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**SOME FAMILIES OF MEROMORPHIC MULTIVALENT FUNCTIONS
INVOLVING CERTAIN LINEAR OPERATOR**

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Abstract. Let Σ_p denote the class of functions of the form

$$f(z) = z^{-p} + \sum_{k=0}^{\infty} a_k z^k \quad (p \in N = \{1, 2, 3, \dots\})$$

which are analytic and p -valent in the punctured disc $D = \{z : 0 < |z| < 1\}$. We introduce and study some new families of meromorphic multivalent functions defined by certain linear operator. A number of useful characteristics of functions in these families are obtained. In particular, some properties of neighborhoods of functions in these families are given.

Key words and phrases. Meromorphic, neighborhood, operator, partial sum.

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1. Introduction

Let Σ_p denote the class of functions of the form

$$f(z) = z^{-p} + \sum_{k=0}^{\infty} a_k z^k \quad (p \in N = \{1, 2, 3, \dots\}) \tag{1.1}$$

which are analytic and p -valent in the punctured disc $D = \{z : 0 < |z| < 1\}$. Define a linear operator as following

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = z^{-p} + (p+1)a_0 + (p+2)a_1 z + (p+3)a_2 z^2 + \dots$$

$$= \frac{(z^{p+1} f(z))'}{z^p}$$

$$D^2 f(z) = D(D^1 f(z))$$

and for $n = 1, 2, \dots$

$$D^n f(z) = D(D^{n-1} f(z)) = z^{-p} + \sum_{k=0}^{\infty} (p+k+1)^n a_k z^k$$

$$= \frac{(z^{p+1} D^{n-1} f(z))'}{z^p}. \quad (1.2)$$

It is easy to see that

$$z(D^n f(z))' = D^{n+1} f(z) - (p+1)D^n f(z). \quad (1.3)$$

When $p = 1$, Uralegaddi and Somanatha [20] investigated certain properties of the operator D^n . Recently, Aouf and Hossen [2] showed some results of the operator D^n for $p \in N = \{1, 2, 3, \dots\}$.

Let $-1 \leq B < A \leq 1$. A function $f(z) = z^{-p} + \sum_{k=0}^{\infty} a_k z^k \in \Sigma_p$ is said to be in the class $T_n(A, B)$ if it satisfies the condition

$$\left| \frac{z(D^n f(z))' + pD^n f(z)}{Bz(D^n f(z))' + ApD^n f(z)} \right| < 1 \quad (1.4)$$

for all $z \in E = \{z : |z| < 1\}$.

Furthermore, a function $f(z) = z^{-p} + \sum_{k=p}^{\infty} |a_k| z^k \in \Sigma_p$ is said to be in the class $T_n^*(A, B)$ if it satisfies the condition (1.4).

It should be remarked in passing that the definition (1.4) is motivated essentially by the recent work of Mogra [13] and Liu and Srivastava [12]. The special class $T_0(A, B)$ was studied by Mogra [13]. Another subclass associated with the linear operator D^n was considered recently by Liu and Srivastava [12].

In recent years, many important properties and characteristics of various interesting

subclasses of the class Σ_p of meromorphically p -valent functions were investigated extensively by (among others) Aouf et al. ([2] and [3]), Joshi and Srivastava [8], Kulkarni et al.[9], Liu and Srivastava ([10],[11] and [12]), Mogra ([13] and [14]), Owa et al.[15], Srivastava et al.[18], Uralegaddi and Somanatha ([20] and [21]), and Yang ([22]). The main object of this paper is to present several inclusion and other properties of functions in the classes $T_n(A,B)$ and $T_n^*(A,B)$ which we have introduced here. We also apply the familiar concept of neighborhoods of analytic functions (see [6] and [16]) to meromorphically p -valent functions in the class Σ_p .

2. Properties of the class $T_n(A,B)$

In proving our results, we shall need the following lemma which is due to Jack [7].

Lemma. Let $w(z)$ be non-constant analytic in $E = \{z : |z| < 1\}$, $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at z_0 , we have $z_0 w'(z_0) = kw(z_0)$, where k is a real number and $k \geq 1$.

Theorem 2.1. Let $1 + B \geq p(A - B)$, then $T_{n+1}(A, B) \subset T_n(A, B)$.

Proof. Let $f(z) \in T_{n+1}(A, B)$. Suppose that

$$\frac{z(D^n f(z))'}{D^n f(z)} = -p \frac{1 + Aw(z)}{1 + Bw(z)}, \quad (2.1)$$

where $w(0) = 0$.

By using (2.1) and (1.3), we have

$$\frac{z(D^{n+1} f(z))'}{D^{n+1} f(z)} = -p \frac{1 + Aw(z)}{1 + Bw(z)} - \frac{p(A - B)zw'(z)}{(1 + Bw(z))\{1 + [B - p(A - B)]w(z)\}}. \quad (2.2)$$

Suppose now that, for $z_0 \in E$, $\max_{|z| \leq z_0} |w(z)| = |w(z_0)| = 1$. Applying Lemma, we have

$z_0 w'(z_0) = kw(z_0)$, $k \geq 1$. Writing $w(z_0) = e^{i\theta}$ and putting $z = z_0$ in (2.2), we obtain

$$\left| \frac{z(D^{n+1} f(z))' + pD^{n+1} f(z)}{Bz(D^{n+1} f(z))' + ApD^{n+1} f(z)} \right|_{z=z_0}^2 - 1$$

$$\begin{aligned}
&= \left| \frac{(k+1) + [B - p(A-B)]e^{i\theta}}{1 - [B(k-1) + p(A-B)]e^{i\theta}} \right|^2 - 1 \\
&= \frac{H_1(\cos\theta)}{|1 - [B(k-1) + p(A-B)]e^{i\theta}|^2}, \tag{2.3}
\end{aligned}$$

where

$$H_1(\cos\theta) = k^2(1 - B^2) + 2k[1 + B^2 - pB(A - B)] + 2k[2B - p(A - B)]\cos\theta.$$

Since $1 + B \geq p(A - B)$, then

$$H_1(1) = k^2(1 - B^2) + 2k(1 + B)[(1 + B) - p(A - B)] \geq 0$$

and

$$H_1(-1) = k^2(1 - B^2) + 2k(1 - B)[(1 - B) + p(A - B)] \geq 0.$$

This shows that $H(\cos\theta) \geq 0$ for all θ ($0 \leq \theta < 2\pi$) and it follows that (2.3)

contradicts the hypothesis $f(z) \in T_{n+1}(A, B)$. Hence $|w(z)| < 1$ for all $z \in E$ and

(2.1) shows that $f(z) \in H_n(A, B)$. \square

Theorem 2.2 Let $\alpha > 0$ and let λ be a complex number such that $\operatorname{Re} \lambda \geq p\alpha \frac{1+A}{1+B}$. If $f(z) \in T_n(A, B)$, then the function $g(z)$ defined by

$$D^n g(z) = \left\{ \frac{\lambda - p\alpha}{z^\lambda} \int_0^z t^{\lambda-1} (D^n f(t))^\alpha dt \right\}^{\frac{1}{\alpha}} \tag{2.4}$$

also belongs to $T_n(A, B)$.

Proof. Put

$$\frac{z(D^n g(z))'}{D^n g(z)} = -p \frac{1 + Aw(z)}{1 + Bw(z)}, \tag{2.5}$$

where $w(0) = 0$.

Using (2.4) and (2.5) together with some computations, it follows that

$$\frac{z(D^n f(z))'}{D^n f(z)} = -p \frac{1 + Aw(z)}{1 + Bw(z)} - \frac{p(A - B)zw'(z)}{(1 + Bw(z))[(\lambda - p\alpha) + (B\lambda - Ap\alpha)w(z)]}. \tag{2.6}$$

The remaining part of the proof of Theorem 2.2 is similar to that of Theorem 2.1 and so is omitted. \square

3. Properties of the class $T_n^*(A, B)$

In this section, we assume that $A + B \leq 0$.

Theorem 3.1 Let $f(z) = z^{-p} + \sum_{k=p}^{\infty} |a_k| z^k$ be analytic and p -valent in

$D = \{z : 0 < |z| < 1\}$. Then $f(z) \in T_n^*(A, B)$ if and only if

$$\sum_{k=p}^{\infty} [p(1-A) + k(1-B)](p+k+1)^n |a_k| \leq p(A-B). \quad (3.1)$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^{-p} + \frac{p(A-B)}{(p+k+1)^n [p(1-A) + k(1-B)]} z^k \quad (k \geq p). \quad (3.2)$$

Proof. Let $f(z) = z^{-p} + \sum_{k=p}^{\infty} |a_k| z^k \in T_n^*(A, B)$. Then

$$\left| \frac{z(D^n f(z))' + pD^n f(z)}{Bz(D^n f(z))' + ApD^n f(z)} \right| = \left| \frac{\sum_{k=p}^{\infty} (p+k)(p+k+1)^n |a_k| z^{k+p}}{p(A-B) + \sum_{k=p}^{\infty} (Ap+Bk)(p+k+1)^n |a_k| z^{k+p}} \right| < 1. \quad (3.3)$$

Since $|\operatorname{Re} z| \leq |z|$ for any z , choosing z to be real and letting $z \rightarrow 1^-$ through real values, (3.3) yields

$$\sum_{k=p}^{\infty} (p+k)(p+k+1)^n |a_k| \leq p(A-B) + \sum_{k=p}^{\infty} (Ap+Bk)(p+k+1)^n |a_k|,$$

which gives (3.1).

On the other hand, we have that

$$\left| \frac{z(D^n f(z))' + pD^n f(z)}{Bz(D^n f(z))' + ApD^n f(z)} \right| \leq \frac{\sum_{k=p}^{\infty} (p+k)(p+k+1)^n |a_k|}{p(A-B) + \sum_{k=p}^{\infty} (Ap+Bk)(p+k+1)^n |a_k|} < 1.$$

This shows that $f(z) \in T_n^*(A, B)$. \square

Next, we prove the following growth and distortion property for the class $T_n^*(A, B)$.

Theorem 3.2 Let $0 \leq m < p$. If $f(z) \in T_n^*(A, B)$, then for $0 < |z| = r < 1$,

$$\left\{ \frac{(p+m-1)!}{(p-1)!} - \frac{(A-B)p!}{(2-(A+B))(2p+1)^n(p-m)!} r^{2p} \right\} r^{-(p+m)}$$

$$\leq |f^{(m)}(z)|$$

$$\leq \left\{ \frac{(p+m-1)!}{(p-1)!} + \frac{(A-B)p!}{(2-(A+B))(2p+1)^n(p-m)!} r^{2p} \right\} r^{-(p+m)}.$$

The results are sharp for the function $f(z)$ given by

$$f(z) = z^{-p} + \frac{A-B}{(2-(A+B))(2p+1)^n} z^p.$$

Proof. Let $f(z) \in T_n^*(A, B)$. Then we find from Theorem 3.1 that

$$\frac{p(2-(A+B))(2p+1)^n}{p!} \sum_{k=p}^{\infty} k! |a_k| \leq \sum_{k=p}^{\infty} [p(1-A) + k(1-B)](p+k+1)^n |a_k|$$

$$\leq p(A-B),$$

which yields

$$\sum_{k=p}^{\infty} k! |a_k| \leq \frac{(A-B)p!}{(2-(A+B))(2p+1)^n}. \quad (3.4)$$

Now by differentiating $f(z)$ m times, we have

$$f^{(m)}(z) = (-1)^m \frac{(p+m-1)!}{(p-1)!} z^{-p-m} + \sum_{k=p}^{\infty} \frac{k!}{(k-m)!} |a_k| z^{k-m}. \quad (3.5)$$

Hence Theorem 3.2 would follow from (3.4) and (3.5). \square

Finally, we determine the radius of meromorphically p -valent starlikeness and convexity for functions in the class $T_n^*(A, B)$.

Theorem 3.3 Let $f(z) \in T_n^*(A, B)$. Then

(i) $f(z)$ is meromorphically p -valent starlike of order δ in $|z| < r_1$, that is

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} < -p\delta \quad (|z| < r_1) \quad (3.6)$$

where $0 \leq \delta < 1$ and

$$r_1 = \inf_{k \geq p} \left\{ \frac{(1-\delta)[p(1-A) + k(1-B)](p+k+1)^n}{(A-B)(k+p\delta)} \right\}^{\frac{1}{k+p}}$$

(ii) $f(z)$ is meromorphically p -valent convex of order δ in $|z| < r_2$, that is

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < -p\delta \quad (|z| < r_2), \quad (3.7)$$

where $0 \leq \delta < 1$ and

$$r_2 = \inf_{k \geq p} \left\{ \frac{p(1-\delta)[p(1-A) + k(1-B)](p+k+1)^n}{k(k+p\delta)(A-B)} \right\}^{\frac{1}{k+p}}$$

Each of these results is sharp for the function $f(z)$ given by (3.2).

Proof. (i) From Theorem 3.1, we have

$$\begin{aligned} \sum_{k=p}^{\infty} \frac{k+p\delta}{p(1-\delta)} |a_k| |z|^{k+p} &< \sum_{k=p}^{\infty} \frac{[p(1-A) + k(1-B)](p+k+1)^n}{p(A-B)} |a_k| \\ &\leq 1 \quad (|z| < r_1). \end{aligned}$$

Therefore for $|z| < r_1$

$$\begin{aligned} \left| \frac{zf'(z)/f(z) + p}{zf'(z)/f(z) - p(1-2\delta)} \right| &\leq \frac{\sum_{k=p}^{\infty} (k+p) |a_k| |z|^{k+p}}{2p(1-\delta) - \sum_{k=p}^{\infty} [k - p(1-2\delta)] |a_k| |z|^{k+p}} \\ &< 1, \end{aligned}$$

which shows that (3.6) is true.

(ii) It follows from Theorem 3.1 that

$$\begin{aligned} \sum_{k=p}^{\infty} \frac{k(k+p\delta)}{p^2(1-\delta)} |a_k| |z|^{k+p} &< \sum_{k=p}^{\infty} \frac{[p(1-A) + k(1-B)](p+k+1)^n}{p(A-B)} |a_k| \\ &\leq 1 \quad (|z| < r_2). \end{aligned}$$

Thus for $|z| < r_2$, we obtain

$$\begin{aligned} \left| \frac{1 + zf''(z)/f'(z) + p}{1 + zf''(z)/f'(z) - p(1-2\delta)} \right| &\leq \frac{\sum_{k=p}^{\infty} k(k+p) |a_k| |z|^{k+p}}{2p^2(1-\delta) - \sum_{k=p}^{\infty} k[k - p(1-2\delta)] |a_k| |z|^{k+p}} \\ &< 1, \end{aligned}$$

which shows that (3.7) is true.

Sharpness can be verified easily. \square

4. Neighborhoods and partial sums

Following the earlier works (based upon the familiar concept of neighborhoods of analytic functions) by Goodman[6] and Ruscheweyh[16], and (more recently) by Altintas and Owa [1] and Liu and Srivastava ([10] and [12]), we begin by introducing here the δ -neighborhood of a function $f \in \Sigma_p$ of the form (1.1) by means of the definition:

$$N_\delta(f) = \left\{ g(z) = z^{-p} + \sum_{k=0}^{\infty} b_k z^k \in \Sigma_p : \sum_{k=0}^{\infty} \frac{[p(1-A) + k(1-B)]}{p(A-B)} (p+k+1)^n |b_k - a_k| \leq \delta, -1 \leq B < A \leq 1; \delta \geq 0 \right\}.$$

Making use of the definition, we now prove

Theorem 4.1 Let $\delta > 0$ and $-1 < A \leq 0$. If $f(z) = z^{-p} + \sum_{k=0}^{\infty} a_k z^k \in \Sigma_p$ satisfies

the condition

$$\frac{f(z) + \varepsilon z^{-p}}{1 + \varepsilon} \in T_n(A, B) \quad (4.1)$$

for any complex number ε such that $|\varepsilon| < \delta$, then $N_\delta(f) \subset T_n(A, B)$.

Proof. It is obvious from (1.4) that $g(z) \in T_n(A, B)$ if and only if for any complex number σ with $|\sigma| = 1$

$$\frac{z(D^n g(z))' + pD^n g(z)}{Bz(D^n g(z))' + ApD^n g(z)} \neq \sigma \quad (z \in E),$$

which is equivalent to

$$\frac{g(z) * h(z)}{z^{-p}} \neq 0 \quad (z \in E), \quad (4.2)$$

where

$$\begin{aligned} h(z) &= z^{-p} + \sum_{k=0}^{\infty} c_k z^k \\ &= z^{-p} + \sum_{k=0}^{\infty} \frac{[(p+k) - \sigma(pA + kB)]}{p\sigma(B-A)} (p+k+1)^n z^k \end{aligned} \quad (4.3)$$

From (4.3), we have

$$\begin{aligned} |c_k| &= \left| \frac{[(p+k) - \sigma(pA+kB)]}{p\sigma(B-A)} (p+k+1)^n \right| \\ &\leq \frac{p(1-A) + k(1-B)}{p(A-B)} (p+k+1)^n. \end{aligned}$$

If $f(z) = z^{-p} + \sum_{k=0}^{\infty} a_k z^k \in \sum_p$ satisfies the condition (4.1), then (4.2) yields

$$\left| \frac{f(z) * h(z)}{z^{-p}} \right| \geq \delta \quad (z \in E). \quad (4.4)$$

Now let $p(z) = z^{-p} + \sum_{k=0}^{\infty} b_k z^k \in N_{\delta}(f)$, then

$$\begin{aligned} \left| \frac{(p(z) - f(z)) * h(z)}{z^{-p}} \right| &= \left| \sum_{k=0}^{\infty} (b_k - a_k) c_k z^{k+p} \right| \\ &\leq |z| \sum_{k=0}^{\infty} \frac{[p(1-A) + k(1-B)]}{p(A-B)} (p+k+1) |b_k - a_k| \\ &< \delta. \end{aligned}$$

Thus for any complex number σ such that $|\sigma|=1$, we have

$$\frac{p(z) * h(z)}{z^{-p}} \neq 0 \quad (z \in E),$$

which implies that $p(z) \in T_n(A, B)$. \square

Theorem 4.2 Let $-1 < A \leq 0$. Let $f(z) = z^{-p} + \sum_{k=0}^{\infty} a_k z^k \in \sum_p$, $s_1(z) = z^{-p}$

and $s_m(z) = z^{-p} + \sum_{k=0}^{m-2} a_k z^k$ ($m \geq 2$). Suppose that

$$\sum_{k=0}^{\infty} c_k |a_k| \leq 1 \quad (4.5)$$

where

$$c_k = \frac{p(1-A) + k(1-B)}{p(A-B)} (p+k+1)^n.$$

Then we have

(i) $f(z) \in T_n(A, B)$;

$$(ii) \operatorname{Re} \left\{ \frac{f(z)}{s_m(z)} \right\} > 1 - \frac{1}{c_{m-1}} \quad (4.6)$$

and

$$\operatorname{Re} \left\{ \frac{s_m(z)}{f(z)} \right\} > \frac{c_{m-1}}{1 + c_{m-1}}. \quad (4.7)$$

The estimates are sharp.

Proof. (i) It is obvious that $z^{-p} \in T_n(A, B)$. Thus from Theorem 4.1 and the condition (4.5), we have $N_1(z^{-p}) \subset T_n(A, B)$. This gives $f(z) \in T_n(A, B)$.

(ii) It is easy to see that $c_{k+1} > c_k > 1$. Thus

$$\sum_{k=0}^{m-2} |a_k| + c_{m-1} \sum_{k=m-1}^{\infty} |a_k| \leq \sum_{k=0}^{\infty} c_k |a_k| \leq 1. \quad (4.8)$$

Let

$$\begin{aligned} h_1(z) &= c_{m-1} \left\{ \frac{f(z)}{s_m(z)} - \left(1 - \frac{1}{c_{m-1}}\right) \right\} \\ &= 1 + \frac{c_{m-1} \sum_{k=m-1}^{\infty} a_k z^{k+p}}{1 + \sum_{k=0}^{m-2} a_k z^{k+p}}. \end{aligned}$$

It follows from (4.8) that

$$\left| \frac{h_1(z) - 1}{h_1(z) + 1} \right| \leq \frac{c_{m-1} \sum_{k=m-1}^{\infty} |a_k|}{2 - 2 \sum_{k=0}^{m-2} |a_k| - c_{m-1} \sum_{k=m-1}^{\infty} |a_k|} \leq 1 \quad (z \in E).$$

From this we obtain the inequality (4.6).

If we take

$$f(z) = z^{-p} - \frac{z^{m-1}}{c_{m-1}}, \quad (4.9)$$

then

$$\frac{f(z)}{s_m(z)} = 1 - \frac{z^{p+m-1}}{c_{m-1}} \rightarrow 1 - \frac{1}{c_{m-1}} \quad \text{as } z \rightarrow 1^-.$$

This shows that the bound in (4.6) is best possible for each m .

Similarly, if we take

$$h_2(z) = (1 + c_{m-1}) \left\{ \frac{s_m(z)}{f(z)} - \frac{c_{m-1}}{1 + c_{m-1}} \right\}$$

$$= 1 - \frac{(1 + c_{m-1}) \sum_{k=m-1}^{\infty} a_k z^{k+p}}{1 + \sum_{k=0}^{\infty} a_k z^{k+p}},$$

then we deduce that

$$\left| \frac{h_2(z) - 1}{h_2(z) + 1} \right| \leq \frac{(1 + c_{m-1}) \sum_{k=m-1}^{\infty} |a_k|}{2 - 2 \sum_{k=0}^{m-2} |a_k| + (1 - c_{m-1}) \sum_{k=m-1}^{\infty} |a_k|}$$

$$\leq 1 \quad (z \in E),$$

which yields (4.7). The estimate (4.7) is sharp with the extremal function $f(z)$ given by (4.9). \square

For $\delta \geq 0$, $-1 \leq B < A \leq 1$ and $f(z) = z^{-p} + \sum_{k=0}^{\infty} |a_k| z^k \in \Sigma_p$, we define

neighborhood of $f(z)$ by

$$N_{\delta}^*(f) = \left\{ g(z) = z^{-p} + \sum_{k=p}^{\infty} |b_k| z^k \in \Sigma_p : \right.$$

$$\left. \sum_{k=p}^{\infty} \frac{[p(1-A) + k(1-B)]}{p(A-B)} (p+k+1)^n |b_k| - |a_k| \leq \delta \right\}.$$

Theorem 4.3 Let $A+B \leq 0$. If $f(z) = z^{-p} + \sum_{k=p}^{\infty} |a_k| z^k \in T_{n+1}^*(A, B)$, then

$N_{\delta}^*(f) \subset T_n^*(A, B)$, where $\delta = \frac{2p}{1+2p}$. The result is sharp.

Proof. Using the same method as in Theorem 4.1, we would have

$$h(z) = z^{-p} + \sum_{k=p}^{\infty} c_k z^k$$

$$= z^{-p} + \sum_{k=p}^{\infty} \frac{(p+k) - \sigma(pA+kB)}{p\sigma(B-A)} (p+k+1)^n z^k.$$

Under the hypothesis $A+B \leq 0$, we obtain that

$$\begin{aligned} \left| \frac{f(z) * h(z)}{z^{-p}} \right| &= \left| 1 + \sum_{k=p}^{\infty} c_k |a_k| z^{k+p} \right| \\ &\geq 1 - \frac{1}{1+2p} \sum_{k=p}^{\infty} \frac{[p(1-A) + k(1-B)]}{p(A-B)} (p+k+1)^{n+1} |a_k|. \end{aligned}$$

From Theorem 3.1, we get

$$\left| \frac{f(z) * h(z)}{z^{-p}} \right| \geq \frac{2p}{1+2p} = \delta.$$

The remaining part of the proof is similar to that of Theorem 4.1.

To show the sharpness, we consider the function

$$f(z) = z^{-p} + \frac{A-B}{(2-(A+B))(1+2p)^{n+1}} z^p \in T_{n+1}^*(A, B)$$

and

$$g(z) = z^{-p} + \left[\frac{A-B}{(2-(A+B))(1+2p)^{n+1}} + \frac{(A-B)\delta'}{(2-(A+B))(1+2p)^n} \right] z^p,$$

where $\delta' > \frac{2p}{1+2p}$. Then the function $g(z)$ belongs to $N_{\delta'}^*(f)$.

On the other hand, we find from Theorem 3.1 that $g(z)$ is not in $T_n^*(A, B)$. Now the proof is complete. \square

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