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Toroidal groups without non-constant meromorphic functions

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1 Introduction

Gheradelli and Andreotti [4] obtained fibration theorem for quasi-Abelian varieties. Abe [1] proved the fibration theorems by getting some standard forms of period matrices. Umeno [8] characterized the quasi-Abelian varieties by the above standard forms of period matrices.

We shall study meromorphic functions on a toroidal group by using period matrices for quasi-Abelian varieties. In the section 2, we shall give the conditions that a toroidal group has no non-constant meromorphic functioins. In the section 3, we shall discuss the example given by Abe and Kopfermann using the recent results of Umeno [8].

2 Meromorphic functions on a toroidal group

In this section, we discuss meromorphic functions on a toroidal group. Before proceeding, we introduce some definitions and terminologies.

A connected complex Lie group X is called a toroidal group if every holomorphic function on X is constant.

Since any toroidal group is an abelian Lie group, there exists a discrete subgroup Γ of $\mathbb{C}^n$ such that X is isomorphic to $\mathbb{C}^n/\Gamma$. Let $X=\mathbb{C}^n/\Gamma$ be a toroidal group and $\Gamma=\mathbb{Z}\{\lambda_1,\cdots,\lambda_{n+q}\}, 0<q\leq n$ be a discrete subgroup of $\mathbb{C}^n$ generated by $\mathbb{R}$-linearly independent vectors $\lambda_1,\cdots,\lambda_{n+q}$. The matrix $P=[\lambda_1,\cdots,\lambda_{n+q}]$ is called a period matrix for $X=\mathbb{C}^n/\Gamma$. We sometimes write $\Gamma=\mathbb{Z}\{P\}$ instead of $\Gamma=\mathbb{Z}\{\lambda_1,\cdots,\lambda_{n+q}\}$. Let $\mathbb{R}_\Gamma=\mathbb{R}\{\lambda_1,\cdots,\lambda_{n+q}\}$ be the $\mathbb{R}$-span of Γ. We denote by $\mathbb{C}_\Gamma=\mathbb{R}_\Gamma\cap\sqrt{-1}\mathbb{R}_\Gamma$ the maximal complex subspace of $\mathbb{R}_\Gamma$.

Definition 2.1 A toroidal group $\mathbb{C}^n/\Gamma$ is of type $q\,(q>0)$ if

$$\dim_{\mathbb{C}}\mathbb{C}_\Gamma = q.$$

Definition 2.2 A toroidal group $\mathbb{C}^n/\Gamma$ is a quasi-Abelian variety, if there exists a Hermitian form H on $\mathbb{C}^n \times \mathbb{C}^n$ such that

$H \mid \mathbb{C}_\Gamma \times \mathbb{C}_\Gamma > 0$ and

$E := \operatorname{Im} H \mid \Gamma \times \Gamma$ is a $\mathbb{Z}$-valued skew-symmetric form.

A Hermitian form H is called an ample Riemann form which defines a quasi-Abelian structure on $X = \mathbb{C}^n/\Gamma$. Let $f(z)$ be a meromorphic function on $\mathbb{C}^n$. A period of f is a vector $\lambda \in \mathbb{C}^n$ such that $f(z+\lambda) = f(z)$ for all $z \in \mathbb{C}^n$ and the period group of f is the set $G(f)$ of all periods of f.

For later use, we first consider the following([3]):

Theorem 2.1 *Let $X = \mathbb{C}^n/\Gamma$ be a toroidal group and f be a meromorphic function on $\mathbb{C}^n$ with $\Gamma \subset G(f)$. Then there exist $p, q \in H^0(\mathbb{C}^n, \mathcal{O})$ with $(p,q) = 1$ and $f = p/q$, and there exist linear polynomials $l_\lambda(\lambda \in \Gamma)$ such that*

$$\begin{aligned} p(z+\lambda) &= p(z)\exp(l_\lambda(z)) \text{ and} \\ q(z+\lambda) &= q(z)\exp(l_\lambda(z)), \end{aligned}$$

for all $z \in \mathbb{C}^n$ and $\lambda \in \Gamma$.

Next, let us set $el_\lambda(z) := \exp(l_\lambda(z))$. Then we see

$$el_{\lambda'}(z+\lambda)el_\lambda(z) = el_\lambda(z+\lambda')el_{\lambda'}(z),$$

since $el_{\lambda'+\lambda}(z) = el_{\lambda'}(z+\lambda)el_\lambda(z)$.

Definition 2.3 A system of holomorphic functions $e_\lambda \in H^0(\mathbb{C}^n, \mathcal{O}^*)$ satisfying

$$e_{\lambda'}(z+\lambda)e_\lambda(z) = e_\lambda(z+\lambda')e_{\lambda'}(z)$$

is said to be multipliers.

We have already known the following(cf.[6]):

Proposition 2.1 *Let $X = \mathbb{C}^n/\Gamma$ be a toroidal group and $L \longrightarrow X$ be a complex line bundle. Then, for each $\lambda \in \Gamma$, there exist multipliers e_λ such that*

$$L \cong \mathbb{C}^n \times \mathbb{C}/\Gamma$$

where Γ acts on $\mathbb{C}^n \times \mathbb{C}$ by $\lambda \circ (z, \xi) = (z + \lambda, e_\lambda(z)\xi)$ for $\lambda \in \Gamma$.

Set $e_\lambda(z) = \exp(2\pi\sqrt{-1}f_\lambda(z))$ where $f_\lambda \in H^0(\mathbb{C}^n, \mathcal{O})$. For the line bundle L defined by $e_\lambda \in H^0(\mathbb{C}^n, \mathcal{O}^*)$, we see the following([6]):

Proposition 2.2 *Let L be a line bundle on a toroidal group $X = \mathbb{C}^n/\Gamma$ defined by $e_\lambda(z) = \exp(2\pi\sqrt{-1}f_\lambda(z))$ such that $c_1(L) = E$. Then*

$$E(\lambda_1, \lambda_2) = f_{\lambda_2}(z + \lambda_1) + f_{\lambda_1}(z) - f_{\lambda_1}(z + \lambda_2) - f_{\lambda_2}(z) \quad \textit{for} \quad z \in \mathbb{C}^n, \text{ and } \lambda_i \in \Gamma$$

Here, we recall the definition of Néron-Severi group of X. Let X be a toroidal group. The Néron-Severi group $NS(X)$ of X is defined by

$NS(X) = \{E : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R} | E$: an alternating form with $E(\Gamma \times \Gamma) \subseteq \mathbb{Z}$ and $E(\sqrt{-1}\lambda, \sqrt{-1}\mu) = E(\lambda, \mu)\}$.

Definition 2.4 A toroidal group $\mathbb{C}^n/\Gamma$ is called of cohomologically finite type if

$$\dim H^1(\mathbb{C}^n/\Gamma, \mathcal{O}) < +\infty$$

Now, we state our main theorem.

Theorem 2.2 *Let $X = \mathbb{C}^n/\Gamma$ be a toroidal group of cohomologically finite type. Suppose that the Néron-Severi group $NS(X)$ is zero. Then X has no non-constant meromorphic functions.*

To prove theorem 2.2, we need some results. So, we first consider the following result well known in classical complex torus theory such as Appell-Humbert decomposition [7].

Theorem 2.3 *Let $X = \mathbb{C}^n/\Gamma$ be a toroidal group, $L \longrightarrow X$ a complex line bundle such that $c_1(L) = E \in H^2(X,\mathbb{Z})$ and H a Hermitian form on $\mathbb{C}^n$ such that* $\operatorname{Im} H \mid \Gamma \times \Gamma = E$.

Then there exists a map $\alpha : \Gamma \longrightarrow \mathbb{C}_1^* = \{z \in \mathbb{C}^* \| |z| = 1\}$ such that

$$\begin{aligned}\alpha(\lambda_1, \lambda_2) &= \alpha(\lambda_1)\alpha(\lambda_2)\exp(\pi\sqrt{-1}E(\lambda_1,\lambda_2)) \text{ for all } \lambda_1, \lambda_2 \in \Gamma \text{ and} \\ e_\lambda(z) &:= \alpha(\lambda)\exp(\pi H(z,\lambda) + \frac{\pi}{2}H(\lambda,\lambda))\end{aligned}$$

are multipliers which define a complex line bundle $L^0 \longrightarrow X$ satisfying $c_1(L^0) = E$.

Proof

Let us set $g_\lambda(z) = \frac{1}{2\sqrt{-1}}H(z,\lambda) + \beta_\lambda$ for any constants β_λ. Then, we have

$$\begin{aligned}& g_{\lambda_2}(z+\lambda_1) + g_{\lambda_1}(z) - g_{\lambda_1}(z+\lambda_2) - g_{\lambda_2}(z) \\ =\ & \frac{1}{2\sqrt{-1}}(H(z+\lambda_1,\lambda_2) + \beta_{\lambda_2} + H(z,\lambda_1) + \beta_{\lambda_1} - H(z+\lambda_2,\lambda) \\ & -\beta_{\lambda_1} - H(z,\lambda_2) - \beta_{\lambda_2}) \\ =\ & \frac{1}{2\sqrt{-1}}(H(\lambda_1,\lambda_2) - H(\lambda_2,\lambda_1)) \\ =\ & \operatorname{Im} H(\lambda_1,\lambda_2) \\ =\ & E(\lambda_1,\lambda_2)\end{aligned}$$

for all $\lambda_1, \lambda_2 \in \Gamma$ and $z \in \mathbb{C}^n$.

Suppose that $e_\lambda^0(z) := \exp(2\pi\sqrt{-1}g_\lambda(z))$ are multipliers. Then

$$e^0_{\lambda_2}(z+\lambda_1)e^0_{\lambda_1}(z) = e^0_{\lambda_1+\lambda_2}(z)$$

for all $\lambda_1, \lambda_2 \in \Gamma$ and $z \in \mathbb{C}^n$.

Then, we see that

$$\frac{1}{2\pi\sqrt{-1}}(\log e^0_{\lambda_2}(z+\lambda_1) + \log e^0_{\lambda_1}(z) - \log e^0_{\lambda_1+\lambda_2}(z)) \in \mathbb{Z}.$$

So, from this fact,

$$
\begin{aligned}
& g_{\lambda_2}(z+\lambda_1)+g_{\lambda_1}(z)-g_{\lambda_1+\lambda_2}(z) \\
= \ & \frac{1}{2\sqrt{-1}}H(z+\lambda_1,\lambda_2)+\beta_{\lambda_2}+\frac{1}{2\sqrt{-1}}H(z,\lambda_1)+\beta_{\lambda_1} \\
& -\frac{1}{2\sqrt{-1}}H(z,\lambda_1+\lambda_2)-\beta_{\lambda_1+\lambda_2} \\
= \ & \frac{1}{2\sqrt{-1}}H(\lambda_1,\lambda_2)+\beta_{\lambda_1}+\beta_{\lambda_2}-\beta_{\lambda_1+\lambda_2}\in\mathbb{Z}
\end{aligned}
$$

for all $\lambda_1,\lambda_2\in\Gamma$.

Thus, we get

$$\frac{1}{2}H(\lambda_1,\lambda_2)+\sqrt{-1}\beta_{\lambda_1}+\sqrt{-1}\beta_{\lambda_2}-\sqrt{-1}\beta_{\lambda_1+\lambda_2}\in\sqrt{-1}\mathbb{Z}.$$

Next, setting $\sqrt{-1}\beta_\lambda=\gamma_\lambda+\frac{1}{4}H(\lambda,\lambda)$ for any constants γ_λ, we reduce the above equation to

$$
\begin{aligned}
& \frac{1}{2}H(\lambda_1,\lambda_2)+\gamma_{\lambda_1}+\frac{1}{4}H(\lambda_1,\lambda_1)+\gamma_{\lambda_2}+\frac{1}{4}H(\lambda_2,\lambda_2) \\
& -\gamma_{\lambda_1+\lambda_2}-\frac{1}{4}H(\lambda_1+\lambda_2,\lambda_1+\lambda_2) \\
= \ & \frac{1}{4}(H(\lambda_1,\lambda_2)-H(\lambda_2,\lambda_1)+\gamma_{\lambda_1}+\gamma_{\lambda_2}-\gamma_{\lambda_1+\lambda_2} \\
= \ & \gamma_{\lambda_1}+\gamma_{\lambda_2}-\gamma_{\lambda_1+\lambda_2}+\frac{\sqrt{-1}}{2}E(\lambda_1,\lambda_2)\in\sqrt{-1}\mathbb{Z}.
\end{aligned}
$$

Then, from this fact, we see that $\operatorname{Re}\gamma_\lambda$ is additive in Γ , that is, $\operatorname{Re}\gamma_\lambda\in\operatorname{Hom}(\Gamma,\mathbb{R})$.

Hence, $\operatorname{Re}\gamma_\lambda$ extends to an $\mathbb{R}$-linear function $\mu:\mathbb{C}^n\longrightarrow\mathbb{R}$ such that $\mu\mid\Gamma=\operatorname{Re}\gamma_\lambda$, and there is a $\mathbb{C}$-linear form $l:\mathbb{C}^n\longrightarrow\mathbb{C}$ defined by $l(z)=\mu(z)-\sqrt{-1}\mu(\sqrt{-1}z)$ with $\operatorname{Re} l=\mu$.

Now, setting $\gamma_\lambda'=\gamma_\lambda-l(z)$, $\beta_\lambda'=\frac{1}{\sqrt{-1}}(\gamma_\lambda'+\frac{1}{4}H(\lambda,\lambda))$ and $h_\lambda(z)=\frac{1}{2\sqrt{-1}}H(z,\lambda)+\beta_\lambda'$, we calculate

$$\begin{aligned}
& h_{\lambda_2}(z+\lambda_1) + h_{\lambda_1}(z) - h_{\lambda_1+\lambda_2}(z) \\
= & \frac{1}{2\sqrt{-1}}(H(z+\lambda_1,\lambda_2) + H(z,\lambda_1) - H(z,\lambda_1+\lambda_2)) + \beta'_{\lambda_1} + \beta'_{\lambda_2} - \beta'_{\lambda_1+\lambda_2} \\
= & \frac{1}{\sqrt{-1}}(\frac{1}{2}H(\lambda_1,\lambda_2) - \frac{1}{4}(H(\lambda_1,\lambda_2) + H(\lambda_2,\lambda_1)) + (\gamma'_{\lambda_1} + \gamma'_{\lambda_2} - \gamma'_{\lambda_1+\lambda_2})) \\
= & \frac{1}{2}\operatorname{Im} H(\lambda_1,\lambda_2) + \frac{1}{\sqrt{-1}}(\gamma'_{\lambda_1} + \gamma'_{\lambda_2} - \gamma'_{\lambda_1+\lambda_2}) \\
= & \frac{1}{2}E(\lambda_1,\lambda_2) + \frac{1}{\sqrt{-1}}(\gamma'_{\lambda_1} + \gamma'_{\lambda_2} - \gamma'_{\lambda_1+\lambda_2}) \in \mathbb{Z}.
\end{aligned}$$

Thus, it follows from this result that $\exp(2\pi\sqrt{-1}h_\lambda(z))$ are multipliers.

Next, to complete our proof of theorem, it suffices to show that $e^0_\lambda(z)$ and $\exp(2\pi\sqrt{-1}h_\lambda(z))$ are equivalent in $H^1(X,\mathcal{O}^*)$. Since

$$\begin{aligned}
\exp(2\pi\sqrt{-1}h_{\lambda(z)}) &= \exp(\pi H(z,\lambda))\exp(2\pi\sqrt{-1}\beta'_\lambda) \\
&= \exp(\pi H(z,\lambda))\exp(2\pi(\gamma'_\lambda + \frac{1}{4}H(\lambda,\lambda)) \\
&= \exp(\pi H(z,\lambda))\exp(2\pi\sqrt{-1}\beta_\lambda)\exp(-l(\lambda)) \\
&= \exp(2\pi\sqrt{-1}(\frac{1}{2\sqrt{-1}}H(z,\lambda)) + \beta_\lambda))\exp(-l(\lambda)) \\
&= e^0_\lambda(z)\exp(-l(z+\lambda))\exp(-l(\lambda))^{-1},
\end{aligned}$$

so we obtain that $e^0_\lambda(z)$ is equivalent to $\exp(2\pi\sqrt{-1}h_\lambda(z))$ in $H^1(X,\mathcal{O}^*)$. We may assume that γ'_λ is pure imaginary.

Then, setting $\alpha(\lambda) = \exp(2\pi\gamma'_\lambda)$, we see that $|\alpha(\lambda)| = 1$.

Then, since

$$\gamma'_{\lambda_1} + \gamma'_{\lambda_2} - \gamma'_{\lambda_1+\lambda_2} + \frac{\sqrt{-1}}{2}E(\lambda_1,\lambda_2) \in \sqrt{-1}\mathbb{Z}$$

for all $\lambda_1, \lambda_2 \in \Gamma$,

$$\begin{aligned}
\frac{\alpha(\lambda_1+\lambda_2)}{\alpha(\lambda_1)\alpha(\lambda_2)} &= \exp(2\pi(\gamma'_{\lambda_1+\lambda_2} - \gamma'_{\lambda_1} - \gamma'_{\lambda_2})) \\
&= \exp(2\pi(\frac{\sqrt{-1}}{2}E(\lambda_1,\lambda_2) - \sqrt{-1}n)) \\
&= \exp(\pi\sqrt{-1}E(\lambda_1,\lambda_2)), \quad n \in \mathbb{Z}.
\end{aligned}$$

Therefore $e_\lambda^0(z)$ is equivalent to $\alpha(\lambda)\exp(\pi H(z,\lambda)+\frac{\pi}{2}H(\lambda,\lambda))$, and hence the proof of theorem is completed.

A theta-function for Γ is a holomorphic function $\theta \in H^0(\mathbb{C}^n, \mathcal{O})$ such that there exist linear polynomials $l_\lambda(z)$ which define multipliers $el_\lambda(z)$ satisfying

$$\theta(z+\lambda) = \theta(z)el_\lambda(z)$$

for all $z \in \mathbb{C}^n$.

Definition 2.5 Let $X = \mathbb{C}^n/\Gamma$ be a toroidal group. A multipliers is said to be a theta factor or linearizable if it is given by exponential system of linear polynomials. A line bundle L on X is a theta bundle or linearizable, if it can be given by a theta factor.

For an additive group $\mathcal{F}$, we denote by $C^p(\Gamma,\mathcal{F})$ the group of p-cochains with values in $\mathcal{F}$, $Z^p(\Gamma,\mathcal{F})$ the group of p-cocycles with values in $\mathcal{F}$ and $B^p(\Gamma,\mathcal{F})$ the group of p-coboundaries with values in $\mathcal{F}$.

The following theorem was first proved by Vogt([9]).

Theorem 2.4 *Let $\mathbb{C}^n/\Gamma$ be a toroidal group of a cohomologically finite type. Then every complex line bundle L on $\mathbb{C}^n/\Gamma$ is a theta bundle.*

Proof By theorem 2.3, we have a theta bundle L_0 on $\mathbb{C}^n/\Gamma$ which is defined by $\alpha'(\lambda)\exp(\pi H(z,\lambda) + H(\lambda,\lambda))$ such that $c_1(L_0) = c_1(L) = E$, where $E = \operatorname{Im} H|\Gamma\times\Gamma$. Put $L_1 := L\otimes {L_0}^{-1}$. Then L_1 is topologically trivial. Let $\exp(2\pi\sqrt{-1}g_\lambda)$ $(g_\lambda \in H^0(\mathbb{C}^n,\mathcal{O}))$ be multipliers for L_1.

So, we see that

$$c_1(L_1)(\lambda_1,\lambda_2) = g_{\lambda_2}(z+\lambda_1) - g_{\lambda_1+\lambda_2}(z) + g_{\lambda_1}(z) \in B^2(\Gamma,\mathbb{Z})$$

for all $\lambda_1,\lambda_2 \in \Gamma$

This means that there exist $\alpha_\lambda \in C^1(\Gamma,\mathbb{Z})$ such that

$$g_{\lambda_2}(z+\lambda_1) - g_{\lambda_1+\lambda_2}(z) + g_{\lambda_1}(z) = \alpha_{\lambda_2} - \alpha_{\lambda_1+\lambda_2} + \alpha_{\lambda_1}.$$

Next, replacing g_λ by $g_\lambda - \alpha_\lambda$, then we get

$$g_{\lambda_2}(z+\lambda_1) - g_{\lambda_1+\lambda_2}(z) + g_{\lambda_1}(z) = 0.$$

Thus, from the above equation, we see that $g_\lambda \in Z^1(\Gamma, \mathcal{H})$, where $\mathcal{H} = H^0(\mathbb{C}^n, \mathcal{O})$.

So, according to our assumption that $\mathbb{C}^n/\Gamma$ is a cohomologically finite type, the map

$$H^1(\mathbb{C}^n/\Gamma, \mathbb{C}) \longrightarrow H^1(\mathbb{C}^n/\Gamma, \mathcal{O}) \quad \text{is surjective}$$

and also the map

$$H^1(\Gamma, \mathbb{C}) \longrightarrow H^1(\Gamma, \mathcal{H}) \quad \text{is surjective.}$$

Then, there exist $c_\lambda \in Z^1(\Gamma, \mathbb{C})$ such that

$$g_\lambda(z) - c_\lambda(z) = h(z+\lambda) - h(z), \quad \text{for some } h \in C^0(\Gamma, \mathcal{H}).$$

From the above equation, we get

$$\exp(2\pi\sqrt{-1}g_\lambda(z)) = \exp(2\pi\sqrt{-1}c_\lambda(z))\exp(h(z+\lambda))\exp(h(z))^{-1}.$$

This implies that $\exp(2\pi\sqrt{-1}c_\lambda(z))$ are the multipliers for L_1. Since the line bundle L_1 is topologically trivial, so $c_\lambda \in \mathrm{Hom}(\Gamma, \mathbb{C})$. Therefore, there exists a $\mathbb{C}$-linear form $\varphi : \mathbb{C}^n \longrightarrow \mathbb{C}$ satisfying

$$\mathrm{Im}\,\varphi \mid \Gamma = \mathrm{Im}\, c_\lambda.$$

So we get

$$\exp 2\pi\sqrt{-1}(c_\lambda - \varphi(\lambda)) = \exp 2\pi\sqrt{-1}(c_\lambda(z))\exp(2\pi\sqrt{-1}(-\varphi(z+\lambda)+\varphi(z))).$$

This then means that $\exp 2\pi\sqrt{-1}(c_\lambda - \varphi(\lambda))$ are also the multipliers for L_1.

On the other hand, we see $c_\lambda - \varphi(\lambda) \in \mathbb{R}$ since $\mathrm{Im}\,(c_\lambda - \varphi(\lambda)) = 0$ on Γ.

Setting $\exp 2\pi\sqrt{-1}(c_\lambda - \varphi(\lambda)) = \psi(\lambda)$ and $\alpha(\lambda) = \psi(\lambda)\alpha'(\lambda)$, since $L_1 := L \otimes {L_0}^{-1}$, then $\alpha(\lambda)\exp(\pi H(z,\lambda) + \frac{\pi}{2}H(\lambda,\lambda))$ are the multipliers for

L. Therefore, it follows from this result that L is represented by linear polynomial, and hence we complete the proof of theorem.

For a proof of main theorem, we need the following notations.

Let $\mathbb{C}^n/\Gamma$ be a toroidal group of type q. After a linear change of coordinates of $\mathbb{C}^n$, we see $\mathbb{C}^n/\Gamma$ has a period matrix of the form $P = [I_n, V]$, where $I_n = [e_1, \ldots, e_n]$ is the $n \times n$ unit matrix and $V = [v_{ij}; 1 \le i \le n, 1 \le j \le q] = [v_1, \ldots, v_q]$ is a $n \times q$ matrix. Put $V_1 = [v_{ij}; 1 \le i, j \le q]$, and $V_2 = [v_{ij}; q+1 \le i \le n, 1 \le j \le q]$. We may assume det $(\operatorname{Im} V_1) \ne 0$. We put $v_i = \sqrt{-1}e_i$ for $q+1 \le i \le n$, and $\beta_i = \operatorname{Im} v_i$ for $1 \le i \le n$. Then $\beta_1, \ldots, \beta_n$ are lineary independent over $\mathbb{C}$. Put

$$z = z_1\beta_1 + \cdots + z_n\beta_n.$$

Then we have $\mathbb{C}_\Gamma = \mathbb{C}\{\beta_1, \cdots, \beta_n\}$.

We have the following(cf. [10])

Lemma 2.1 *Let L be a topologically trivial line bundle on a toroidal group $X = \mathbb{C}^n/\Gamma$ of cohomologically finite type. If there exists $s \in H^0(X, \mathcal{O}(L))$ which is not identically zero, then L is analytically trivial.*

Proof

By Theorem 2.4, L is defined by multipliers

$$\alpha(\lambda)\exp(\pi H(z, \lambda) + \frac{\pi}{2}H(\lambda, \lambda)),$$

where $\operatorname{Im} H|\Gamma \times \Gamma = c_1(L)$.

Since $c_1(L) = 0$, we may assume $H = 0$. Then the holomorphic section $s(z)$ is a holomorphic function on $\mathbb{C}^n$ satisfying

$s(z + \lambda) = \alpha(\lambda)s(z)$, for $z \in \mathbb{C}^n$ and $\lambda \in \Gamma$.

Hence

$|s(z + \lambda)| = |s(z)|$, for $z \in \mathbb{C}^n$ and $\lambda \in \Gamma$.

Then $|s(z)|$ is bounded on the maximal compact subgroup $\mathbb{R}_\Gamma/\Gamma$ of $\mathbb{C}^n/\Gamma$. Hence $s(z)$ is a bounded holomorphic function on $\mathbb{C}_\Gamma$. Then $s(z) =$ constant on $\mathbb{C}_\Gamma$.

Let $\mathbb{C}^n/\Gamma$ has a period matrix of the form $P = [I_n, V]$. Then $s(z)$ is holomorphic function of $z_{q+1}, \cdots, z_n$. Put

$z' = {}^t(z_1, \cdots, z_q) \in \mathbb{C}^q$, $z'' = {}^t(z_{q+1}, \cdots, z_n) \in \mathbb{C}^{n-q}$, $\pi'(z) = z'$ and $\pi''(z) = z''$, for $z \in \mathbb{C}^n$.

For any vectors $\mu_1, \cdots, \mu_r$ in $\mathbb{C}^n$ and matrix $M = [\mu_1, \cdots, \mu_r]$, we write $M' = \pi' M = [\mu'_1, \cdots, \mu'_r]$. Similarly we write M''.
We have a holomorphic function $\hat{s}(z'')$ on $\mathbb{C}^{n-q}$ such that $s(z) = \hat{s}(\pi''(z))$.

Suppose there exists $z^0 \in \mathbb{C}^n$ such that $s(z^0) = 0$. We may assume $z^0 = 0$. Then $s(\lambda) = \hat{s}(\pi''(\lambda)) = 0$, for all $\lambda \in \Gamma$. Put

$V = \alpha + \sqrt{-1}\beta$,

Then

$P'' = [-\beta''\beta'^{-1}, I_{n-q}, \alpha'' - \beta''\beta^{-1}\alpha']$,

where I_{n-q} is the identity matrix of degree $n-q$. Put

$\hat{P}'' = [I_{n-q}, R]$,

where $R = [-\beta''\beta'^{-1}, \alpha'' - \beta''\beta^{-1}\alpha']$.
Since $\mathbb{C}^n/\Gamma$ is toroidal, ${}^t\sigma R \notin {}^t\mathbb{Z}^{2q}$ for any $\sigma \neq 0 \in \mathbb{Z}^{n-q}$.
Hence $\mathbb{Z}\{P''\}$ is dense in $\mathbb{R}^{n-q}$.
Since

$s(\lambda) = \hat{s}(\lambda'') = 0$ for all $\lambda \in \Gamma$, and $\lambda'' \in \mathbb{Z}\{P''\}$,

$\hat{s}(x) = 0$ for all $x \in \mathbb{R}^{n-q}$,

then

$\hat{s}(z'') = 0$ for all $z'' \in \mathbb{C}^{n-q}$. Hence $s(z) = 0$ for all $z \in \mathbb{C}^n$.

But this is a contradiction. Hence the lemma is proved.

Now we return to prove theorem 2.2.

Proof Let f be a meromorphic function on $\mathbb{C}^n$ with $\Gamma \subset G(f)$.

Then, there exist $p, q \in H^0(\mathbb{C}^n, \mathcal{O})$ with $f = p/q$ and $(p, q) = 1$. Moreover there exist linear polynomials $l_\lambda(z)$ such that

$$p(z + \lambda) = el_\lambda(z)p(z) \text{ and } q(z + \lambda) = el_\lambda(z)q(z),$$

for all $z \in \mathbb{C}^n$. By the assumption $NS(X) = 0$. Hence $p(z)$ and $q(z)$ are the holomorphic sections of topologically trivial line bundle on X. Since $p(z)$ and $q(z)$ are not identically zero, these are the sections of analytically trivial line bundle. Since X is toroidal $p(z)$ and $q(z)$ are constant. Hence there are no non-constant meromorphic functions on X and theorem is proved.

3 Existence of non-constant meromorphic functions on X

In this section we shall discuss the example given by Abe and Kopfermann. They gave an example [2] of a non-compact toroidal group which has only constants as meromorphic functions. It is a toroidal group $X = \mathbb{C}^n/\Gamma$, where $\Gamma = \mathbb{Z}\{P\}$ and

$$P = \begin{bmatrix} 1 & 0 & 0 & i & \sqrt{2}i \\ 0 & 1 & 0 & \sqrt{3}i & \sqrt{5}i \\ 0 & 0 & 1 & \sqrt{7}i & i \end{bmatrix}$$

They asserted that all meromorphic functions on X are constant. However, by using the recent results of Umeno [8], we can see that there exist non-constant meromorphic functions on X.

Next, for later use, we shall state the following results proved in [8].

Theorem 3.1 ([8], Theorem 3.1) *Let $X = \mathbb{C}^n/\Gamma$ be a toroidal group of type q, with a period matrix of the form $P = [\lambda_1, \cdots, \lambda_{n+q}] = [I_n, V]$.*

(1) *If $\mathbb{C}^n/\Gamma$ is a quasi-Abelian variety with an ample Riemann form H, then $E := ImH \mid \Gamma \times \Gamma$ satisfies the following conditions:*

$$R1 = {}^tVE_1V + {}^tE_2V - {}^tVE_2 + E_3 = 0$$

$$R2 = \tfrac{\sqrt{-1}}{2}({}^t\bar{V}E_1V + {}^tE_2V - {}^t\bar{V}E_2 + E_3) > 0,$$

where $E = \begin{bmatrix} E_1 & E_2 \\ -{}^tE_2 & E_3 \end{bmatrix}$, $E_1 \in \mathbb{Z}^{n+n}$, *and* $E_3 \in \mathbb{Z}^{q\times q}$.

(2) *Conversely, if we have a $\mathbb{Z}$-valued skew-symmetric matrix $E = [E(\lambda_i, \lambda_j); 1 \le i, j \le n+q] \in \mathbb{Z}^{(n+q)\times(n+q)}$, which satisfies R1 and R2, then $X = \mathbb{C}^n/\Gamma$ is a quasi-Abelian variety with an ample Riemann form H satisfying $ImH \mid \Gamma \times \Gamma = E$.*

The following result is about a period matrix which characterize a quasi-Abelian variety.

Theorem 3.2 ([8],Theorem 3.4) *Let $X = \mathbb{C}^n/\Gamma$ be a toroidal group. Then $X = \mathbb{C}^n/\Gamma$ is a quasi-Abelian variety of type q if and only if there exist a* *basis* $\lambda_1, \cdots, \lambda_{n+q}$ *for* Γ *and a complex basis* $e_1, \cdots, e_n$ *for* $\mathbb{C}^n$ *such that the period matrix*

$$P = [\lambda_1, \cdots, \lambda_{n+q}] = [\Delta(q,n), W],$$

where $\Delta(q,n) := [\delta_1 e_1, \cdots, \delta_q e_q, e_{q+1}, \cdots, e_n] \in \mathbb{Z}^{n+n}$, *with positive integers* $\delta_1|\delta_2|\cdots|\delta_q$ *and* $W = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \in \mathbb{C}^{n\times q}$ *satisfying* $W_1 \in \mathbb{C}^{q\times q}$ *is symmetric and* $\mathrm{Im}W_1 > 0$.

To do our goal, we have only to find E satisfying the conditions R1 and R2 of theorem 3.1. Then, we have the following:

Proposition 3.1 *Let* $\mathbb{C}^3/\Gamma$, *where* $\Gamma = \mathbb{Z}\{P\}$ *be a toroidal group of type 2 with a period matrix of the form* $P = [I_3, V] = \begin{bmatrix} 1 & 0 & 0 & \mathrm{i} & \sqrt{2}\mathrm{i} \\ 0 & 1 & 0 & \sqrt{3}\mathrm{i} & \sqrt{5}\mathrm{i} \\ 0 & 0 & 1 & \sqrt{7}\mathrm{i} & \mathrm{i} \end{bmatrix}$.

Then we get a $\mathbb{Z}$*-valued skew-symmetric form* E *such that satisfies*

$$^tVE_1V + {}^tE_2V - {}^tVE_2 + E_3 = 0 \qquad (1)$$

$$\frac{\sqrt{-1}}{2}({}^t\bar{V}E_1V + {}^tE_2V - {}^t\bar{V}E_2 + E_3) > 0, \qquad (2)$$

Proof We first recall the peroid matrix of the form $P = [I_3, V] = \begin{bmatrix} 1 & 0 & 0 & \mathrm{i} & \sqrt{2}\mathrm{i} \\ 0 & 1 & 0 & \sqrt{3}\mathrm{i} & \sqrt{5}\mathrm{i} \\ 0 & 0 & 1 & \sqrt{7}\mathrm{i} & \mathrm{i} \end{bmatrix}$.

Then, we set E as the following form: $E = \begin{bmatrix} E_1 & E_2 \\ -{}^tE_2 & E_3 \end{bmatrix}$, where

$$E_1 = \begin{bmatrix} 0 & -p & -a \\ p & 0 & -b \\ a & b & 0 \end{bmatrix}, E_2 = \begin{bmatrix} -e & -h \\ -f & -i \\ -g & -j \end{bmatrix}, \text{ and } E_3 = \begin{bmatrix} 0 & -c \\ c & 0 \end{bmatrix}.$$

We note E is a $\mathbb{Z}$-valued skew-symmetric form.

Substituting E into R1 and R2, then we have $R1 = \begin{bmatrix} 0 & r \\ -r & 0 \end{bmatrix}$, where

$$\begin{aligned} r =& (a - \sqrt{14}a + \sqrt{3}b - \sqrt{35}b - c + \sqrt{5}p - \sqrt{6}p) \\ &+ i(-\sqrt{2}e - \sqrt{5}f - g + h + \sqrt{3}i + \sqrt{7}j), \text{ where } a,\ b, \cdots,\ p \in \mathbb{Z} \end{aligned}$$

and $R2 = \begin{bmatrix} \sqrt{7}g & g \\ g & \sqrt{2}g \end{bmatrix}$.

Then, for satisfying the conditions (1) and (2), $a = b = c = p = 0, e = f = i = j = 0$ and $g = h$, where $g > 0$.

Therefore, we get $E = \begin{bmatrix} 0 & 0 & 0 & 0 & -g \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -g & 0 \\ 0 & 0 & g & 0 & 0 \\ g & 0 & 0 & 0 & 0 \end{bmatrix}$, where $g(> 0) \in \mathbb{Z}$ which satisfies the conditions (1) and (2). The proof is completed.

Then, the above proposition implies that $\mathbb{C}^3/\Gamma$ is a quasi-Abelian variety from Theorem 3.1.

After a linear change of coordinates, by setting

$$\lambda_1 = ge'_1 = e_3, \lambda_2 = ge'_2 = e_1, \lambda_3 = e'_3 = e_2, \lambda_4 = v_1, \lambda_5 = v_2,$$

we get an alternating form $[E(\lambda_i, \lambda_j); 1 \leq i, j \leq 5] = \begin{bmatrix} 0 & 0 & -\Delta(g) \\ 0 & 0 & 0 \\ \Delta(g) & 0 & 0 \end{bmatrix}$,
where $\Delta(g) = \mathrm{diag}(g, g)$.

Thus, it follows from the same way that we get the period matrix

$$P' = [\Delta(g), V'] = \begin{bmatrix} g & 0 & 0 & \sqrt{7}gi & gi \\ 0 & g & 0 & gi & \sqrt{2}gi \\ 0 & 0 & 1 & \sqrt{3}i & \sqrt{5}i \end{bmatrix}$$

from the period matrix P, where V' is a represention of V with respect to a new basis e'_1, e'_2, e'_3 for $\mathbb{C}^3$. Then

$$V' = \begin{bmatrix} V'_1 \\ V'_2 \end{bmatrix} = \begin{bmatrix} \sqrt{7}gi & gi \\ gi & \sqrt{2}gi \\ \sqrt{3}i & \sqrt{5}i \end{bmatrix},$$ where $V'_1 \in \mathbb{C}^{2\times 2}$ and $g(> 0) \in \mathbb{Z}$

satisfies that V'_1 is symmetric and $\mathrm{Im}V'_1$ is positive definite.

Hence $\mathbb{C}^3/\Gamma'$, where $\Gamma' = \mathbb{Z}\{P'\}$ is a quasi-Abelian variety of type 2 from the Theorem 3.2.

Then, to make sure the result, we project the period matrix P' to $\mathbb{C}^2$. It suffices to show that the 2-dimensional torus group generated by P'^* is an abelian variety. Here, the period matrix P'^* is of the form

$$\begin{bmatrix} g & 0 & \sqrt{7}gi & gi \\ 0 & g & gi & \sqrt{2}gi \end{bmatrix} = [\Delta(g), Z].$$

Then Z is symmetric and ImZ is positive definite.

Therefore, from the Riemann conditions III [5], $\mathbb{C}^2/\Gamma'^*$, where $\Gamma'^* = \mathbb{Z}\{P'^*\}$ is an abelian variety.

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