connecting the *initial point* a and the *end point* b is a (continuous) polygonal line from the initial point a to the end point b which consists of (i), (ii) or (iii) in Fig. 3

locally; here by the initial point a we mean the middle point of the south edge of $(a,0)$-box and by the end point b the middle point of the north edge of (b,M)-box. For example, the left part in Fig. 4 shows a path connecting the initial point 1 and the end point 1, and the right part shows a path from 5 to 2.

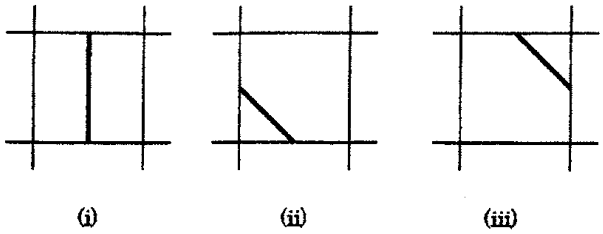


Figure 3: A line can pass through a box in three possible ways.

There is a natural correspondence between $\prod_{i=1}^{M-\ell+1} x_{n-h_i+i-1,h_i}$ $(0 \le h_1 < h_2 < \cdots < h_{M-\ell+1} \le M)$ and a path on $C_{N,M+1}$. To put it concretely, we draw the line (i) on $(n-h_i+i-1,h_i)$-box $(i=1,2,\dots,M-\ell+1)$. For each r, $h_i < r < h_{i+1}$, we draw the line (ii) on $(n+i-r,r)$-box and the line (iii) on $(n+i-r-1,r)$-box where $h_0=-1$ and $h_{M-\ell+2}=M+1$; then we obtain the path. For example, for $N=8$ and $M=5$, $x_{1,0}x_{1,1}x_{1,2}x_{1,3}x_{1,4}x_{1,5}$ and $x_{5,0}x_{4,2}x_{2,5}$ correspond to the paths in Fig. 4 respectively.

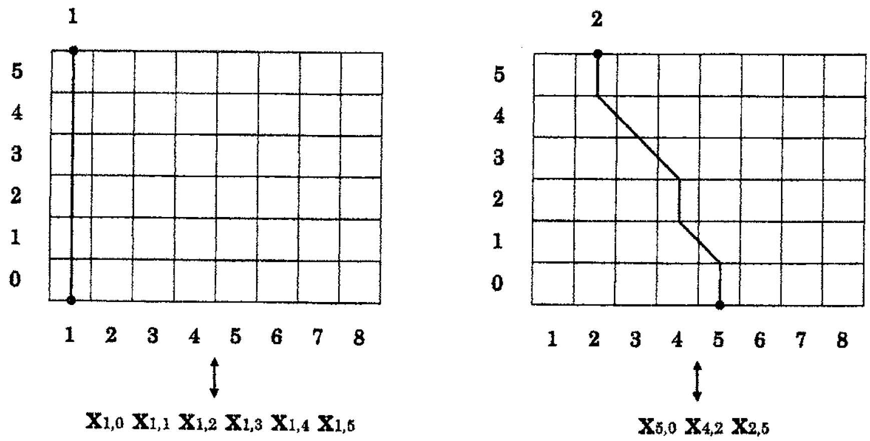


Figure 4: Paths corresponding to $x_{1,0}x_{1,1}x_{1,2}x_{1,3}x_{1,4}x_{1,5}$ and $x_{5,0}x_{4,2}x_{2,5}$.

Let $X=\{d_1,d_2,\dots,d_{N-k}\}$ $(1 \le d_1 < d_2 < \cdots < d_{N-k} \le N)$, and we denote by $\mathcal{P}(d;\sigma)$ the set of all paths which connect the initial point d and the end point $\sigma(d)$ $(d \in X, \sigma \in S_X$; cf. Fig. 5). Define $\xi_{n,m} : \mathcal{P}(d;\sigma) \to$

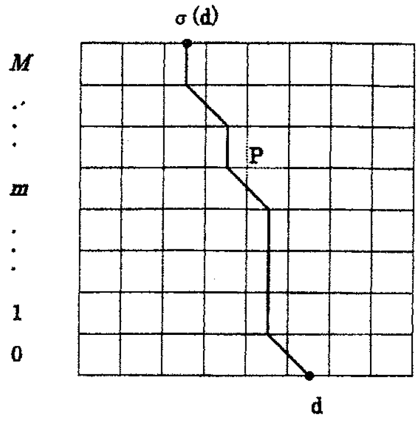


Figure 5: A path $P \in \mathcal{P}(d;\sigma)$.

$\{x_{n,m}, 1\}$ as

$$\xi_{n,m}(P) := \begin{cases} x_{n,m} & \left(\begin{array}{l} P \text{ has the vertical line} \\ \text{on the } (n,m)\text{-box of } C_{N,M+1}. \end{array}\right), \\ 1 & \text{(otherwise)} \end{cases} \tag{2.4}$$

where $P \in \mathcal{P}(d;\sigma)$.

We introduce

$$\mathcal{P}^{(j)}(d_1,\dots,d_{N-k}) := \left\{ (P_1,\dots,P_{N-k}) \;\middle|\; \begin{array}{l} P_i \in \mathcal{P}(d_i;\sigma) \;\; (i=1,2,\dots,N-k). \\ P_i \text{ and } P_j \text{ are nonintersecting} \\ \text{for any } i \text{ and } j,\ i \neq j. \end{array} \right\} \tag{2.5}$$

The upper bound of the summation over j is $\left[\frac{N-k}{N}(M+1)\right]$ where $[\ell]$ denotes the largest integer which does not exceed ℓ.

Then, we obtain the following theorem.

Theorem 2.1
For B defined in (2.1), it holds that

$$\det(\lambda I - B) = \sum_{k=0}^{N}(-1)^{N-k}\lambda^k \sum_{j=0}^{\left[\frac{(N-k)(M+1)}{N}\right]} (-1)^{j(N-k-1)}\mu^j \sum_{\substack{1\le d_1<d_2<\cdots \\ \cdots<d_{N-k}\le N}} \sum_{\substack{(P_1,\dots,P_{N-k}) \\ \in\mathcal{P}^{(j)}(d_1,\dots,d_{N-k})}} \prod_{i=1}^{N-k}\prod_{n=1}^{N}\prod_{m=0}^{M}\xi_{n,m}(P_i)$$

where $\xi_{n,m}$ *and* $\mathcal{P}^{(j)}(d_1, \ldots, d_{N-k})$ *are defined in (2.4) and (2.5) respectively.* □

3 GPBBS and ndKP equation

In order to describe the dynamics of the GPBBS, we introduce a new independent variable s ($s \in \mathbb{Z}$). As any integer s can be uniquely expressed as $s = Mt + j$ ($t \in \mathbb{Z}$, $1 \leq j \leq M$), we denote by u_n^s the number of balls with index $j \equiv s \mod M$ in the nth box at time step $t \equiv \left[\frac{s-1}{M}\right]$, where $[x]$ denotes the largest integer which does not exceed x. In other words, the new *time* variable s is a refinement of the original time, indicating explicitly when balls with index j move.

We assume that θ_n and u_n^s satisfy the relation

$$\sum_{n=1}^{N} \theta_n - \sum_{j=1}^{M} \sum_{n=1}^{N} u_n^j \geq \sum_{n=1}^{N} u_n^k \quad (k = 1, 2, \ldots, M). \tag{3.1}$$

The first and second terms of the left-hand side of (3.1) represent the number of spaces and the number of balls in the GPBBS respectively, hence the left-hand side is the total number of free spaces of the GPBBS. The right-hand side of (3.1) is the number of balls with index k. Thus (3.1) requires the total number of free spaces of the GPBBS to be larger than the number of copies of any type of ball in the time evolution process.

Let us consider the process at time s, *i.e.*, the movement of the balls with index j at time step t where $s = Mt + j$; we often use s instead of j, *i.e.* we treat the indices modulo M. If we define κ_n^s, which denotes the number of spaces of the nth box at s, by

$$\kappa_n^s := \theta_n - \left(u_n^s + u_n^{s-1} + \cdots + u_n^{s-M+1}\right),$$

condition (3.1) is rewritten as

$$\sum_{n=1}^{N} \kappa_n^s \geq \sum_{n=1}^{N} u_n^{s-M+k} \quad (k = 1, 2, \ldots, M).$$

Then we have the following theorem.

Theorem 3.1 ([9])
The time evolution of the GPBBS is described by an ultradiscrete equation:

$$u_n^s - \kappa_n^{s-1} = \max_{k=1,\dots,N}\left[\sum_{j=1}^{k} u_{n-j}^{s-M} - \kappa_{n-j+1}^{s-1}\right] - \max\left[0, \max_{k=1,\dots,N-1}\left[\sum_{j=1}^{k} u_{n-j}^{s-M} - \kappa_{n-j+1}^{s-1}\right]\right]. \quad (3.2)$$

□

The ndKP equation is obtained from the generating formula of the KP hierarchy [12, 13]. The ndKP equation is given as

$$\begin{aligned}(b(m) - c(n))\,\tau(l+1,m,n)\tau(l,m+1,n+1)\\ + (c(n) - a(l))\,\tau(l,m+1,n)\tau(l+1,m,n+1)\\ + (a(l) - b(m))\,\tau(l,m,n+1)\tau(l+1,m+1,n) = 0,\end{aligned} \quad (3.3)$$

where $l, m, n \in \mathbb{Z}$ are independent variables, the tau function $\tau : \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ (or $\mathbb{C}$) is dependent variable and the coefficients $a(l), b(m), c(n)$ are arbitrary functions which depend on the independent variables l, m, n respectively.

In order to relate the ndKP equation to the GPBBS, we take $a(l) = 0, b(m) = 1, c(n) = 1 + \delta_n$ and impose the following constraint on $\tau(l, m, n)$:

$$\tau(l,m,n) = \tau(l-M, m-1, n).$$

If we define $\sigma_n^s := \tau(s-1, 0, n)$, (3.3) turns into

$$\frac{\sigma_{n+1}^{s+M-1}\sigma_{n+1}^{s}}{\sigma_{n+1}^{s+M}\sigma_{n+1}^{s-1}} - (1+\delta_{n+1})\frac{\sigma_n^{s-1}\sigma_{n+1}^{s}}{\sigma_{n+1}^{s-1}\sigma_n^{s}} = -\delta_{n+1}\frac{\sigma_n^{s+M}\sigma_{n+1}^{s}}{\sigma_n^{s}\sigma_{n+1}^{s+M}}. \quad (3.4)$$

Furthermore, we define U_n^s and K_n^s as

$$U_n^s := \frac{\sigma_{n+1}^{s}\sigma_n^{s+1}}{(1+\delta_{n+1})\,\sigma_n^{s}\sigma_{n+1}^{s+1}}, \quad \frac{1}{K_n^s} := \delta_{n+1}\cdot\prod_{j=1}^{M} U_n^{s-j+1}$$

and impose the following periodic condition on U_n^s:

$$U_n^s = U_{n+N}^s.$$

Then, from (3.4), we have

$$\frac{U_n^s}{K_n^{s-1}} = \frac{\displaystyle\sum_{k=1}^{N}\prod_{j=1}^{k}\frac{U_{n-j}^{s-M}}{K_{n-j+1}^{s-1}}}{1+\displaystyle\sum_{k=1}^{N-1}\prod_{j=1}^{k}\frac{U_{n-j}^{s-M}}{K_{n-j+1}^{s-1}}}. \tag{3.5}$$

To take the ultradiscrete limit, we put $U_n^s = e^{u_n^s/\epsilon}, K_n^s = e^{\kappa_n^s/\epsilon}, 1/\delta_{n+1} = e^{\theta_n/\epsilon}$. Then, we found

Theorem 3.2 ([9])
The ultradiscrete limit of the constrained ndKP equation (3.4) with the periodic boundary condition coincides with the time evolution equation of the GPBBS (3.2). □

4 Conserved quantities of ndKP equation

In Ref. [9] we derived the Lax representation for the ndKP equation when it has period N in the spatial variable n. In short, the equation (3.5) is equivalent to the matrix equation

$$\widetilde{M}(s)L(M;s) = L(M;s-1)\widetilde{M}(s)$$

where $\widetilde{M}(s) = G_{U;s} - \widetilde{\Upsilon}$,

$$L(M;s) = \left(-G_{K;s} + \widetilde{\Upsilon}\right)\left(G_{U;s-M+1} - \widetilde{\Upsilon}\right)\left(G_{U;s-M+2} - \widetilde{\Upsilon}\right)\cdots\left(G_{U;s} - \widetilde{\Upsilon}\right), \tag{4.1}$$

$$G_{K;s} := \begin{bmatrix} \frac{1}{K_1^s} & & & & \\ & \frac{1}{K_2^s} & & & \\ & & \frac{1}{K_3^s} & & \\ & & & \ddots & \\ & & & & \frac{1}{K_N^s} \end{bmatrix},$$

$$G_{U;s} := \begin{bmatrix} \frac{1}{U_1^s} & & & & \\ & \frac{1}{U_2^s} & & & \\ & & \frac{1}{U_3^s} & & \\ & & & \ddots & \\ & & & & \frac{1}{U_N^s} \end{bmatrix},$$

$$\widetilde{\Upsilon} := \begin{bmatrix} & & & & (1+\delta_N)\cdot\eta \\ 1+\delta_1 & & & & \\ & 1+\delta_2 & & & \\ & & \ddots & & \\ & & & 1+\delta_{N-1} & \end{bmatrix};$$

here, η is an arbitrary parameter.

This means

$$\det(\lambda I + L(M;s)) = \det(\lambda I + L(M;s-1));$$

therefore, the coefficients e_k of the characteristic polynomial

$$\det(\lambda I + L(M;s)) \\ = \lambda^N + e_{N-1}\lambda^{N-1} + e_{N-2}\lambda^{N-2} + \cdots + e_1\lambda + e_0$$

are conserved in time s. Furthermore, since η is arbitrary and e_k contain η, if we define $e_k^{[j]}$ by

$$e_k = \sum_j e_k^{[j]}\eta^j, \tag{4.2}$$

then $e_k^{[j]}$ are also conserved.

Let $\Delta := \prod_{i=1}^{N}(1+\delta_i)$,

$$\Upsilon := \begin{bmatrix} & & & \eta\Delta \\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix},$$

and

$$D_\delta := \begin{bmatrix} 1 & & & & \\ & 1+\delta_1 & & & \\ & & (1+\delta_1)(1+\delta_2) & & \\ & & & \ddots & \\ & & & & \prod_{i=1}^{N-1}(1+\delta_i) \end{bmatrix}.$$

It follows that

$$\Upsilon = (D_\delta)^{-1}\,\widetilde{\Upsilon}\,D_\delta.$$

Hence, we have the following Lemma.

Lemma 4.1 ([9])
Let

$$L_0(M;s) := (-G_{K;s}+\Upsilon)\,(G_{U;s-M+1}-\Upsilon)\,(G_{U;s-M+2}-\Upsilon)\cdots(G_{U;s}-\Upsilon)\,.$$

Then, it holds that

$$\det(\lambda I + L(M;s)) = \det(\lambda I + L_0(M;s)). \qquad (4.3)$$

□

From theorem 2.1, we obtain a combinatorial formula for $e_k^{[j]}$ immediately.

Theorem 4.1
Put

$$x_{n,0} = \frac{1}{K_n^s}, \qquad x_{n,m} = \frac{1}{U_n^{s-M+m}} \quad (m \neq 0)$$

in (2.4) and put $\mu = \eta\Delta$. Then, for $k = 0, 1, \ldots, N$ and $j = 0, 1, \ldots, \left[\frac{(N-k)(M+1)}{N}\right]$, it holds that

$$e_k^{[j]} = (-1)^{\ell(k,j)}\Delta^j \sum_{\substack{1\le d_1<d_2<\cdots\\ \cdots<d_{N-k}\le N}} \sum_{\substack{(P_1,\ldots,P_{N-k})\\ \in\mathcal{P}^{(j)}(d_1,\ldots,d_{N-k})}} \prod_{i=1}^{N-k}\prod_{n=1}^{N}\prod_{m=0}^{M} \xi_{n,m}(P_i),$$

where $\ell(k,j) := (j+1)N - (k+j+kj)$. □

5 Conserved quantities of GPBBS

Using the results in section 4, we construct the conserved quantities of the GPBBS.

For $k = 0, 1, \ldots, N$ and $j = 0, 1, \ldots, \left[\frac{(N-k)(M+1)}{N}\right]$, the ultradiscrete limit of $e_k^{[j]}$ is

$$\begin{aligned} ue_k^{[j]} &:= -\lim_{\epsilon \to +0} \epsilon \log \left((-1)^{\ell(k,j)} e_k^{[j]} \right) \\ &= -\lim_{\epsilon \to +0} \epsilon \log \left(\Delta^j \sum_{\substack{1 \le d_1 < d_2 < \cdots \\ \cdots < d_{N-k} \le N}} \sum_{\substack{(P_1, \ldots, P_{N-k}) \\ \in \mathcal{P}^{(j)}(d_1, \ldots, d_{N-k})}} \prod_{i=1}^{N-k} \prod_{n=1}^{N} \prod_{m=0}^{M} \xi_{n,m}(P_i) \right). \end{aligned}$$

Since θ_n is the capacity of the nth box,

$$\begin{aligned} \lim_{\epsilon \to +0} \epsilon \log \Delta^j &= j \cdot \lim_{\epsilon \to +0} \epsilon \log \prod_{j=1}^{N} \left(1 + e^{-\theta_j/\epsilon} \right) = j \cdot \sum_{j=1}^{N} \max[0, -\theta_j] \\ &= 0. \end{aligned}$$

Therefore, from theorem 4.1, $ue_k^{[j]}$ is given by

Theorem 5.1
Put

$$x_{n,0} = \kappa_n^s, \qquad x_{n,m} = u_n^{s-M+m} \quad (m \neq 0)$$

in (2.4). Then, for $k = 0, 1, \ldots, N$ *and* $j = 0, 1, \ldots, \left[\frac{(N-k)(M+1)}{N}\right]$, *it holds that*

$$ue_k^{[j]} = \min_{\substack{1 \le d_1 < d_2 < \cdots \\ \cdots < d_{N-k} \le N}} \left[\min_{\substack{(P_1, \ldots, P_{N-k}) \\ \in \mathcal{P}^{(j)}(d_1, \ldots, d_{N-k})}} \left[\sum_{i=1}^{N-k} \sum_{n=1}^{N} \sum_{m=0}^{M} \xi_{n,m}(P_i) \right] \right].$$

□

Remark 5.1
The conserved quantity $ue_k^{[0]}$ $(0 \le k \le N)$ *is trivial. Since* $j = 0$, *all paths are vertical lines. Hence we have*

$$\begin{aligned} ue_k^{[0]} &= \min_{\substack{1 \le d_1 < d_2 < \cdots \\ \cdots < d_{N-k} \le N}} \left[\sum_{i=1}^{N-k} \left(\kappa_{d_i}^s + \sum_{m=1}^{M} u_{d_i}^{s-M+m} \right) \right] \\ &= \min_{\substack{1 \le d_1 < d_2 < \cdots \\ \cdots < d_{N-k} \le N}} \left[\sum_{i=1}^{N-k} \theta_{d_i} \right]. \end{aligned}$$

As θ_n is the capacity of the nth box, $ue_k^{[0]}$ dose not depend on the time steps. So we are not interested in them.

Remark 5.2
If a quantity $A_1(t)$ is conserved in terms of the original time step t, that is, $^\forall t$, $A_1(t) = A_1(0)$, we can always associate it with the quantities $B_k(s)$ $(k = 1, 2, \ldots, M)$ which are conserved in s, that is, $^\forall s$, $B_k(s) = B_k(0)$ $(k = 1, 2, \ldots, M)$. The reason is as follows. First we note that $A_1(t) = F(\{u_n^{Mt+s}\}_{s=1}^{M}) = F(\{u_n^s\}_{s=1}^{M})$. If we put $A_j(t) := F(\{u_n^{Mt+s}\}_{s=j}^{j+M})$, since the time evolution rule for the GPBBS at time step s is just the same as that at $s+1$, $A_j(t)$ is also conserved in terms of the original time step t. Hence if we denote by $B_k(s)$ the kth elementary symmetric polynomial of $\{A_j(t)\}_{j=1}^{M}$, $B_k(s)$ is a conserved quantity in terms of s. We should also note that the number of independent conserved quantities in $\{A_j\}_{j=1}^{M}$ is equal to that of $\{B_k\}_{k=1}^{M}$.

An easy way to read off $\sum_{i=1}^{N-k}\sum_{n=1}^{N}\sum_{m=0}^{M}\xi_{n,m}(P_i)$ is as follows: Associate values $\kappa_n^s, u_n^s, \ldots, u_n^{s-M+1}$ with boxes of $C_{N,M+1}$ as shown in Fig. 6. For $(P_1, \ldots, P_{N-k}) \in \mathcal{P}^{(j)}(d_1, \ldots, d_{N-k})$, summing up the values corresponding to the vertical lines of the paths, we get the value $\sum_{i=1}^{N-k}\sum_{n=1}^{N}\sum_{m=0}^{M}\xi_{n,m}(P_i)$.

M	u_1^s	u_2^s	u_3^s	$\cdots$	u_{N-1}^s	u_N^s
M-1	u_1^{s-1}	u_2^{s-1}	u_3^{s-1}	$\cdots$	u_{N-1}^{s-1}	u_N^{s-1}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2	u_1^{s-M+2}	u_2^{s-M+2}	u_3^{s-M+2}	$\cdots$	u_{N-1}^{s-M+2}	u_N^{s-M+2}
1	u_1^{s-M+1}	u_2^{s-M+1}	u_3^{s-M+1}	$\cdots$	u_{N-1}^{s-M+1}	u_N^{s-M+1}
0	κ_1^s	κ_2^s	κ_3^s	$\cdots$	κ_{N-1}^s	κ_N^s
	1	*2*	*3*	$\cdots$	*N-1*	*N*

Figure 6:

Example 5.1
For a state in Fig. 7 ($N = 10, M = 5$), we obtain a table in Fig. 8. For

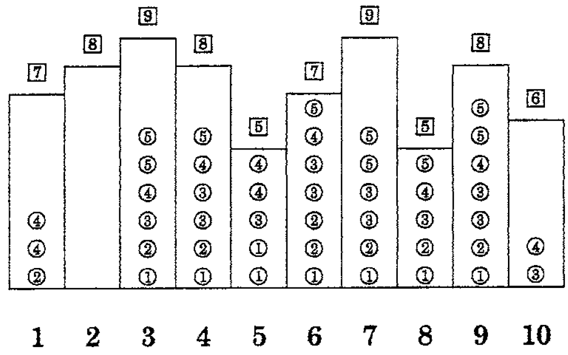


Figure 7: A state of the GPBBS.

	1	2	3	4	5	6	7	8	9	10
5	0	0	2	1	0	1	2	1	2	0
4	2	0	1	1	2	1	0	1	1	1
3	0	0	1	2	1	2	2	1	2	1
2	1	0	1	1	0	2	1	1	1	0
1	0	0	1	1	2	1	1	1	1	0
0	4	8	3	2	0	0	3	0	1	4

Figure 8:

paths shown in Fig. 9,

$$\begin{aligned}\sum_{i=1}^{N-k}\sum_{n=1}^{N}\sum_{m=0}^{M}\xi_{n,m}(P_i)&\\ &=(0+0+1)+(0+1+0+0)+(0+0+0+0)\\ &\quad+(0+1+0)+(0+0+1)+(0+1+0)+(1+0)\\ &=6.\end{aligned}$$

Occasionally these paths minimize $\sum_{i=1}^{N-k}\sum_{n=1}^{N}\sum_{m=0}^{M}\xi_{n,m}(P_i)$*; thus,* $ue_3^{[2]}$ *is* 6.

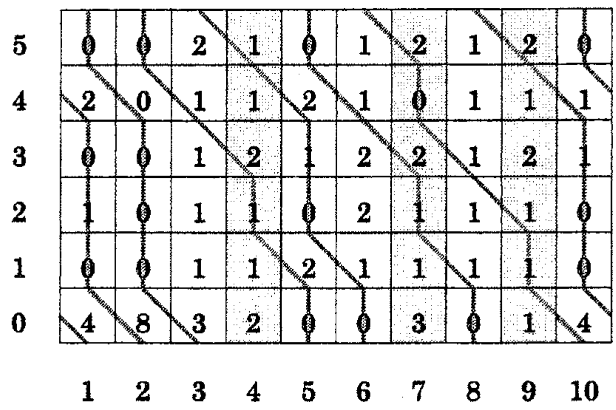


Figure 9:

6 Affine Wyle group, integrable lattice models and GPBBS

Let $\mathcal{T}$ be the set of $N\times(M+1)$ rectangular tableaux with integer entries, and s_ℓ $(\ell\in\mathbb{Z}/(M+1)\mathbb{Z})$ and π be mappings: $\mathcal{T}\to\mathcal{T}$. For a tableau Y,

$$Y=\begin{array}{|c|c|c|c|}\hline y_{1,M} & y_{2,M} & \cdots & y_{N,M}\\ \hline \vdots & \vdots & \cdots & \vdots\\ \hline y_{1,1} & y_{2,1} & \cdots & y_{N,1}\\ \hline y_{1,0} & y_{2,0} & \cdots & y_{N,0}\\ \hline\end{array},$$

these mappings are given as

$$s_\ell(Y) = \begin{array}{|c|c|c|c|}\hline s_\ell(y_{1,M}) & s_\ell(y_{2,M}) & \cdots & s_\ell(y_{N,M}) \\ \hline \vdots & \vdots & \cdots & \vdots \\ \hline s_\ell(y_{1,1}) & s_\ell(y_{2,1}) & \cdots & s_\ell(y_{N,1}) \\ \hline s_\ell(y_{1,0}) & s_\ell(y_{2,0}) & \cdots & s_\ell(y_{N,0}) \\ \hline \end{array},$$

$$\pi(Y) = \begin{array}{|c|c|c|c|}\hline \pi(y_{1,M}) & \pi(y_{2,M}) & \cdots & \pi(y_{N,M}) \\ \hline \vdots & \vdots & \cdots & \vdots \\ \hline \pi(y_{1,1}) & \pi(y_{2,1}) & \cdots & \pi(y_{N,1}) \\ \hline \pi(y_{1,0}) & \pi(y_{2,0}) & \cdots & \pi(y_{N,0}) \\ \hline \end{array},$$

where

$$\begin{aligned} s_\ell(y_{n,m}) &= y_{n,m+1} + Q_{n,m} - Q_{n-1,m} \quad (m \equiv \ell \mod M+1), \\ s_\ell(y_{n,m+1}) &= y_{n,m} + Q_{n-1,m} - Q_{n,m} \quad (m \equiv \ell \mod M+1), \\ s_\ell(y_{n,m}) &= y_{n,m} \quad (m \not\equiv \ell, \ell+1 \mod M+1), \\ \pi(y_{n,m}) &= y_{n,m+1}, \end{aligned}$$

and

$$Q_{n,m} = \max_{1 \le h \le N} \left[\sum_{k=1}^{h-1} y_{n+k,m+1} + \sum_{k=h+1}^{N} y_{n+k,m} \right].$$

Here we extend the indices n, m of $y_{n,m}$ for $n, m \in \mathbb{Z}$ by the condition $y_{n+N,m} = y_{n,m+M+1} = y_{n,m}$.

For example,

$$s_0(Y) = \begin{array}{|c|c|c|c|}\hline y_{1,M} & y_{2,M} & \cdots & y_{N,M} \\ \hline \vdots & \vdots & \cdots & \vdots \\ \hline y_{1,2} & y_{2,2} & \cdots & y_{N,2} \\ \hline y_{1,0} + Q_{N,0} - Q_{1,0} & y_{2,0} + Q_{1,0} - Q_{2,0} & \cdots & y_{N,0} + Q_{N-1,0} - Q_{N,0} \\ \hline y_{1,1} + Q_{1,0} - Q_{N,0} & y_{2,1} + Q_{2,0} - Q_{1,0} & \cdots & y_{N,1} + Q_{N,0} - Q_{N-1,0} \\ \hline \end{array},$$

$$\pi(Y) = \begin{array}{|c|c|c|c|}\hline y_{1,M-1} & y_{2,M-1} & \cdots & y_{N,M-1} \\ \hline \vdots & \vdots & \cdots & \vdots \\ \hline y_{1,0} & y_{2,0} & \cdots & y_{N,0} \\ \hline y_{1,M} & y_{2,M} & \cdots & y_{N,M} \\ \hline \end{array}.$$

The following theorem is proved by direct calculations.

Theorem 6.1 ([14, 15])
The mappings s_ℓ ($\ell \in \mathbb{Z}/(M+1)\mathbb{Z}$) and π defined as above give a realization of the affine Weyl group $\widetilde{W}(A_M^{(1)})$. □

Remark 6.1
The affine Weyl group $\widetilde{W}(A^{(1)}_{n-1})$ is defined as the group generated by the simple reflections $s_0, s_1, \ldots, s_{n-1}$ and diagram rotation π subject to the fundamental relations

$$\begin{aligned}
&s_i^2 = 1,\\
&s_i s_j = s_j s_i \quad (j \not\equiv i, i \pm 1 \mod n),\\
&s_i s_j s_i = s_j s_i s_j \quad (j \equiv i \pm 1 \mod n),\\
&\pi s_i = s_{i+1}\pi,
\end{aligned}$$

where we understand the indices for s_i as elements of $\mathbb{Z}/(M+1)\mathbb{Z}$.

When we put

$$y_{n,0} = \kappa_n^s, \qquad y_{n,m} = u_n^{s-M+m} \quad (m \neq 0), \tag{6.1}$$

we get the following theorem which gives a relation between the GPBBS and the affine Weyl group.

Theorem 6.2
$\pi s_{M-1} s_{M-2} \cdots s_0$ gives the time evolution which concerns with time t, i.e.

$$\pi s_{M-1} s_{M-2} \cdots s_0 \left(\begin{array}{|c|c|c|c|} \hline u_1^s & u_2^s & \cdots & u_N^s \\ \hline \vdots & \vdots & \cdots & \vdots \\ \hline u_1^{s-M+1} & u_2^{s-M+1} & \cdots & u_N^{s-M+1} \\ \hline \kappa_1^s & \kappa_2^s & \cdots & \kappa_N^s \\ \hline \end{array} \right)$$
$$= \begin{array}{|c|c|c|c|} \hline u_1^{s+M} & u_2^{s+M} & \cdots & u_N^{s+M} \\ \hline \vdots & \vdots & \cdots & \vdots \\ \hline u_1^{s+1} & u_2^{s+1} & \cdots & u_N^{s+1} \\ \hline \kappa_1^{s+M} & \kappa_2^{s+M} & \cdots & \kappa_N^{s+M} \\ \hline \end{array} .$$

□

The BBSs can be reformulated as integrable lattice models at temperature zero from the crystal theory and the combinatorial R matrix [17, 18]. The PBBS with one kind of ball and box capacity one has also been reformulated into two types of lattice models, a periodic $A^{(1)}_1$ crystal lattice and a twisted $A^{(1)}_{N-1}$ crystal chain, where N denotes the number of the boxes in the system [8]. It is straightforward to extend this result to the case of the GPBBS. Here, we will briefly show how the GPBBS is reinterpreted as some integrable lattice systems. Since the proofs for the statements below are almost the same as those in Ref. [8], we will omit them here.

Let B_k be the classical crystal of $U'_q(A^{(1)}_M)$ corresponding to the k-fold symmetric tensor representation of $U_q(A_M)$. As a set it consists of the single row semistandard tableaux of length k on letters $\{1, 2, \ldots, M+1\}$:

$$B_k := \left\{ \boxed{i_1}\boxed{i_2}\boxed{\cdots}\boxed{i_k} \,\middle|\, 1 \le i_1 \le i_2 \le \cdots \le i_k \le M+1 \right\}.$$

An element b

$$b = \boxed{i_1}\boxed{i_2}\boxed{\cdots}\boxed{i_k} \quad \in \ B_k$$

is also denoted as a series of $M+1$ integers $b \equiv (x^{(M+1)}, x^{(M)}, \ldots, x^{(2)}, x^{(1)})$, where $x^{(j)}$ is the number of letters j in b. A state $|\psi\rangle_t$ of the GPBBS is naturally identified with

$$\begin{aligned} |\psi\rangle_t &\cong b^t_1 \otimes b^t_2 \otimes \cdots \otimes b^t_N \\ &\in B_{\theta_1} \otimes B_{\theta_2} \otimes \cdots \otimes B_{\theta_N}, \end{aligned}$$

where

$$b^t_n = (\kappa^s_n, u^{s-M+1}_n, u^{s-M+2}_n, \ldots, u^s_n) \quad (n = 1, 2, \ldots, N).$$

For the BBS without the periodic boundary condition, time evolution is given by the isomorphism induced by the combinatorial R matrices:

$$\begin{aligned} \mathcal{T}: &\ B_\infty \otimes (B_{\theta_1} \otimes B_{\theta_2} \otimes \cdots \otimes B_{\theta_N}) \to (B_{\theta_1} \otimes B_{\theta_2} \otimes \cdots \otimes B_{\theta_N}) \otimes B_\infty, \\ \mathcal{T}: &\ |\{0\}\rangle \otimes |\psi\rangle_t \to |\psi\rangle_{t+1} \otimes |\{0\}\rangle \end{aligned}$$

where $|\{0\}\rangle$ is the highest weight vector of B_∞. For the GPBBS, by taking the trace of the auxiliary state in B_∞, $T := \mathrm{Tr}_{B_\infty}\mathcal{T}$, we have the time evolution

$$\begin{aligned} T: &\ B_{\theta_1} \otimes B_{\theta_2} \otimes \cdots \otimes B_{\theta_N} \to B_{\theta_1} \otimes B_{\theta_2} \otimes \cdots \otimes B_{\theta_N}, \\ T: &\ |\psi\rangle_t \to |\psi\rangle_{t+1}. \end{aligned}$$

As the $A^{(1)}_1$ crystal, the operator T maps $|\psi\rangle_t$ to the unique tensor product of $A^{(1)}_M$ crystal that exactly corresponds to the state of the GPBBS at $t+1$.

The GPBBS is also reformulated as a twisted lattice of M vertical axes in terms of $U'_q(A^{(1)}_{N-1})$ crystals. In this case, a state $|\psi\rangle_t$ is identified

$$\begin{aligned} |\psi\rangle_t &\cong b^t_\kappa \otimes (b^t_{u_1} \otimes b^t_{u_2} \otimes \ldots b^t_{u_M}) \\ &\in B_k \otimes (B_{n_1} \otimes B_{n_2} \otimes \ldots B_{n_M}) \end{aligned}$$

where

$$\begin{aligned} b^t_\kappa &= (\kappa^s_N, \kappa^s_{N-1}, \ldots, \kappa^s_1), \\ b^t_{u_j} &= (u^{s-M+j}_N, u^{s-M+j}_{N-1}, \ldots, u^{s-M+j}_1) \quad (j = 1, 2, \ldots, M), \end{aligned}$$

$k := \sum_{n=1}^{N} \kappa_n^s$ and $n_j := \sum_{n=1}^{N} u_n^{s-M+j}$. The time evolution is determined by the isomorphism induced by the combinatorial R-matrix for $A_{N-1}^{(1)}$ crystal:

$$b_\kappa^t \otimes (b_{u_1}^t \otimes b_{u_2}^t \otimes \ldots b_{u_M}^t) \cong (b_{u_1}^{t+1} \otimes b_{u_2}^{t+1} \otimes \ldots b_{u_M}^{t+1}) \otimes b_\kappa^{t+1}.$$

In Fig. 10, we schematically show the twisted crystal lattice associated with the GPBBS.

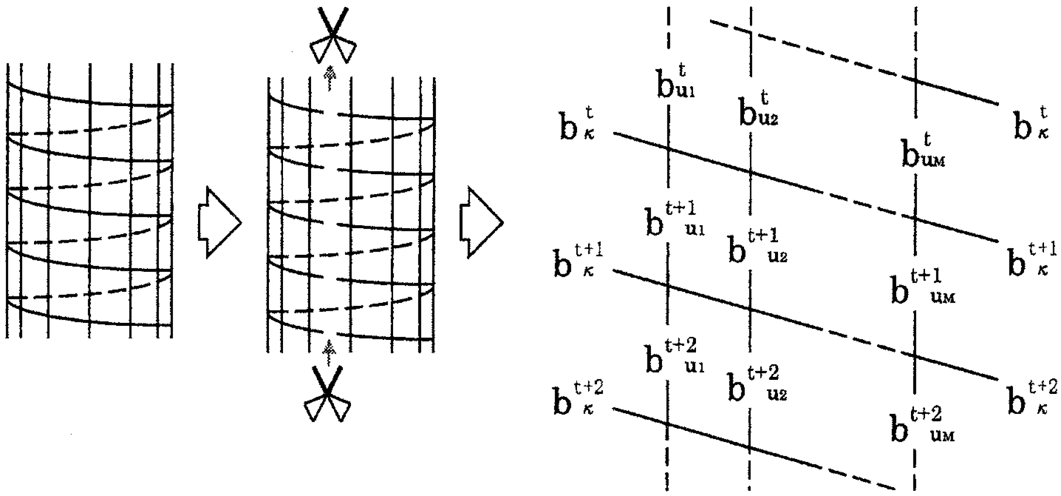


Figure 10: The twisted crystal lattice associated with the GPBBS.

7 Concluding remarks

In this article, using a path description of characteristic polynomial of particular matrices and an algorithm to construct the conserved quantities using the Lax representation of the ndKP equation, we showed explicit form of the conserved quantities of the GPBBS. Relations to the affine Weyl group action and the crystal theory were also clarified. An advantage to reformulate the PBBS as crystal lattices is that we can extend it to the crystals associated with other root systems.

Since the GPBBS is composed of a finite number of boxes and balls, it can only take on a finite number of patterns. Hence its trajectory is always periodic and a fundamental cycle, *i.e.* the shortest period of the periodic motion, exists for any given initial state. In the case where the box capacity is one everywhere and only one kind of ball exists, the formula used to calculate the fundamental cycle is explicitly obtained using the conserved quantities and some rescaling properties of the states [16]. Hence, using the results in this article, we may get the formula to calculate the fundamental cycle for the GPBBS, which is a problem we wish to address in the future.

References

[1] D. Takahashi and J. Satsuma, "A soliton cellular automaton," J. Phys. Soc. Jpn. **59**, 3514–3519 (1990).

[2] D. Takahashi, "One some soliton systems defined by using boxes and balls," Proceedings of the International Symposium on Nonlinear Theory and Its Applications, NOLTA'93, 555–558 (1993).

[3] T. Miwa, M. Jimbo and E. Date, "Solitons – Differential equations, symmetries and infinite dimensional algebras," Cambridge University Press (2000).

[4] R. Hirota, "Chokusetsuhou ni yoru soliton no suuri," (in Japanese) Iwanami (1992).

[5] T. Tokihiro, D. Takahashi, J. Matsukidaira, and J. Satsuma, "From soliton equations to integrable cellular automata through a limiting procedure," Phys. Rev. Lett. **76**, 3247–3250 (1996).

[6] R. Hirota and S. Tsujimoto, "Conserved quantities of a class of nonlinear difference-difference equations," J. Phys. Soc. Jpn. **64**, 3125–3127 (1995).

[7] A. Nagai, D. Takahashi and T. Tokihiro, "Soliton cellular automaton, Toda molecule equation and sorting algorithm" Phys. Lett. A **255**, 265–271 (1999).

[8] F. Yura and T. Tokihrio, "On a periodic soliton cellular automaton", J. Phys. A : Math. Gen. **35**, 3787–3801 (2002).

[9] J. Mada, M. Idzumi and T. Tokihiro, "Conserved quantities of generalized periodic box-ball systems constructed from the ndKP equation," J. Phys. A : Math. Gen. **37**, 6531–6556 (2004).

[10] R. Willox, T. Tokihiro and J. Satsuma, "Darboux and binary Darboux transformations for the nonautonomous discrete KP equation," J. Math. Phys. **38**, 6455–6469 (1997).

[11] M. Torii, D. Takahashi and J. Satsuma, "Combinatorial representation of invariants of a soliton cellular automaton," Physica D **92**, 209–220 (1996).

[12] M. Sato and Y. Sato, "Soliton Equations as Dynamical Systems on Infinite Dimensional Grassmann Manifold," *Nonlinear PDE in Applied Science. U.S.-Japan Seminar, Tokyo, 1982* Lecture Notes Num. Appl. Anal. **5**, 259 (1982).

[13] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, "Transformation groups for soliton equations," in *Proceedings of RIMS symposium on Non-linear Integrable Systems-Classical Theory and Quantum Theory, Kyoto, Japan May 13 – ay 16, 1981*, ed. by M. Jimbo, T. Miwa (World Scientific Publ. Co., Singapore 1983). 39–119.

[14] K. Kajiwara, M. Noumi and Y. Yamada, "Discrete Dynamical Systems with $W(A^{(1)}_{m-1} \times A^{(1)}_{n-1})$ Symmetry," Lett. in Math. Phys. **60**, 211–219 (2002).

[15] M. Noumi and Y. Yamada, "Tropical Robinson-Schensted-Knuth correspondence and birational Weyl group actions," Advanced Studies in Pure Mathematics **40** (2004), Representation Theory of Algebraic Groups and Quanatum Groups, 371–442 (math-ph/0203030).

[16] D. Yoshihara, F. Yura and T. Tokihiro, "Fundamental cycle of a periodic box-ball system," J. Phys. A : Math. Gen. **36**, 99–121 (2003).

[17] K. Fukuda, M. Okado, and Y. Yamada, "Energy function in box ball systems," Int. J. Mod. Phys. **A 15**, 1379–1392 (2000).

[18] G. Hatayama, K. Hikami, R. Inoue, A. Kuniba, T. Takagi, and T. Tokihiro, "The $A^{(1)}_M$ automata related to crystals of symmetric tensors", J. Math. Phys. **42**, 274–308 (2001).