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The number of columns $(1, k-2, 1)$ in the intersection array of a distance-regular graph

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1 Definitions

Let $\Gamma = (V\Gamma, E\Gamma)$ be a connected graph with usual distance ∂_Γ.
Let $d = \max\{\partial_\Gamma(x, y) \mid x, y \in V\Gamma\}$ denote the diameter of Γ. For vertex u, let $\Gamma_j(u) := \{x \in V\Gamma \mid \partial_\Gamma(u, x) = j\}$ the set of vertices which are at distance j from u.

For $x, y \in V\Gamma$ with $\partial_\Gamma(x, y) = i$, let

$$\begin{aligned} C(x,y) &:= \Gamma_{i-1}(x) \cap \Gamma_1(y), \\ A(x,y) &:= \Gamma_i(x) \cap \Gamma_1(y) \\ \text{and} \qquad B(x,y) &:= \Gamma_{i+1}(x) \cap \Gamma_1(y). \end{aligned}$$

A graph Γ is said to be *distance-regular* if

$$c_i = |C(x,y)|, \quad a_i = |A(x,y)| \quad \text{and} \quad b_i = |B(x,y)|$$

depend only on $i = \partial_\Gamma(x, y)$ rather than individual vertices.

Then c_i, a_i, b_i are called the *intersection numbers* of Γ and the array

$$\iota(\Gamma) = \left\{ \begin{array}{cccccccc} * & c_1 & c_2 & \cdots & c_j & \cdots & c_{d-1} & c_d \\ a_0 & a_1 & a_2 & \cdots & a_j & \cdots & a_{d-1} & a_d \\ b_0 & b_1 & b_2 & \cdots & b_j & \cdots & b_{d-1} & * \end{array} \right\}$$

is called the *intersection array* of Γ. Let

$$\ell(c, a, b) := |\{i \mid (c_i, a_i, b_i) = (c, a, b)\}|.$$

For example the dodecahedron is a distance-regular graph with

$$\iota(\Gamma) = \left\{ \begin{array}{cccccc} * & 1 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 & 1 & * \end{array} \right\}$$

and

$$\ell(1, 0, 2) = 1, \quad \ell(1, 1, 1) = 2, \quad \ell(2, 0, 1) = 1.$$

2 Bannai-Ito conjecture

The following are well-known basic properties of the intersection numbers.

Lemma 1 *A distance-regular graph* Γ *is a regular graph of valency* $k = b_0$. *And :*
(1) $k = b_0 \geq b_1 \geq \cdots \geq b_{d-1} \geq 1$.
(2) $1 = c_1 \leq c_2 \leq \cdots \leq c_d \leq k$.
(3) $c_i + a_i + b_i = k$ *for all* i.

The reader is referred to [1] and [4] for general theory of distance-regular graphs.

Conjecture. (Bannai-Ito) For fixed integer k with $k \geq 3$ there are only finitely many distance-regular graphs of valency k.

This conjecture is equivalent to construction of a diameter bound for distance-regular graphs in terms of valency k.

Fix an integer k with $k \geq 3$. Let Γ be a distance-regular graph with valency k. From the above lemma there are only finitely many kind of column vectors ${}^t(c, a, b)$ with $c + a + b = k$. To solve the conjecture we have to bound $\ell(c, a, b)$ by a function of k or absolute constant.

Bannai and Ito proved a lot of interesting results concerning this problem by using eigenvalue technique. For example:

Result 2 [2, Bannai-Ito (1987)]

$$\ell(c, a, c) \leq 10k2^k.$$

For the special case of the Bannai-Ito's result, we have the following results.

Result 3 [7, Higashitani-Suzuki (1992)]
If $k \geq 5$, *then*

$$\ell(1, k-2, 1) \leq 46\sqrt{k-3}.$$

Result 4 [9, Hiraki (1996)]

$$\ell(1, k-2, 1) \leq 20.$$

For the small valency we have:

Result 5 [3, Biggs-Boshier and Shawe-Taylar (1986)] [5, Brouwer-Koolen (1999)]

$$\ell(1, 1, 1) \leq 3 \quad \text{and} \quad \ell(1, 2, 1) \leq 1.$$

We recall the sketch of the proof of the above results.

Let Γ be a distance-regular graph with :

$$\left\{\begin{array}{cccccccccccc} * & c_1 & \cdots & c_1 & c_{r+1} & \cdots & c_s & c & \cdots & c & c' & \cdots \\ 0 & a_1 & \cdots & a_1 & a_{r+1} & \cdots & a_s & a & \cdots & a & a' & \cdots \\ k & b_1 & \cdots & b_1 & b_{r+1} & \cdots & b_s & b & \cdots & b & b' & \cdots \end{array}\right\}$$
$$\underbrace{\qquad\quad}_{r} \qquad\qquad\qquad \underbrace{\qquad\quad}_{t}$$

where $r := \ell(c_1, a_1, b_1)$, $t = \ell(c, a, b)$ and $s = \min\{j \mid (c_{j+1}, a_{j+1}, b_{j+1}) = (c, a, b)\}$.

Let $A := A(\Gamma)$ be adjacency matrix of Γ. Let

$$k = \theta_0 > \theta_1 > \theta_2 > \cdots > \theta_d$$

be the eigenvalues of A, and $m(\theta_j)$ denote the multiplicity of θ_j in A.

It is known that the $m(\theta_j)$ is bounded from bellow by a function $f_1(r, k)$ of r and k. (See [13] and [2].) On the other hand by Biggs formula for $m(\theta_1)$ it can be bounded from above by a function $f_2(k, s, t)$ of k, s and t. Next we bound s by a function $f_3(k, r)$ of k and r. Combining these we have

$$f_1(k, r) \leq m(\theta_1) \leq f_2(k, s, t) \leq f_4(k, r, t).$$

For our cases the above inequality gives us an upper bound for t by a function of k or absolute constant.

The bound for s that they used are as follows.

		$k = 5$	$k = 9$	$k = 17$	$k \sim 2^n$
B-I	$2^{k-3} r$	$4r$	$64r$	$2^{14}r$	$2^{2^n}r$
H-S	$(k-3)\, r$	$2r$	$6r$	$14r$	$2^n r$
H	$(r+1)\log_2(k-1)$	$2(r+1)$	$3(r+1)$	$4(r+1)$	$n(r+1)$

In [10] we have the following result.

Lemma 6 *Let Γ be a distance-regular graph of valency $k \geq 3$ with $r := \ell(1, a_1, b_1) \geq 2$. Then*

$$\max\{i \mid c_i = 1\} \leq 2r + 2.$$

□

Suppose $t := \ell(1, k-2, 1) \geq 2$. Then we have $s + t = \max\{i \mid c_i = 1\} \leq 2r + 2$ from the above lemma. Apply this new bound for s to Bannai-Ito's method. We have the following improvement of the bound for t

Theorem 7 *Let Γ be a distance-regular graph of valency $k \geq 5$ and $t := \ell(1, k-2, 1)$. Then the following hold.*
(1) If $k = 5$, then $t \leq 14$.
(2) If $k \geq 6$, then $t \leq 7$.
(3) If $k \geq 9$, then $t \leq 6$.
(4) If $k \geq 12$, then $t \leq 4$.
(5) If $k \geq 20$, then $t \leq 3$.
(6) If $k \geq 30$, then $t \leq 2$.
(7) If $k \geq 58$, then $t \leq 1$.

The detailed proof of the theorem will be found in [11]. □

3 Sketch of the proof of Lemma 6.

Let Γ be a distance-regular graph. A subgraph Δ is called *strongly closed* if

$$C(x,y) \cup A(x,y) \subseteq \Delta \quad \text{for any} \quad x, y \in \Delta.$$

Then $c_i(\Delta) = c_i(\Gamma)$ and $a_i(\Delta) = a_i(\Gamma)$. If Δ is regular of valency k_Δ, then $b_i(\Delta) = k_\Delta - c_i(\Delta) - a_i(\Delta)$. Hence a strongly closed subgraph of a distance-regular graph is also distance-regular if it is regular.

There are examples of a non-regular strongly closed subgraph of a distance-regular graph. However Suzuki proved the following result.

Result 8 [12, Suzuki (1995)]
Let Γ be a distance-regular graph with $r := \ell(c_1, a_1, b_1)$. A strongly closed subgraph of diameter q with $r + 1 \leq q$ is one of the following graph :
(i) *A distance-regular graph. In particular, $b_{q-1} > b_q$.*
(ii) *A distance-biregular graph.*
(iii) *The 3-subdivision graph of a complete graph, or of a Moore graph. In particular, $r \in \{3, 6\}$.*

On the other hand we have the following existence condition for strongly closed subgraphs of a distance-regular graph.

Result 9 [10, Hiraki (2001)]
Let Γ be a distance-regular graph of diameter d and $r := \ell(c_1, a_1, b_1)$. Let q be an integer with $r+1 \leq q \leq d-1$. If $b_{q-1} > b_q$ and $c_{q+r} = 1$, then there exists a sequence of strongly closed subgraphs

$$\Delta_{r+1} \subseteq \Delta_{r+2} \subseteq \cdots \subseteq \Delta_q,$$

where the diameter of Δ_i is i. Moreover Δ_q is distance-regular.

Proof of Lemma 6. Suppose $c_{2r+3} = 1$ to derive a contradiction.

Since $b_r > b_{r+1}$ and $c_{2r+1} = 1$, there exists a strongly closed subgraph Δ of diameter $r+1$ with

$$\iota(\Delta) = \left\{ \begin{array}{ccccc} * & 1 & \cdots & 1 & 1 \\ 0 & a_1 & \cdots & a_1 & a_{r+1} \\ k' & b'_1 & \cdots & b'_1 & * \end{array} \right\}.$$

Since $r \geq 2$, a graph Δ has to be an ordinary polygon. (See [4, Theorem 6.8.1].) Hence we have $(a_1, a_{r+1}) = (0, 1)$ and thus $\ell(1, 1, k-2) \in \{1, 2\}$ from [8].

Suppose $\ell(1, 1, k-2) = 1$. Then $b_{r+1} > b_{r+2}$ and $c_{2r+2} = 1$. There exists a strongly closed subgraph Δ' of diameter $r+2$ with

$$\iota(\Delta') = \left\{ \begin{array}{cccccc} * & 1 & \cdots & 1 & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 & a \\ a+1 & a & \cdots & a & a-1 & * \end{array} \right\}.$$

But no such Δ' exists from [6]. We have a contradiction.

Suppose $\ell(1, 1, k-2) = 2$. Then $b_{r+1} = b_{r+2} > b_{r+3}$ and $c_{2r+3} = 1$. Thus there exists a sequence $\Delta_{r+1} \subsetneq \Delta_{r+2} \subsetneq \Delta_{r+3}$ of strongly closed subgraphs. Since $b_{r+1} = b_{r+2}$ and $a_{r+1} = a_{r+2} = 1$, a strongly closed subgraph Δ_{r+2} of diameter $r+2$ have to be the 3-subdivision graph of a complete graph, or of a Moore graph from Result 8. In particular, we have $r \in \{3, 6\}$. By counting the number of the 3-subdivision graphs of a complete graph, or of a Moore graph in Δ_{r+3}, we have $a+2$ is a divisor of 216. So there are only finitely many possible parameters for (a, r). Each of them are ruled out by integrality condition of $m(\theta_1)$.

The lemma is proved. □

参考文献

[1] E. Bannai and T. Ito, *Algebraic Combinatorics I;* Benjamin-Cummings, California 1984.

[2] E. Bannai and T. Ito, On distance-regular graph with fixed valency, Graphs and Combin. **3,** pp 95–109, (1987).

[3] N. L. Biggs, A. G. Boshier and J. Shawe-Taylor, Cubic distance-regular graphs. J. Combin. Theory Ser. B. **44,** pp 147–153, (1986).

[4] A. E. Brouwer, A. M. Cohen and A. Neumaier, *Distance-Regular Graphs,* Springer Verlag, Berlin, Heidelberg, 1989.

[5] A. E. Brouwer and J. H. Koolen, The distance-regular graphs of valency four, J. Alg. Combin. **10** pp 5–24, (1999).

[6] Y. L. Chen and A. Hiraki, On the non-existence of certain distance-regular graphs, *Kyushu J. Math.* **53** pp 1–15, (1999).

[7] N. Higashitani and H. Suzuki, Bounding the number of columns $(1, k-2, 1)$ in the intersection array of a distance-regular graph. Math. Japonica **37,** pp 487–494, (1992).

[8] A. Hiraki, An improvement of the Boshier-Nomura bound, *J. of Comb. Theory, Ser. B* **61** pp1–4, (1994).

[9] A. Hiraki, A constant bound on the number of columns $(1, k-2, 1)$ in the intersection array of distance-regular graphs, Graphs and Combin. **12** pp 23–37, (1996).

[10] A. Hiraki, A distance-regular graph with strongly closed subgraphs, J. of Alg combin. **14** pp 127–131, (2001).

[11] A. Hiraki, The number of columns $(1, k-2, 1)$ in the intersection array of a distance-regular graph, preprint.

[12] H. Suzuki, On strongly closed subgraphs of highly regular graphs, *Europ. J. Combin.* **16** pp197–220, (1995).

[13] P. Terwilliger, Eigenvalue multiplicities of highly symmetric graphs. Discreat Math. **41** pp 295–302, (1989).