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Minimax Theorems of Convexlike Functions

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1 Introduction

In this study we consider fuzzy numbers with bounded supports due to [3] and we treat some type of fuzzy optimization problems, which arise from linear optimization problems and are analyzed under assumptions of the fuzzy goal and fuzzy constraints of decision makers. [5] gives an existence criterion for optimal solutions of the fuzzy optimization problems. In Section 2 the existence of optimal solutions means that there exists at least one solution for systems of inequalities concerning concave functions by applying Ky Fan's theorem. In Section 3 we show an extension of Ky Fan's theorem, in which functions are not convex but quasiconvex. In proving the extension we apply fixed point theorems for set-valued mappings. In Section 4 we deal with definitions of convexlike or concavelike functions in the similar way to Chapter 6 in [7] as well as we get minimax theorems under conditions that functions of two variables are lower semi-continuous and quasiconvex in one variable and concavelike in the other.

2 Existence Criterion

Let us denote by $\mathbf{R}$ the set of real numbers and $I=[0,1]$. In [3] the set of fuzzy numbers is characterized by membership functions as follows:

Definition 1 *Let $\mathcal{F}(\mathbf{R})$ be the set of fuzzy numbers $u:\mathbf{R}\to I$ satisfying the following conditions* (i) - (iii) (*see* [3]):

(i) $u(\cdot)$ *is upper semi-continuous on* $\mathbf{R}$;

(ii) *the* $\alpha-$*cut set* $L_\alpha(u)=\{y\in\mathbf{R}:u(y)\geq\alpha\}$ *is bounded for* $\alpha>0$ *and* $L_0(u)=\overline{\cup_{0<\alpha\leq 1}L_\alpha(u)}$ *is bounded ;*

(iii) $u(\cdot)$ *is fuzzy convex, i.e.,*

$$u(\lambda y_1+(1-\lambda)y_2)\geq\min[u(y_1),u(y_2)]$$

for $y_i\in\mathbf{R}, i=1,2$ *and* $\lambda\in\mathbf{R}$ *with* $0\leq\lambda\leq 1$;

(iv) *there exists one and only one* $m\in\mathbf{R}$ *such that* $u(m)=1$.

The $\alpha-$cut set $L_\alpha(u)$ is compact in $\mathbf{R}$ for each $\alpha\in I$ from the above Conditions (i) and (ii), since (i) means that $L_\alpha(u)$ is closed for $\alpha\in I$.

Remark 1 *Under the above Conditions* (ii) *and* (iv) *the following statements* (a)-(d) *concerning the the function* $u : \mathbf{R} \to I$ *are equivalent each other:*

(a) $u(\cdot)$ *is fuzzy convex;*

(b) $L_\alpha(u)$ *is convex for any* $\alpha \in I$;

(c) $u(\cdot)$ *is non-decreasing on* $(-\infty, m]$ *and that* $u(\cdot)$ *is non-increasing on* $[m, \infty)$;

(d) $L_\alpha(u) \subset L_\beta(u)$ *for* $\alpha > \beta$.

From (a) *it is clear that* (b) *holds. If we suppose that* (a) *doesn't hold but* (b) *hold, this leads to a contradiction. It can be seen that* (c) *leads to* (d) *and the converse holds. Suppose that for any* $m_1 \in \mathbf{R}$ *with* $m_1 > m$ *there exist* $y_1 < y_2 \leq m_1$ *such that* $u(y_1) > u(y_2)$ *under Condition* (ii) *and* (a). *Then it leads to a contradiction. From* (c), *it follows that* (a) *holds.*

In the following definition we give the quasiconvexity of functions.

Definition 2 *Let* C *be a convex set in a linear space and* f *a mapping from* C *to* $\mathbf{R}$. *It is said that* f *is quasiconcave if* $f(\lambda y_1 + (1-\lambda)y_2) \geq \min[f(y_1), f(y_2)]$ *for* $y_i \in C, i = 1, 2$ *and* $0 \leq \lambda \leq 1$. *It is said that* f *is quasiconvex if*

$$f(\lambda y_1 + (1-\lambda)y_2) \leq \max[f(y_1), f(y_2)]$$

for $y_i \in C, i = 1, 2$ *and* $0 \leq \lambda \leq 1$.

Remark 2 *In the same way as in Remark 2.1 it is easily seen that* $f : C \to \mathbf{R}$ *is quasiconvex if and only if the lower level set* $L(f;\gamma) = \{x \in C : f(x) \leq \gamma\}$ *is convex for any* $\gamma \in \mathbf{R}$.

Next we consider the following linear optimization problem (e.g. [4]):

$$a_0^T x \preceq b_0 \quad \text{subject to} \quad a_i^T x \preceq b_i, \tag{2.1}$$

$$i = 1, 2, \cdots, m, \quad x \geq 0, \tag{2.2}$$

where the symbol " $\preceq$ " denotes a relaxed or fuzzy version of the ordinary inequality " $\leq$ ". The first fuzzy inequality (fuzzy goal) means that " the objective function $a_0^T x$ should be essentially smaller than or equal to an aspiration level $b_0 \in \mathbf{R}$ of the decision maker (DM)" and the second (fuzzy constraints of DM) means that " the constraints $a_i^T x$ should be essentially smaller than or equal to $b_i \in \mathbf{R}, i = 1, \cdots, m$". Membership functions $u_i \in \mathcal{F}(\mathbf{R})$, $i = 0, 1, \cdots, m$, and it follows that $u_i(y)$ is non-decreasing in $y \in [C_i, b_i]$, non-increasing in $y \in [b_i, D_i]$ and $u_i(y) \equiv 0$ elsewhere. Here $C_i \leq b_i \leq D_i$ are constants. Let u_i be concave on the set $[C_i, D_i]$. Put $S_i = \{x \in \mathbf{R}^n : C_i \leq a_i^T x \leq D_i\}$ and $S = \cap_{i=0}^n S_i$.

Then, in order to solve the above problem, we have the following optimization problem:

$$\text{maximize } u(x), \tag{2.3}$$

$$\text{where} \quad u(x) = \min_{0 \leq i \leq m} [u_i(a_i^T x)]. \tag{2.4}$$

In [5] we showed the existence criterion for optimal solutions of fuzzy optimization problems as follows:

Theorem 1 *Let* $u_i(\cdot) \in \mathcal{F}$ *for* $i = 0, 1, \cdots, m$. *The following statements* (i) *and* (ii) *hold;*

(i) *Let* $\mu_0 = \max_x \min_i u_i(a_i^T x)$. *Then we have*

$$\begin{aligned} \mu_0 &= \max\{0 < \alpha \leq 1 : \cap_{i=0}^m L_\alpha(u_i) \neq \emptyset\} \\ &= \sup\{0 < \alpha \leq 1 : \cap_{i=0}^m L_\alpha(u_i) \neq \emptyset\}. \end{aligned}$$

(ii) *We have at least one optimal solution* x^* *for* ((2.3),(2.4)), *if and only if there exists an* $\alpha_0 > 0$ *such that*

$$\cap_{i=0}^{m} L_{\alpha_0}(u_i) \neq \emptyset.$$

The above condition (ii) can be reduced to another type of condition by applying Ky Fan's theorem in [2] as follows:

Theorem K *Let* C *be a compact and convex set in a topological linear space. Suppose that functions* $f_i : C \to \mathbf{R}, i = 1, 2, \cdots, n$, *are lower semi-continous and convex. Let* $d \in \mathbf{R}$. *Then the following* (i) *and* (ii) *are equivalent each other:*

(i) *There exists an* $x_0 \in C$ *such that*

$$f_i(x_0) \leq d$$

for $i = 1, 2, \cdots, n$;

(ii) *for* $c = (c_1, \cdots, c_n)$ *such that* $c_i \geq 0, i = 1, 2, \cdots, n$, *and* $\sum_{i=1}^{n} c_i = 1$, *there exists a* $y_c \in C$ *satisfying*

$$\sum_{i=1}^{n} c_i f_i(y_c) \leq d.$$

From the above theorem, Problem ((2.3),(2.4)) has an optimal solution x^* if and only if there exist $0 < \alpha_0 \leq 1$ and x_0 such that

$$u_i(a_i^T x_0) \geq \alpha_0$$

for $i = 0, 1, \cdots, m$.

Theorem 2 *Let* $S = \cap_{i=0}^{n} S_i$ *be non-empty and* $u_i(\cdot)$ *concave on* $[C_i, D_i]$ *for* $i = 0, 1, \cdots, m$. *Then Problem* ((2.3),(2.4)) *has an optimal solution* x^*, *if and only if for some* α_0 *with* $0 < \alpha_0 \leq 1$ *and* $c = (c_0, \cdots, c_m) \in \mathbf{R}^{m+1}$ *with* $c_i \geq 0, i = 0, 1, \cdots, m$, *there exists a* $y_c \in S$ *such that*

$$\sum_{i=0}^{m} c_i u_i(a_i^T y_c) \geq \alpha_0.$$

3 Quasiconvex Functions

In this section we suppose the quasiconvexity of membership functions and we show an extension of Ky Fan's theorem by applying the following lemma.

Lemma 1 *Let* C *be a compact and convex set in a topological linear space* E. *Suppose that a set* $A \subset C \times C$ *satisfies the following conditions* (a) - (c):

(a) *The set* $\{x \in C : (x, y) \in A\}$ *is closed for any* $y \in C$;

(b) *the set* $\{y \in C : (x, y) \notin A\}$ *is convex for any* $x \in C$;

(c) *for* $x \in C$, *the point* $(x, x) \in A$.

Then there exists some $x_0 \in C$ *such that* $\{x_0\} \times C \subset A$.

The above Lemma can be proved by applying the following type of fixed points theorem for a class of set-valued mappings (e.g., Theorem 10.3.6 in [1]).

Theorem 3 *Let* E *be a topological linear space and* C *a non-empty, compact and convex set in* E. *Let* T *be a mapping from* C *to the set of all subsets of* C. *Assume that the image* $T(x)$ *is non-empty and convex for* *each $x \in C$. If for each $y \in C$, the inverse $T^{-1}(y) = \{x \in C : T(x) \ni y\}$ is open, then T has a fixed point in C, i.e, there exists an $x_0 \in C$ such that $x_0 \in T(x_0)$.*

Proof of Lemma 1

Suppose that for any $x \in C$ there exists a $y \in C$ such that $(x,y) \notin A$. Denote a set-valued mapping T from C to the set of all subsets of C by $T(x) = \{y \in C : (x,y) \notin A\}$. The image $T(x) \subset C$ is non-empty and convex from Condition (b) for any $x \in C$. From Condition (a) the set $T^{-1}(y) = \{x \in C : (x,y) \notin A\}$ is open set in E. Then, by applying Theorem 3, T has a fixed point $x_0 \in C$, i.e., $x_0 \in T(x_0)$. It follows that $(x_0, x_0) \notin A$, which contradicts Condition (c). Thus the conclusion holds.

Q.E.D.

By utilizing the above lemma we think that the following results of an extension of Theorem K can ce shown as the below outline of proof.

Extension of Theorem K(ETK)

- *Let $f_i : C \to \mathbf{R}$ for $i = 1, \cdots, n$, be lower semi-continuous and quasiconvex, where C is a compact and convex set in a topological linear space E and let $d \in \mathbf{R}$. Then the following* (i) *and* (ii) *are equivalent each other:*

  (i) *There exists an $x_0 \in C$ such that*

  $$f_i(x_0) \le d$$

  for $i = 1, 2, \cdots, n$;

  (ii) *for $c = (c_1, \cdots, c_n)$ such that $c_i \ge 0, i = 1, 2, \cdots, n$, and $\sum_{i=1}^{n} c_i = 1$, there exists a $y_c \in C$ such that*

  $$\sum_{i=1}^{n} c_i f_i(y_c) \le d.$$

In the similar way to the discussion of Chapter 6 in [7], we expect that we can prove the above extension.

4 Extensions of Minimax Theorems

[7] gives definitions of convexlike or concavelike functions, which play an important role in proving an extension of minimax theorems under that ETK holds.

Definition 3 *Let C, D be two sets and F a mapping from $C \times D$ to $\mathbf{R}$. It is said that F is concavelike on D for $x \in C$ if for each $y_1, y_2 \in D$ and $0 < \lambda < 1$, there exists an $y_0 \in D$ such that $F(x, y_0) \ge \lambda F(x, y_1) + (1-\lambda)F(x, y_2)$. It is said that F is convexlike on C for $y \in D$ if for each $x_1, x_2 \in C$ and $0 < \lambda < 1$, there exists an $x_0 \in C$ such that $F(x_0, y) \le \lambda F(x_1, y) + (1-\lambda)F(x_2, y)$.*

In what follows we show an extension of minimax theorems concerning concavelike functions.

Extension of Minimax Theorems (EMT)

- *Let C be a convex and compact set in a topological linear space and D an arbitrary non-empty set. A function $F : C \times D \to \mathbf{R}$ satisfies the following conditions* (i) *and* (ii).

(i) $F(\cdot, y)$ *is lower semi-continuous and quasiconvex on* C *for* $y \in D$;

(ii) $F(x, \cdot)$ *is concavelike on* D *for* $x \in C$.

Then it follows that

$$\sup_{y \in D} \min_{x \in C} F(x,y) = \min_{x \in C} \sup_{y \in D} F(x,y).$$

Proof. From (i) and the compactness of C there exists $\min_{x \in C} F(x,y)$. Let $c = \sup_{y \in D} \min_{x \in C} F(x,y) < +\infty$. For any $x \in C$, $\{y_1, y_2, \cdots, y_n\} \subset D$ and $\{\lambda_i \geq 0 : \sum_{i=1}^n \lambda_i = 1\}$, Condition (ii) means that there exists a $y_0 \in D$ such that $\sum_{i=1}^n \lambda_i F(x, y_i) \leq F(x, y_0)$. From (i) there exists an $x_0 \in C$ such that $F(x_0, y_0) = \min_x F(x, y_0) \leq c$ and also we have $\sum_{i=1}^n \lambda_i F(x, y_i) \leq c$ for any $x \in C$. By Condition (i) and ETK, there exists an $x_1 \in C$ such that $F(x_1, y_i) \leq c$ for any i. Then we get $\cap_{i=1}^n \{x \in C : F(x, y_i) \leq c\} \neq \emptyset$. From the compactness of C, we have $\cap_{y \in D} \{x \in C : F(x,y) \leq c\} \neq \emptyset$, which means that there exists an $x_2 \in C$ and any $y \in D$ such that $F(x_2, y) \leq c$, or $\min_x \sup_y F(x,y) \leq \sup_y \min_x F(x,y)$. Since $F(x,y) \geq \min_x F(x,y)$ for $y \in D$, we have $\sup_y F(x,y) \geq \sup_y \min_x F(x,y)$ and also $\min_x \sup_y F(x,y) \geq \sup_y \min_x F(x,y)$. Therefore $\sup_y \min_x F(x,y) = \min_x \sup_y F(x,y)$.
If $\sup_{y \in D} \min_{x \in C} F(x,y) = \infty$, it can be seen that the conclusion holds.

Q.E.D.

The above theorem is an extension of Sion's minimax theorem and Tuy's one. In the following remark an example illustrates EMT.

Remark 3 (a) *In* [6] *Sion assumes that* F *is upper semi-continuous and quasiconcave on* D *under the condition that* D *is compact, in addition to the conditions of EMT. He gets the conclusion that*

$$\min_{x \in C} \max_{y \in D} F(x,y) = \max_{y \in D} \min_{x \in C} F(x,y).$$

Thus EMT is an extension of Sion's theorem.

(b) *Tuy* [8] *assumes that* C *and* D *are convex. He shows that the conclusion*

$$\inf_{x \in C} \sup_{y \in D} F(x,y) = \sup_{y \in D} \inf_{x \in C} F(x,y)$$

under the condition that F *is upper semi-continuous in* y *in addition to conditions of EMT.*

(c) *Let* $F(x,y) = f(x)g(y)$ *for* $(x,y) \in [-n,n] \times (-1,1)$, *where* $n \geq 1$ *is integer,* f *denotes the largest integer which is less than* $|x|$. *Here*

$$g(y) = y^2 + |y \sin \frac{\pi}{2y}|,$$

where $y \in (-1,1)$. *Then function* F *satisfies Conditions* (i) *and* (ii) *of EMT. Since* $\min_x F(x,y) = 0$ *and* $\sup_y F(x,y) = 2f(x)$, *It follows that the conclusion of EMT holds.*

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