Title	BY WHICH KIND OF SOUND, CAN ONE HEAR THE SHAPE OF A DRUM? (Wave phenomena and asymptotic analysis)
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Citation	数理解析研究所講究録 (2003), 1315: 156-175
Issue Date	2003-04
URL	http://hdl.handle.net/2433/42990
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録 1315 巻 2003 年 156-175

BY WHICH KIND OF SOUND, CAN ONE HEAR THE SHAPE OF A DRUM?

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Dedicated to
Professors M. IKAWA and S MIYATAKE

1. INTRODUCTION

In the paper titled "Can one hear the shape of a dram?" [2], M. Kac proposed a problem: Can one know the shape of a domain from the eigenvalues of the Laplacian with the Dirichlet boundary condition? He pointed out that if the eigenvalues of a domain is coincide with those of a disc, it must be the same disc. On the other hand, C. Gordon, D. Webb and S. Wolpert[1] presented a counter-example: there exist two incongruous rectangles with same eigenvalues. However, when a domain has a smooth boundary, we have no counter-example. In order to obtain the final answer to Kac's problem in the case of domains with smooth boundaries, we need find many domains which are determined by the eigenvalues. Recently, K. Watanabe proposed the domains enclosed by "the minimal curvature energy curves", which are determined by the eigenvalues. K. Watanabe analyzed these curves when they are sufficiently close to a disc. We consider a full account of these curves in this paper.

Let Ω be a domain in $\mathbf{R}^2$ and $\partial\Omega$ be its boundary. $\partial\Omega$ is given by $\mathbf{p}(s)$ $(0 \leq s \leq L)$, where s is the arc length parameter and L is the full arc length. Let $\kappa(s)$ be the curvature of $\partial\Omega$. We take $s = 0$ at the maximum of the curvature, $\mathbf{p}(0) =$ *the origin* and $\mathbf{e}_1(0) = d\mathbf{p}/ds(0) = {}^t(1, 0)$. We denote the angle from the positive x axis to $\mathbf{e}_1(s)$ by $\theta(s)$. $d\theta/ds(s) = \kappa(s)$. Let us denote the primitive period of $\kappa(s)$ by L/n, where n is a natural number.

We consider the eigenvalue problem of the Dirichlet problem on the Laplacian:

$$(1.1) \qquad \begin{cases} -\Delta\,\phi(x) &= \lambda\,\phi(x) \quad in\ \Omega\,, \\ \\ \phi(x) &= 0 \qquad\quad on\ \partial\Omega\,. \end{cases}$$

We denote the eigenvalues and the eigenfunctions of the above problem by $\{\lambda_j\}$ and $\{\phi_j(x)\}$. We can take $\{\phi_j(x)\}$ a complete orthonormal system. Using these, we obtain the heat kernel:

$$E(t,x,y) = \sum_{j=1}^{\infty} e^{-\lambda_j t}\,\phi_j(x)\,\phi_j(y).$$

Key words and phrases. Kac' problem, minimal curvature energy curve.

Its trace has an asymptotic expansion near $t = 0$:

(1.2)
$$\int_\Omega E(t,x,y) = \sum_{j=1}^{\infty} e^{-\lambda_j t} \sim \frac{1}{4\pi t}\sum_{k=o}^{\infty} D_k t^{k/2},$$

where

$$D_0 = M = area\ of\ \Omega,$$

$$D_1 = -\frac{\sqrt{\pi}}{2}L, \quad L = length\ of\ \partial\Omega,$$

$$D_2 = \frac{1}{3}\int_{\partial\Omega}\kappa(s)ds,$$

(1.3)
$$D_3 = \frac{\sqrt{\pi}}{4}\int_{\partial\Omega}\kappa(s)^2 ds,$$

$$D_4 = \frac{4}{315}\int_{\partial\Omega}\kappa(s)^3 ds,$$

$$D_5 = \frac{37\sqrt{\pi}}{8192}\int_{\partial\Omega}\kappa(s)^4 ds - \frac{\sqrt{\pi}}{1024}\int_{\partial\Omega}(\kappa(s)')^2 ds,$$

e.t.c.

Under the condition $\pi^2 D_0 = {D_1}^2$ (i.e. $4\pi M = L^2$), the domain is a disc with radius $-\sqrt{\pi}D_0/D_1$. Then, if we find the relation $\pi^2 D_0 = {D_1}^2$ from the asymptotic expansion of the trace of the heat kernel, which is obtained from the eigenvalues, we can say that the domain is a disc.

We pose the following:

Assumption 1.

(1.4)
$$\begin{cases} \pi^2 D_0 & < & {D_1}^2, \\ D_2 & = & 2\pi/3. \end{cases}$$

The former means that the domain is not a disc and the latter means that the winding number is one.

Our question is

Under Assumption 1, if D_3 takes the minimal value, can one determine the domain from the eigenvalues?.

The answer by K. Watanabe is "**Yes**".

Theorem 1. [K. Watanabe[8]]
We pose Assumption 1. We set

$$F = \{\theta \in H^1(0,L); (a)\ x(0) = x(L) = 0,\ (b)\ y(0) = y(L) = 0\,,\ (c)\ \theta(0) = 0\,,\ \theta(L) = 2\pi\,,$$

$$(d)\ \frac{1}{2}\int_0^L \int_0^s \sin(\theta(s) - \theta(\xi))d\xi ds = M\ \}.$$

$E_1(\theta) = \frac{1}{2}\int_0^L (\theta'(s))^2 ds$ possesses a minimizer in F.

Using Lagrange's multiplier,

$$E(\theta) = \frac{\lambda_0}{2}\int_0^L \theta'(s)^2 ds + \lambda_1 \int_0^L \cos\theta(s)ds + \lambda_2 \int_0^L \sin\theta(s)ds + \frac{\lambda_3}{2}\int_0^L \int_0^s \sin(\theta(s) - \theta(\xi))d\xi ds$$

takes a minimizer without restriction condition. This leads us to the Euler equation:

$$\theta''(s) = \mu_1 \sin\theta(s) + \mu_2 \cos\theta(s) + \mu_3 \int_0^s \cos(\theta(s) - \theta(\xi))d\xi\,. \tag{1.5}$$

This implies

$$\kappa''(s) + \frac{1}{2}\kappa(s)^3 + \mu_4\kappa(s) - \mu_3 = 0\,. \tag{1.6}$$

Integrating this, we obtain

$$K(\kappa) + U(\kappa) = \mu_5\,, \tag{1.7}$$

$$K(\kappa) = \frac{1}{2}(\kappa'(s))^2\,, \quad U(\kappa) = \frac{1}{8}\kappa(s)^4 + \frac{\mu_4}{2}\kappa(s)^2 - \mu_3\kappa(s)\,,$$

$$\mu_3 = \frac{1}{L^2 - 4\pi M} \left(\frac{L}{2} \int_0^L \kappa(s)^3 ds - \pi \int_0^L \kappa(s)^2 ds \right),$$

(1.8)
$$\mu_4 = \frac{1}{L^2 - 4\pi M} \left(M \int_0^L \kappa(s)^3 ds - \frac{L}{2} \int_0^L \kappa(s)^2 ds \right),$$

$$\mu_5 = -105\pi\mu_3 + 36 \left(\int_0^L \kappa(s)^2 ds \right) \mu_4 + \frac{3072}{\sqrt{\pi}} D_5 .$$

Theorem 2. [K. Watanabe[8]]

For every $n \geq 2$, the equation (1.7) has a unique C^∞ periodic solution with the prime period L/n (the solution of mode n), which is symmetric in $[O, L/n]$ at $L/2n$, decreasing in $[0, L/2n]$ and increasing in $[L/2n, L/n]$. Then, it hold that $\kappa(0) = \max_{0\leq s \leq L} \kappa(s)$, $\kappa(L/2n) = \min_{0 \leq s \leq L} \kappa(s)$ and $\kappa'(0) = \kappa'(L/2n) = 0$.

Remark 1.1. In order to be a closed curve, n cannot be 1.

Theorem 3. [K. Watanabe[8], [9]]

If $M < L^2/4\pi \leq (49/40)M$, minimizers are simple curves[1] *and domains are determined by those spectrum.*

Further, if $L^2/4\pi$ is sufficiently close to M, the minimizer is unique, of mode 2 and the domain enclosed by this curve is oval.

2. Representation of $\kappa(s)$

By the equation (1.7), in the case of mode n ($n \geq 2$), we have

(2.1)
$$\kappa'(s) = -\frac{1}{2}\sqrt{-\kappa(s)^4 - 4\mu_4\kappa(s)^2 + 8\mu_3\kappa(s) + 8\mu_5} \quad (0 \leq s \leq L/2n),$$

$$\kappa(0) = \max \kappa(s) = p, \ \kappa(L/2n) = \min \kappa(s) = q,$$

$$\kappa(s) = \kappa(L/n - s), \quad (L/2n \leq s \leq L/n).$$

As we orient the curve anti-clockwise, p is positive. We introduce new parameters p, q

[1]K. Watanabe announced $M < L^2/4\pi \leq (25/16)M$, but it seems that his proof holes good under this

and δ, where p and q are those in (1.7):

$$-\kappa(s)^4 - 4\mu_4\kappa(s)^2 + 8\mu_3\kappa(s) + 8\mu_5 = (p-\kappa)(\kappa-q)\{(\kappa+\frac{p+q}{2})^2+4\delta\} \quad (q \le \kappa \le p, \quad p>0), \tag{2.2}$$

$$\begin{cases} \mu_4 = -(1/16)(3p^2+2pq+3q^2-16\delta)\,, \\ \mu_3 = (1/32)(p+q)\{(p-q)^2+16\delta)\}\,, \\ \mu_5 = -(1/32)pq\{(p+q)^2+16\delta)\} \end{cases} \tag{2.3}$$

and

$$\frac{\partial(\mu_4,\,\mu_3,\,\mu_5)}{\partial(p,\,q,\,\delta)} = \frac{1}{4}(p-q)\{(\frac{3p+q}{4})^2+\delta\}\{(\frac{p+3q}{4})^2+\delta\}\,. \tag{2.4}$$

We replace p and q by P and Q, and set $\xi = \kappa + \frac{p+q}{2}$: we have

$$\begin{aligned} &P = (3p+q)/4\,,\; p = (3P-Q)/2\,,\;\; p+q = P+Q\,, \\ &Q = (p+3q)/4\,,\; q = (-P+3Q)/2\,, p-q = 2(P-Q) \end{aligned} \tag{2.5}$$

and

$$\frac{d\kappa}{d\xi} = -\sqrt{(P-\xi)(\xi-Q)(\xi^2+\delta)}\,. \tag{2.6}$$

Thus we arrive at

$$s = \int_{\kappa(0)}^{\kappa(s)} \frac{ds}{d\kappa} d\kappa = \int_{(1/2)\{\kappa(s)+(P+Q)/2\}}^{P} \frac{d\xi}{\sqrt{(P-\xi)(\xi-Q)(\xi^2+\delta)}}\,. \tag{2.7}$$

Changing the the variable of integral:

$$\begin{cases} \xi = Q + (1/\eta)\,, \\ \eta = \{1/(P-Q)\}\{1+\sqrt{(P^2+\delta)/(Q^2+\delta)}\tan^2(\varphi/2)\}\,, \end{cases}$$

(2.7) becomes

$$s = \frac{1}{\sqrt[4]{(P^2+\delta)(Q^2+\delta)}} \int_0^{\varphi(\kappa(s))} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}\,, \tag{2.8}$$

(2.9)
$$\tan^2 \frac{\varphi(\kappa(s))}{2} = \sqrt{\frac{Q^2+\delta}{P^2+\delta}}\,\frac{p-\kappa(s)}{\kappa(s)-q}\ ,$$

(2.10)
$$k^2 = \frac{1}{2}(1 - \frac{(PQ+\delta)}{\sqrt{(P^2+\delta)(Q^2+\delta)}})\ .$$

This means

(2.11)
$$s = \begin{cases} \frac{1}{\sqrt[4]{(P^2+\delta)(Q^2+\delta)}} F(\varphi(\kappa(s));k) & (0 \le s \le L/4n)\,, \\ \frac{1}{\sqrt[4]{(P^2+\delta)(Q^2+\delta)}} \{2K(k) - F(\pi - \varphi(\kappa(s));k)\} & (L/4n \le s \le L/2n)\,, \end{cases}$$

where
$$F(\varphi;k) = \int_0^{\varphi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \quad \text{(the elliptic integral of the first kind)}\,,$$

$$K(k) = F(\pi/2;k) \qquad \text{(the complete elliptic integral of the first kind)}\,.$$

Finally, using the elliptic function, we arrive at

(2.12)
$$\kappa(s) = \frac{p\sqrt{Q^2+\delta} + q\sqrt{Q^2+\delta} + (p\sqrt{Q^2+\delta} - q\sqrt{Q^2+\delta})\ \mathrm{cn}(\hat{K}(k)s;k)}{\sqrt{P^2+\delta} + \sqrt{Q^2+\delta} - (\sqrt{P^2+\delta} - \sqrt{Q^2+\delta})\ \mathrm{cn}(\hat{K}(k)s;k)}\ ,$$

where $\mathrm{sn}(z;k)$ is Jacobi's elliptic function *i.e.* the inverse function of

$$z = \int_0^x \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}} \qquad \text{and} \quad \mathrm{cn}(z;k) = \sqrt{1-\mathrm{sn}^2 z}\ .$$

3. Restriction Condition and New Parameters

3.1. The First Restriction Condition.

Taking $s = L/4n$ in (2.7), we obtain the relation:

(3.1)
$$\sqrt[4]{(P^2+\delta)(Q^2+\delta)} \ = \ \frac{4nK(k)}{L} \ \equiv \ \hat{K}(k)\,,$$

which corresponds to the restriction that the length of the curve is L, that is,

(3.2)
$$\delta = \frac{1}{2}\{-(P^2+Q^2) \pm \sqrt{(P^2-Q^2)^2 + 4\hat{K}(k)^4}\ \}\,.$$

By this restriction, δ is implicitly obtained by P and Q and there rest two free parameters. To see this clearly, let us introduce new parameters k and h:

$$(3.3)\qquad \begin{cases} k^2 = (1/2)\{1-(PQ+\delta)/\sqrt{(P^2+\delta)(Q^2+\delta)}\}, \\ P-Q = hP, \quad (Q=(1-h)P). \end{cases}$$

For the simplicity, we restrict ourselves:

Assumption 2.

$$(3.4)\qquad P \equiv (3p+q)/4 > 0.$$

Under this assumption, h is positive.

By the relations (3.2) and (3.3), P^2 satisfies

$$(3.5)\qquad h^2(1-h)(P^2)^2 - h^2(1-2k^2)\hat{K}(k)^2(P^2) + 4k^2(1-k^2)\hat{K}(k)^4 = 0.$$

[*The case* $0<h<1$]

In order that (3.5) has a positive root, as the product of two roots on P^2 is positive, the discriminant and the sum of both roots must be positive:

$$(3.6)\qquad h > \frac{4k\{-2k(1-k^2)+\sqrt{1-k^2}\}}{(1-2k^2)^2}, \qquad 0<k^2<\frac{1}{2}.$$

Under the conditions (3.6),

$$(3.7)\qquad P^2 = \frac{h(1-2k^2)\pm\sqrt{J}}{2h(1-h)}\hat{K}(k)^2$$

$$= \frac{8k^2(1-k^2)}{h\{h(1-2k^2)+\sqrt{J}\}}\hat{K}(k)^2 \quad \text{(the smaller case)},$$

$$J = h^2(1-2k^2)^2 - 16(1-h)k^2(1-k^2).$$

[*The case* $h=1$]

As the equation (3.5) becomes of degree 1, under the condition $0<k^2<1/2$, it has the positive root:

$$(3.8)\qquad P^2 = \frac{4k^2(1-k^2)}{1-2k^2}\hat{K}(k)^2\,.$$

[*The case* $h > 1$]

In order that (3.5) has a positive root, as the product of two roots on P^2 is negative, the discriminant must be positive:

$$(3.9)\qquad h > \frac{4k\{-2k(1-k^2)+\sqrt{1-k^2}\}}{(1-2k^2)^2}\,.$$

Under the conditions (3.9), (3.5) has a unique positive root:

$$(3.10)\qquad P^2 = \frac{-h(1-2k^2)+\sqrt{J}}{2h(h-1)}\hat{K}(k)^2\,,$$

$$J = h^2(1-2k^2)^2 + 16(h-1)k^2(1-k^2)\,.$$

Remark 3.1. The smaller root in (3.7) is continued to (3.8) and (3.10) as h becomes 1 and greater than 1. The greater root in (3.7) tends to infinity as h tends to 1. The curve bifurcated from a circle is given by the former continued root .

Assumption 3. *When* $0 < h < 1$, *we adopt the smaller root in* (3.7).

For positive h, the Jacobian and δ are positive:

$$(3.11)\qquad \frac{\partial(P,\,Q)}{\partial(h,\,k))} = \frac{1}{2}(P^2)_k = \frac{4k\{(1-2k^2)+4h^2k^2(1-k^2\}}{h\sqrt{J}\{\,(2-h)^2(1-2k^2)+h\sqrt{J}\,\}}\hat{K}(k)^2 > 0\,,$$

$$(3.12)\qquad \delta = \frac{1}{2}(1-2k^2)\,\hat{K}(k)^2 + \frac{\sqrt{J}}{2h} > 0\,.$$

3.2. The Second and Third Restriction Conditions.

As the winding number is one, we obtain the second restriction:

$$(3.13)\qquad \int_0^{L/2n} \kappa(s)ds = \frac{\pi}{n}\,,$$

$$(3.14)\qquad L\hat{K}(k)\{h^2P^2+4k^2\hat{K}(k)^2\}\{h^2P^2+4(1-k^2)\hat{K}(k)^2\}$$

$$-8\pi h^2P^2\{h^2P^2+4(1-k^2)\hat{K}(k)^2\}\,\Pi(\frac{h^2P^2}{4\hat{K}(k)^2}-k^2;k)+4\pi\hat{K}(k)h^2(2-h)P^2 = 0,$$

$$\Pi(c;k)=\int_0^{\pi/2}\frac{d\varphi}{(1+c\sin^2\varphi)\sqrt{1-k^2\sin^2\varphi}}\quad\text{(the elliptic integral of the third kind)}.$$

We replace h by

$$(3.15)\qquad t=\sqrt{J}/k\equiv\sqrt{h^2-4k^2(1-2k^2)(2-h)^2}/k\,,$$

where J is that in (3.7) and (3.10). The relation (3.14) is equivalent to

(3.16)

$$\begin{aligned}
z_n(k,t)\equiv&-[(3-4k^2)\{t^2+16(1-k^2)\}+\{8k(1-k^2)+\sqrt{(1-2k^2)^2t^2+16(1-k^2)}\,\}t]\times\\
&[t^2+\{8k(1-k^2)+\sqrt{(1-2k^2)^2t^2+16(1-k^2)}\,\}t+16(1-k^2)]\hat{K}(k)\\
&-4(1-k^2)\{t^2+16(1-k^2)\}\times\\
&[t^2+\{8k(1-k^2)+\sqrt{(1-2k^2)^2t^2+16(1-k^2)}\,\}t+16(1-k^2)]\times\\
&\Pi(\frac{k^2\{t^2-8k(1-k^2)t+16(1-k^2)-t\sqrt{(1-2k^2)^2t^2+16(1-k^2)}\,\}}{(1-2k^2)t^2+\{8k(1-k^2)+\sqrt{(1-2k^2)^2t^2+16(1-k^2)}\,\}t+16(1-k^2)(1-2k^2)},\,k)\\
&-\frac{\sqrt{2}\pi}{n}\sqrt{(1-k^2)\{t^2+16(1-k^2)\}}\{-kt^2+2\sqrt{(1-2k^2)^2t^2+16(1-k^2)}\,\}\times\\
&\sqrt{(1-2k^2)\{t^2+16(1-k^2)\}+\{8k(1-k^2)+\sqrt{(1-2k^2)^2t^2+16(1-k^2)}\,\}t}\;=0.
\end{aligned}$$

On the other hand, as the area of the domain enclosed by this curve is M, we obtain the third restriction:

$$(3.17)\qquad \frac{L}{2}\,\frac{\mu_4+n\int_0^{L/2n}\hat{K}(k)^2ds}{2\mu_4+n\int_0^{L/2n}\hat{K}(k)^3ds}=M\,.$$

By the relation (1.8) and

$$\text{(3.18)} \qquad \int_0^{L/2n} \hat{K}(k)^2 ds = \frac{L}{8n}\{(2-h)^2P^2 - 16(1-k^2)\hat{K}(k)^2\} + 8\hat{K}(k)E(k)\,,$$

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\varphi}\,d\varphi \quad \text{(the complete elliptic integral of the second kind),}$$

$$\text{(3.19)} \qquad \int_0^{L/2n} \hat{K}(k)^3 ds \quad = \frac{\pi}{4n}\{3(2-h)^2P^2 - 8(1-2k^2)\hat{K}(k)^2\}$$

$$+\frac{L}{8n}(2-h)P\{(h^2+4h-4)P^2 + 4(1-2k^2)\hat{K}(k)^2\}\,,$$

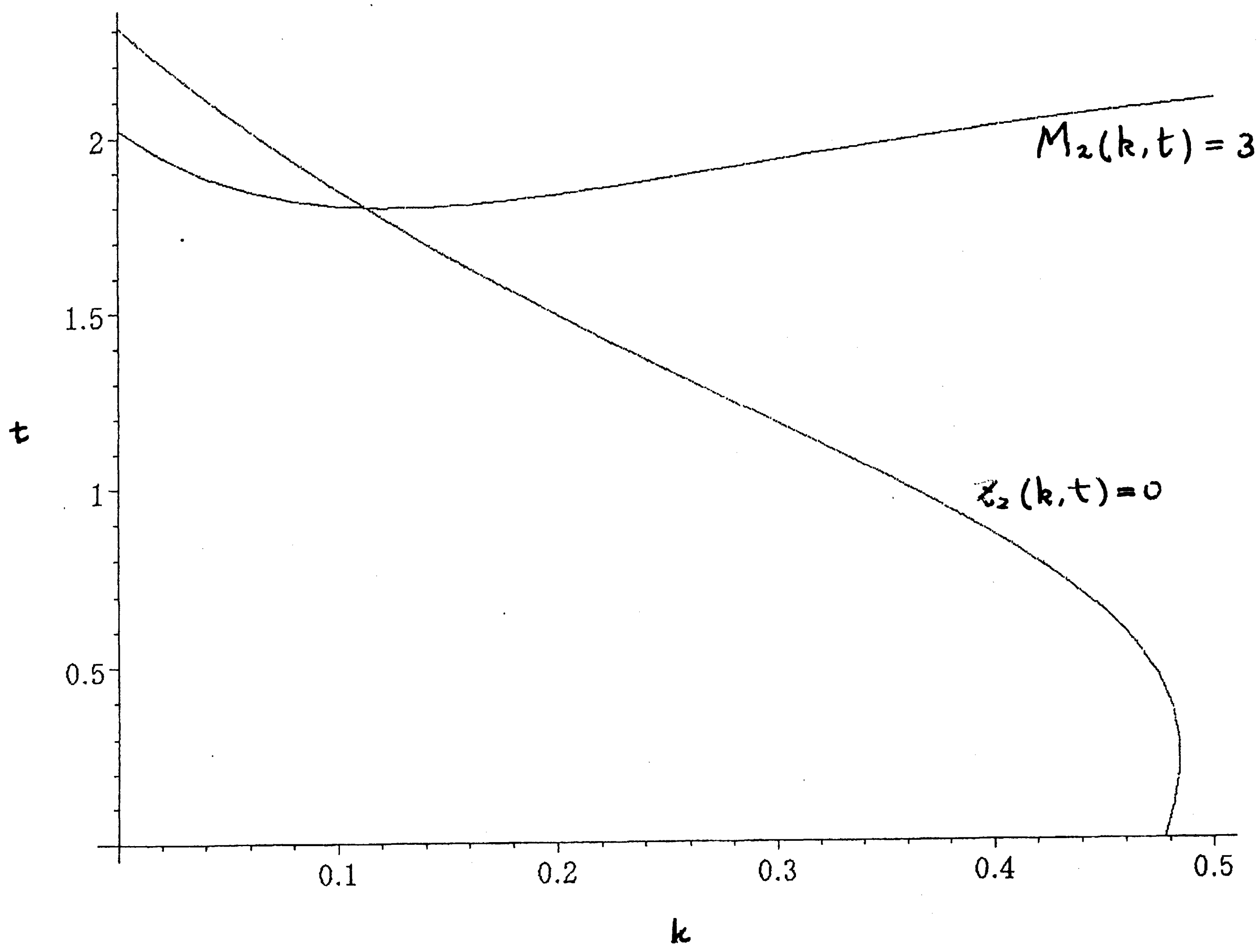
(3.17) is equivalent to

(3.20)

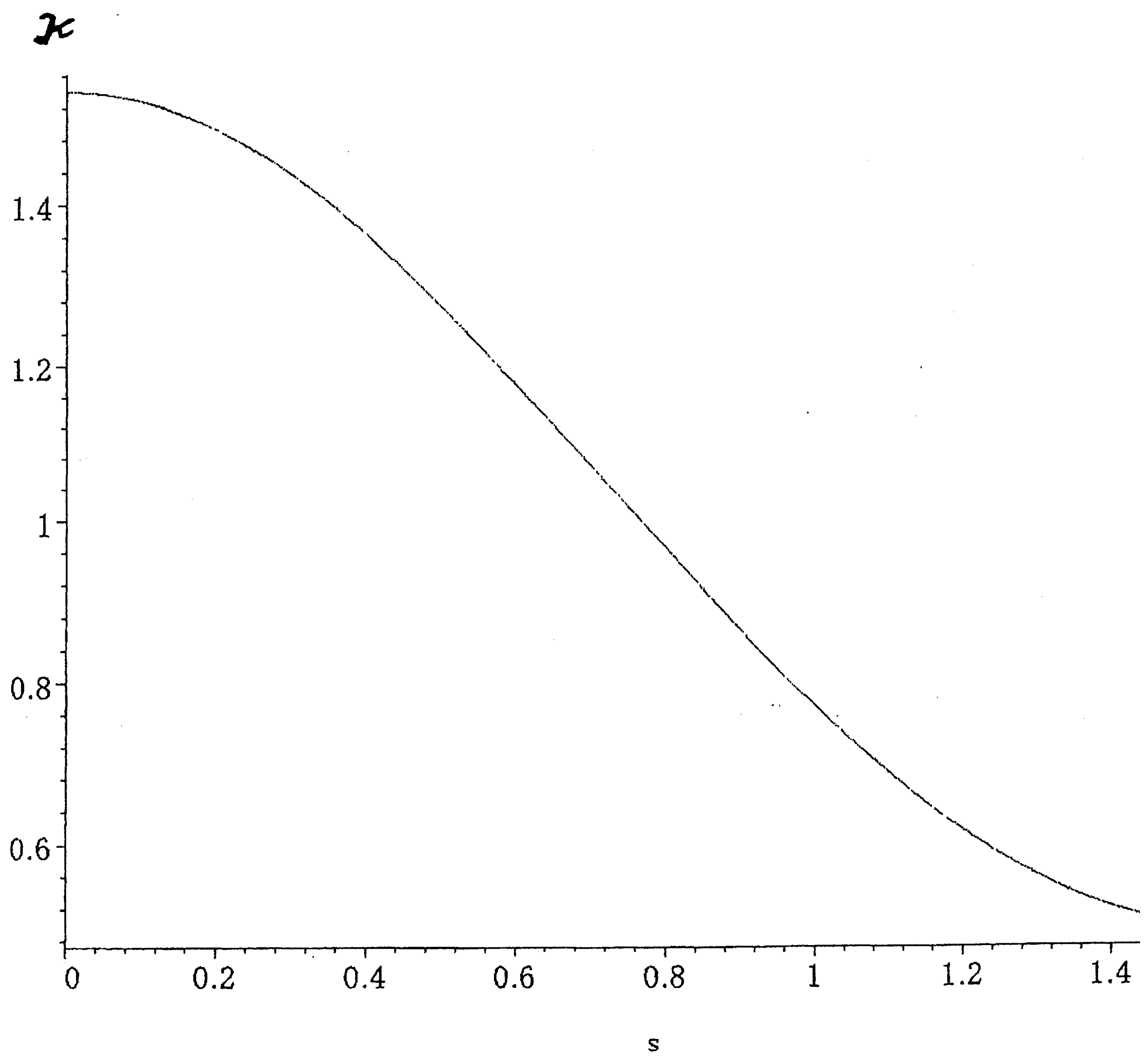
$$\begin{aligned}
&M_n(k,t) \equiv -\sqrt{2\{t^2+16(1-k^2)\}}\times \\
&\quad \sqrt{(1-2k^2)t^2 + \{8k(1-k^2)+\sqrt{(1-2k^2)^2t^2+16(1-k^2)}\,\}t + 16(1-k^2)(1-2k^2)}\times \\
&\quad [\{(1-2k^2)t^4 + 8k(1-k^2)t^3 - 8(1-k^2)(1-2k^2)(2k^2-5)t^2 + 128k(1-k^2)^2t \\
&\qquad - 128(1-k^2)^2(4k^2-3) \\
&\quad + t(t^2-8k(1-k^2)t+16(1-k^2))\sqrt{(1-2k^2)^2t^2+16(1-k^2)}\,\}\hat{K}(k) \\
&\quad - 2\{t^2+16(1-k^2)\}\{(1-2k^2)t^2 + (8k(1-k^2)+\sqrt{(1-2k^2)^2t^2+16(1-k^2)}\,)t \\
&\quad + 16(1-k^2)(1-2k^2)\}E(k)]L^2 \\
&\quad /[16\sqrt{1-k^2}(kt^2 - 2\sqrt{(1-2k^2)^2t^2+16(1-k^2)}\times \\
&\quad \{(-2k^4+2k^2-1)t^4 - 8k(1-k^2)(1-2k^2)t^3 + 8(1-k^2)(1-2k^2)(2k^2-3)t^2 \\
&\qquad - 128(1-k^2)^2(1-2k^2)t - 128k(1-k^2)^2 \\
&\quad - t((1-4k^2)t^2 - 8k(1-k^2)t + 16(1-k^2)(1-2k^2))\sqrt{(1-2k^2)^2t^2+16(1-k^2)}\,\}\times \\
&\qquad n\hat{K}(k)^2] \\
&= M\,.
\end{aligned}$$

Computer-aided, we obtain the graphs of the curves defined by (3.16) and (3.20) :

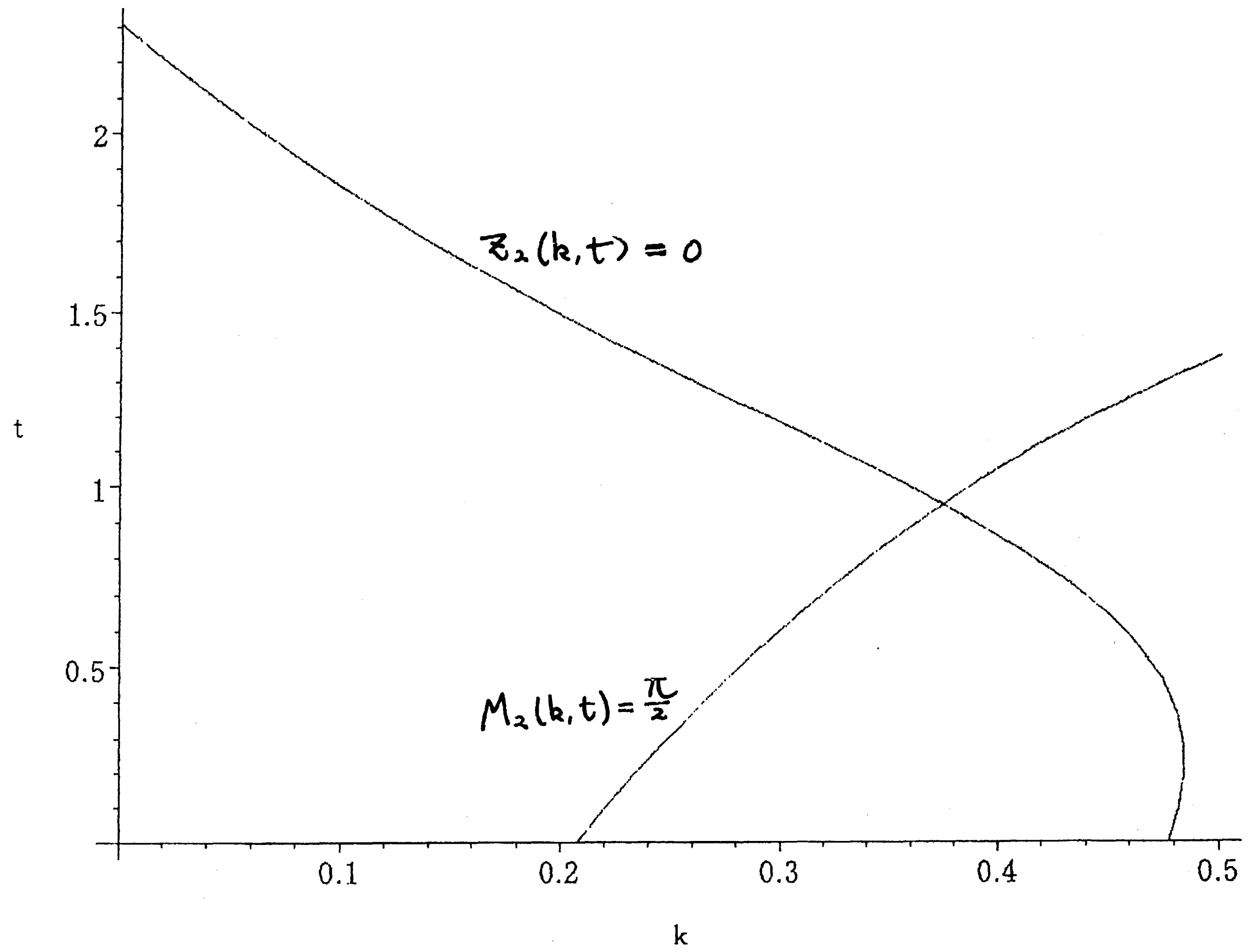
$[Z_2(k,t) = 0$ and $M_2 = 3\ ]$ ($L = 2\pi$)



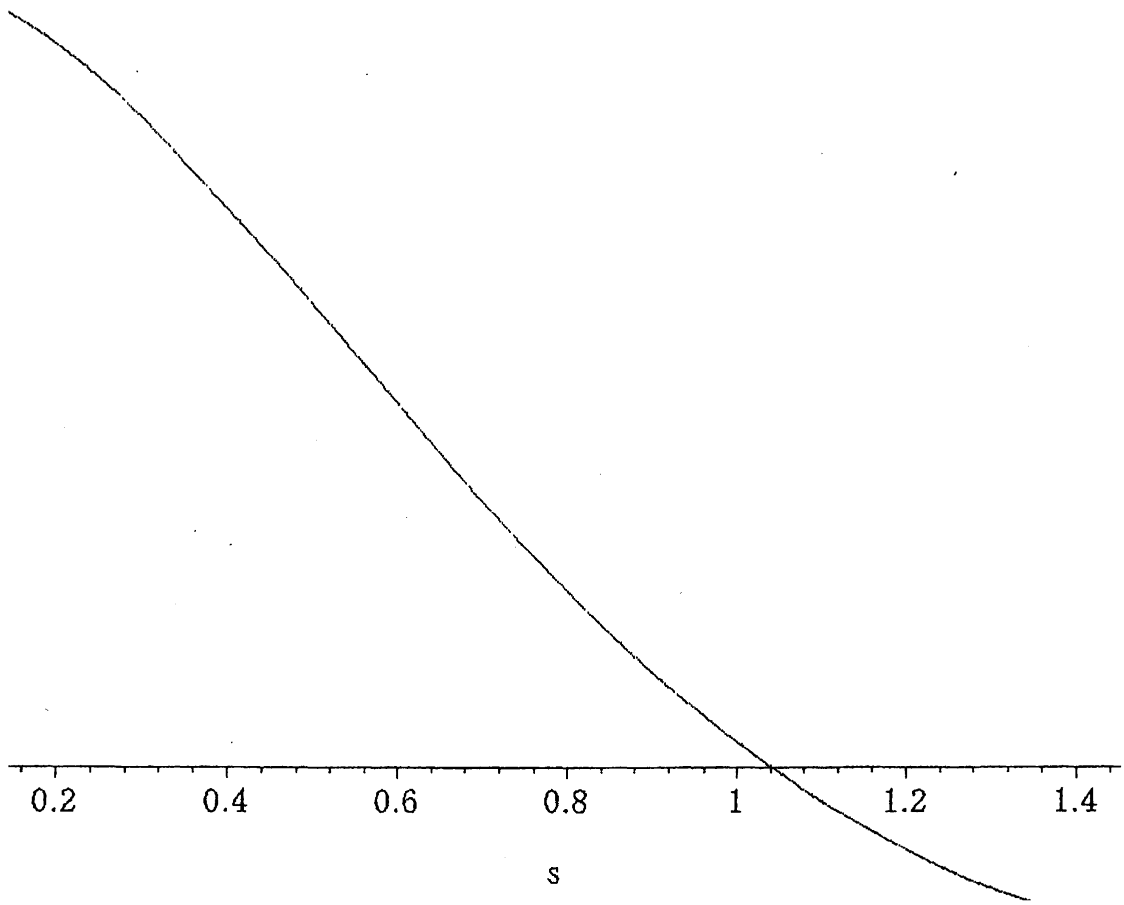
$[\kappa = \kappa(s)]$ ($M_2 = 3$, $L = 2\pi$ and $n = 2$)



$[Z_2(k,t) = 0$ and $M_2 = \pi/2]$ ($L = 2\pi$)

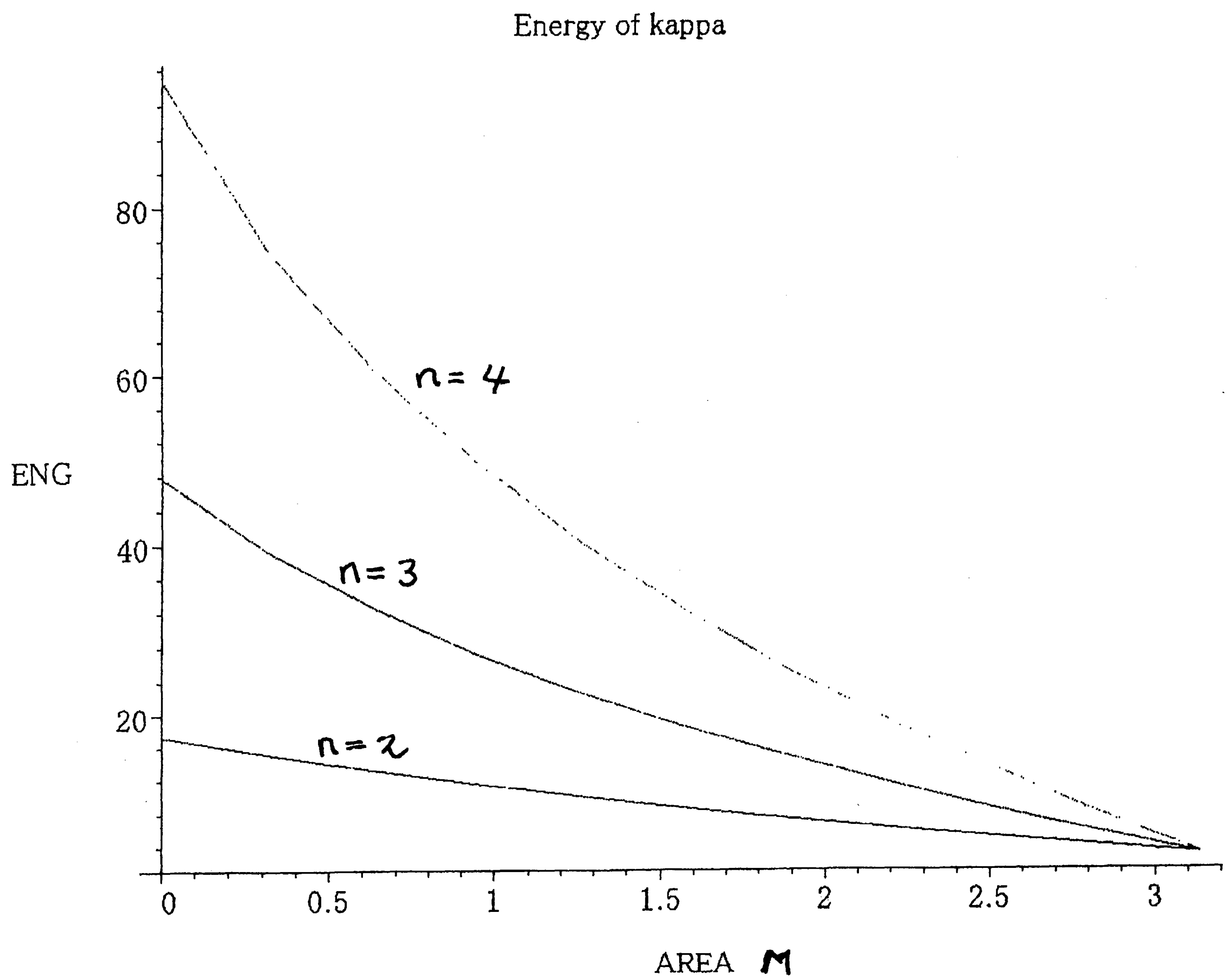


(s)] ($M_2 = \pi/2$, $L = 2\pi$ and $n = 2$)



Computer-aided, we can also obtain the graphs of the curvature energy $\int_0^L \kappa(s; M, n)^2 ds$ with $L = 2\pi$ ($n \geq 2$).

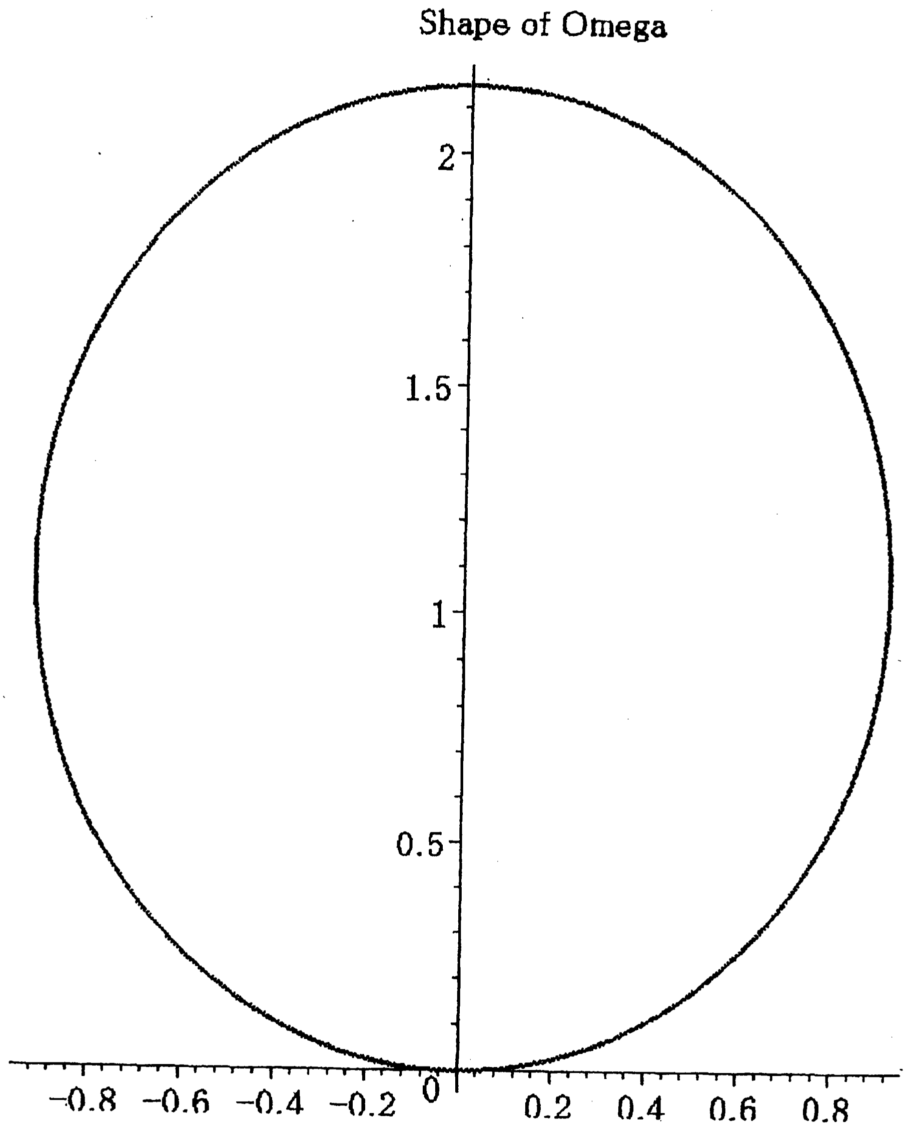
For every $n \geq 2$, as M tends to π, the curvature energy tends to π, which corresponds to the cicle with the radius 1.



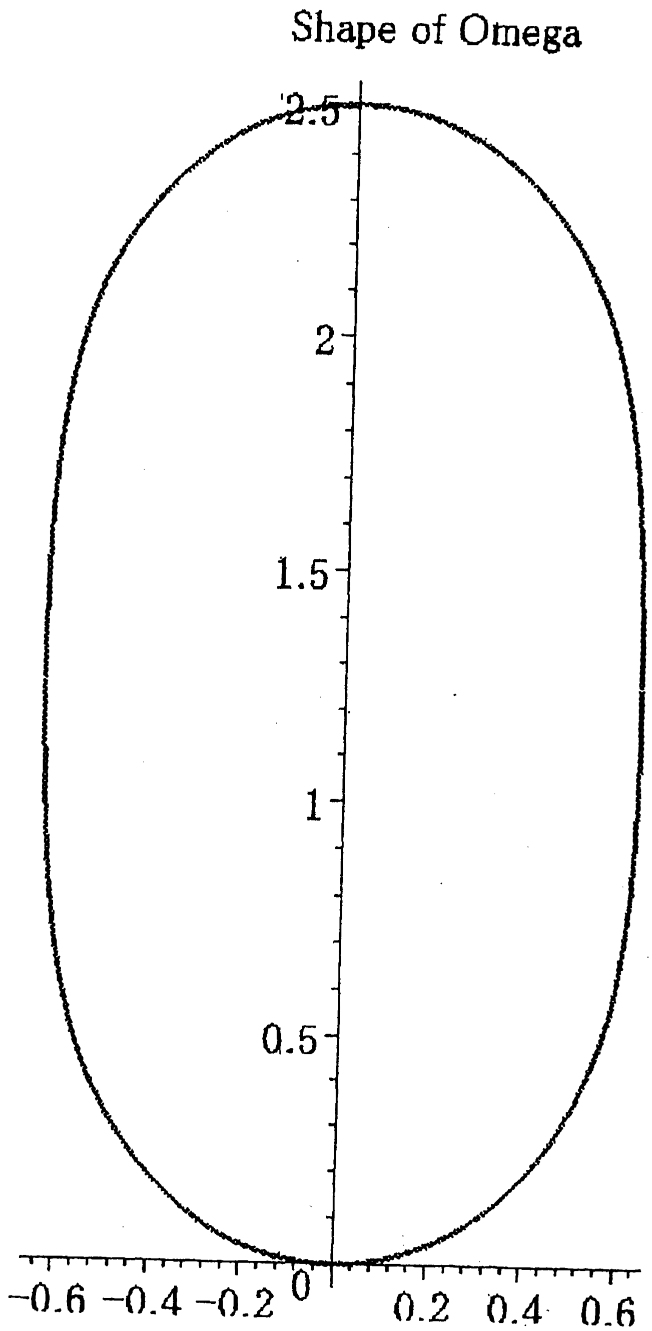
As a computer-aided theorem, we obtain the following

Theorem 4.
The minimizer is of mode 2 and unique.

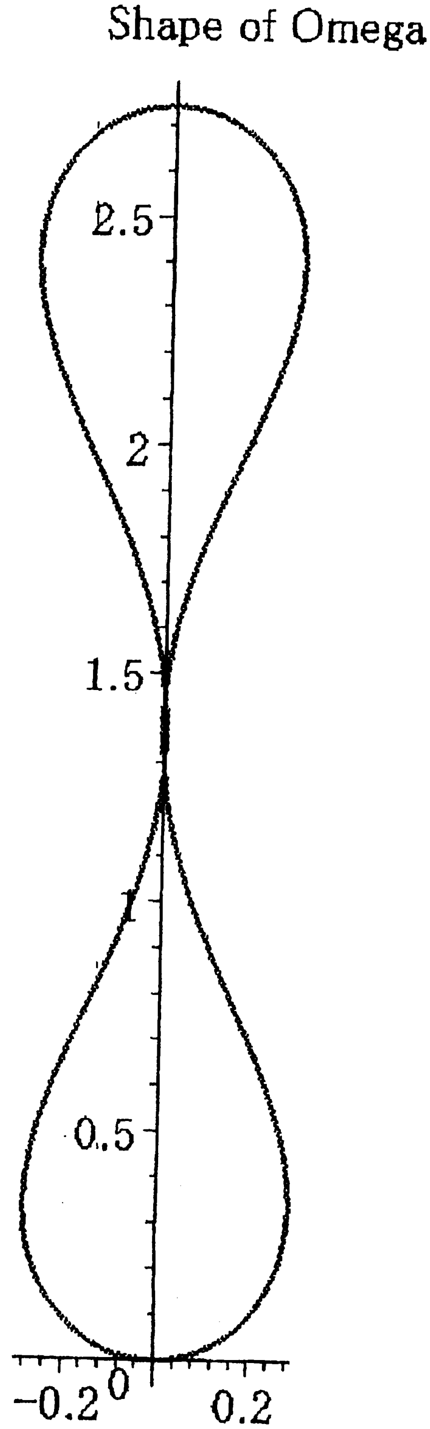
[k= 0.05]



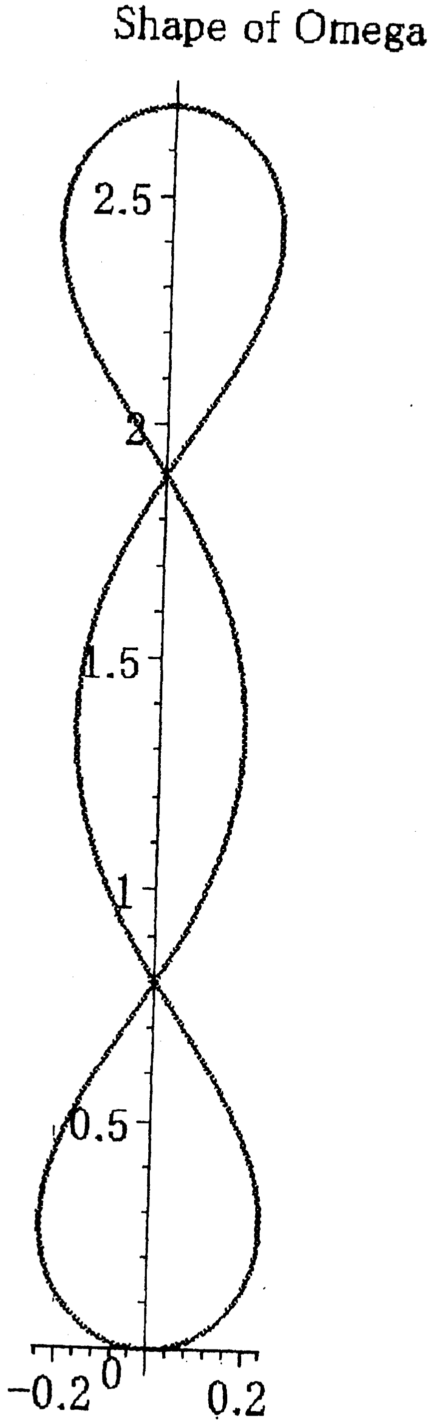
[k= 0.2]



[k= 0.445] (No longer, not simply connected domain)



[k= 0.4776] (No longer, not simply connected domain)



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