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# Shadows of Odd Unimodular Lattices

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This is an exposition of our recent joint paper with G. Nebe: G. Nebe and B. Venkov, “Unimodular lattices with long shadow.” The paper is submitted to Journal of Number Theory. Full text can be found on the web page of G. Nebe.

After Elkies and Gaultier, we study odd unimodular lattice  $\Lambda$  in  $\mathbf{R}^n$ , whose all characteristic vectors  $\gamma$  have big norm  $(\gamma, \gamma) \geq n - 16$ , where  $n = \dim \Lambda$ . We say that such a lattice is an  $(n - 16)$ -lattice. By shadow theory, this condition is equivalent to the fact that the theta series of  $\Lambda$ , which is a polynomial in two standard generators  $\theta_3$  and  $\Delta_8$ , is in fact a sum of three monomials

$$\theta_\Lambda = \theta_3^n + A\theta_3^{n-8}\Delta_8 + B\theta_3\Delta_8^2$$

with two constants  $A$  and  $B$ . If  $\Lambda$  has no elements of norm 1, then  $A = -2n$ . If moreover  $\min \Lambda \geq 3$ , i.e.  $\Lambda$  contains no roots, then  $B$  is also fixed as a function of  $n$ . That fixes  $\theta_\Lambda$  and gives, for example, for the number of elements of norm 3 and 4

$$n_3 = \frac{4}{3}n(n^2 - 69n + 1208),$$

$$n_4 = 2n(n^3 - 94n^2 + 2783n - 24425).$$

Our main result is that such  $(n - 16)$ -lattices without roots can exist only for  $n \leq 46$ . (Previous bound by Gaultier [Gau] for  $(n - 16)$ -lattices, possibly with roots, was  $n \leq 2907$ ). Our bound  $n = 46$  is achieved, because the lattice  $\Lambda_0 = O_{23} \oplus O_{23}$ , where  $O_{23}$  is the shorter Leech lattice, satisfies our conditions.

We also prove the uniqueness of  $\Lambda_0$  for  $n = 46$ , and nonexistence of  $(n - 16)$ -lattice without roots for dimensions  $n = 44$  and  $45$ . We also were able to construct examples of  $(n - 16)$ -lattices without roots for  $n \leq 35$ . It remains unknown, what happens for dimensions  $36 \leq n \leq 44$ .

To prove these results we consider theta series of  $\Lambda$  with harmonic coefficients. For an odd unimodular lattice  $\Lambda$  and a harmonic polynomial  $P$  on  $\mathbb{R}^n$  of degree  $k \equiv 0 \pmod{2}$ , the theta series for  $(\Lambda, P)$  is

$$\theta_{\Lambda, P} = \sum_{\lambda \in \Lambda} P(\lambda) q^{(\lambda, \lambda)}$$

where  $q = e^{\pi iz}$ ,  $\text{Im } z > 0$ . It is a modular form of weight  $n/2 + k$  for the theta group with some character which depend on  $k \pmod{4}$ . It follows that  $\theta_{\Lambda, P}$  is a polynomial in  $\theta_3$  and  $\Delta_8$  if  $k \equiv 0 \pmod{4}$  or is of the form  $\Phi \cdot (\text{polynomial in } \theta_3, \Delta_8)$ , if  $k \equiv 2 \pmod{4}$ . Here

$$\Phi(z) = \theta_2(z)^4 - \theta_3(z)^4.$$

As for usual theta series ( $P = 1$ ), there is an analogue of the shadow theory. If for a modular form  $\phi$  of weight  $m$  we define the shadow transformation by

$$S(\phi)(z) = \left( \sqrt{\frac{i}{z}} \right)^{2m} \phi\left(-\frac{1}{z} + 1\right),$$

then

$$S(\theta_{\Lambda, P}) = (-1)^{k/2} \sum_{\lambda \in S(\Lambda)} P(\lambda) q^{(\lambda, \lambda)},$$

where  $S(\Lambda)$  is the shadow of  $\Lambda$ :  $S(\Lambda) = \Lambda_0^* - \Lambda$ , where  $\Lambda_0$  is the even part of  $\Lambda$ . If our odd lattice  $\Lambda$  has a long shadow, then  $S(\theta_{\Lambda, P})$  starts with a big power of  $q$  and that gives extra information on the theta series with harmonic coefficients. For an  $(n - 16)$ -lattice  $\Lambda$  without roots and for  $k = 2$  that gives

$$\theta_{\Lambda, P_2} = c\Phi\theta_3^{n-16}\Delta_8^2 = 0$$

so all layers of  $\Lambda$  and  $S(\Lambda)$  form spherical 3-designs. From harmonic polynomials of degree  $k = 4$  and  $k = 6$  we get formulas

$$\sum_{v \in \Lambda_4} (v, \alpha)^4 - 2(n - 28) \sum_{u \in \Lambda_3} (u, \alpha)^4 = 24(n - 41)(n - 46)(\alpha, \alpha)^2,$$

$$\sum_{v \in \Lambda_4} (v, \alpha)^6 - 2(n - 40) \sum_{u \in \Lambda_3} (u, \alpha)^6 = 30(\alpha, \alpha) \sum_{u \in \Lambda_3} (u, \alpha)^4 - 240(n - 37)(\alpha, \alpha)^3$$

here  $\alpha \in \mathbf{R}^n$  is an arbitrary element of  $\mathbf{R}^n$ . These formulas permit to find an element  $v_0 \in \Lambda_4$  of norm 4, such that  $|\{u \in \Lambda_3 \mid (u, v_0) = 2\}|$  is big. Considering neighboring lattice to  $\Lambda$  with respect to such a  $v_0$ , we get an  $(n - 1)$ -dimensional unimodular lattice with the root system  $kA_1$  with big  $k$ . For  $n = 46$  this  $k$  happens to be 23. Projecting on this 23-dimensional subspace we get an  $O_{23}$  and it follows that  $\Lambda = O_{23} \oplus O_{23}$ . Similarly one proves impossibility of  $n = 44$  and 45.

## References

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