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A canonical random variable for the q-deformed moments-cumulants formula

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1. Introduction

The cumulants in the usual probability theory are given by the logarithm of Fourier transformation of a probability density function and linearize the usual convolution. They are some invariant for probability distributions and the several important distributions, indeed, are explicitly characterized by them. It is known as the so-called moments-cumulants formula that the nth moment μ_n can be given by the cumulants α_i $(1 \leq i \leq n)$ that

$$\mu_n = \sum_{\substack{k_1,k_2,\ldots,k_n \geq 0 \\ k_1+2k_2+\cdots+nk_n=n}} n! \frac{\left(\frac{\alpha_1}{1!}\right)^{k_1} \left(\frac{\alpha_2}{2!}\right)^{k_2} \cdots \left(\frac{\alpha_n}{n!}\right)^{k_n}}{k_1! k_2! \cdots k_n!},$$

which can be written in terms of the set partitions as

$$\mu_n = \sum_{\substack{\pi \in \mathcal{P}(\{1,2,\ldots,n\}) \\ \pi=\{B_1,B_2,\ldots,B_k\}}} \prod_{i=1}^{k} \alpha_{|B_i|} \quad (n \geq 1), \tag{U}$$

where $\mathcal{P}(\{1,2,...,n\})$ is the set of all the partitions of the ordered set $\{1,2,...,n\}$.

In the free probability theory, Voiculescu invented the R-transform in [Vo] as the free analogue of the cumulants, which linearizes the free additive convolution. His canonical random variable is given of the

$$T = \ell^* + \sum_{i=1}^{\infty} \alpha_i \ell^{i-1}$$

on the full Fock space $\mathcal{F}_0(\mathcal{H})$, where ℓ is the creation operator and ℓ^* is its adjoint, the annihilation, operator. Combinatorial descriptions of the free convolution and the R-transform have been deeply studied by Nica and Speicher in, for instance, [Ni3], [Sp1] and [Sp2]. Namely Speicher has given the free analogue of the moments-cumulants formula in [Sp2] using the noncrossing partitions (the notion was first introduced in [Kr]) that

$$\mu_n = \sum_{\substack{\pi \in \mathcal{NC}(\{1,2,\ldots,n\}) \\ \pi=\{B_1,B_2,\ldots,B_k\}}} \prod_{i=1}^{k} \alpha_{|B_i|} \quad (n \geq 1), \tag{F}$$

which is the same as of the usual formula (U) but the partitions should be restricted to noncrossing ones, that is, $\mathcal{NC}(\{1,2,\ldots,n\})$ is the set of the noncrossing partitions of the ordered set $\{1,2,\ldots,n\}$.

Furthermore, Nica has found in [Ni1] a nice q-analogue of the cumulant generating function $R_q(z)$ which takes Voiculescu's R-transform for the free convolution in case of $q = 0$ and it corresponds to a relative of the logarithm of the Fourier transform, if one takes the limit $q \to 1$. He has adopted as the canonical random variable by the operator

$$T_q = a_q + \sum_{i=1}^{\infty} \alpha_i (a_q^*)^{i-1}$$

on the q-Fock space $\mathcal{F}_q(\mathcal{H})$, where a_q and a_q^* is the q-annihilation and q-creation operators, respectively. He has also introduced the set partition statistics, the left-reduced number of crossings $c_o(\pi)$, in order to evaluate the moments of his canonical random variable T_q. The left-reduced number of crossings has the q-counting which interpolates between usual crossing and noncrossing (See also [Ni2]). If we replace α_n by $\dfrac{\alpha_n}{[n-1]_q!}$ in [Ni1, Theorem 1.2] then we have the q-deformed moments-cumulants formula that

$$\mu_n = \sum_{\substack{\pi \in \mathcal{P}(\{1,2,\ldots,n\}) \\ \pi=\{B_1,B_2,\ldots,B_k\}}} q^{c_o(\pi)} \prod_{i=1}^{k} \alpha_{|B_i|} \quad (n \geq 1), \tag{N}$$

which interpolates between the formula for the usual case (U) at $q = 1$ and one for the free case (F) at $q = 0$, exactly. The above q-deformed formula (N) suggests us another q-deformations by replacing the set partition statistics.

On the q-Fock space, the combinatorics of the operator $(a_q + a_q^*)$ with respect to the vacuum expectation have been studied in [BS1, 3] and [BKS], and it was found that the q-Gaussian distribution can be given as the orthogonalizing probability measure for the continuous q-Hermite polynomials. Inspired by this, we have introduced in [SY1] the q-deformed Poisson distribution of the parameter $\lambda > 0$ as the orthogonalizing probability measure for the q-deformed Charlier polynomials, $\{C_n(X)\}_{n=0}^{\infty}$ defined by the following recurrence relations:

$$C_0(X) = 1, \quad C_1(X) = X - \lambda,$$
$$C_{n+1}(X) = (X - (\lambda + [n]_q))\, C_n(X) - \lambda[n]_q C_{n-1}(X) \quad (n \geq 1),$$

where $[n]_q$ is the q-number. In subsequent paper [SY2], we gave the q-deformed Poisson random variable as an operator on the q-Fock space which is a linear combination of a q-number operator, a q-Gaussian random variable, and a scalar operator,

$$a_q^* a_q + \sqrt{\lambda}\left(a_q^* + a_q\right) + \lambda \cdot 1,$$

where a_q and a_q^* is the q-annihilation and q-creation operators, respectively. It has the same form as in [HP] on the symmetric $(q = 1)$ Fock space, and interpolates between their operator and one of Speicher on the full $(q = 0)$ Fock space in [Sp1].

Using the results on the generating function related to the above q-deformed Charlier polynomials in [Bi], it follows that the nth moment of the q-deformed Poisson distribution, $\mu_n(\mathrm{Po}_q(\lambda))$, can be given in the form

$$\mu_n(\mathrm{Po}_q(\lambda)) = \sum_{k=1}^{n} \mathcal{S}_q(n,k)\lambda^k,$$

where $\mathcal{S}_q(n,k)$ is a kind of the q-Stirling number defined by

$$\mathcal{S}_q(n,k) = \sum_{\substack{\pi \in \mathcal{P}(\{1,2,\ldots,n\}) \text{ s.t.} \\ \pi \text{ has precisely} \\ k \text{ blocks}}} q^{rc(\pi)}.$$

Here $rc(\pi)$ denotes the number of restricted crossings for the partition π introduced in [Bi], of which q-counting also interpolates between usual crossings and noncrossings.

It is natural to consider that the q-deformed Poisson distribution of the parameter λ should be characterized as the distribution all of which cumulants are equal to λ, just as for the usual case. Hence, the above result on the moments of the q-deformed Poisson distribution derives the following another q-deformed moments-cumulants formula:

$$\mu_n = \sum_{\substack{\pi\in\mathcal{P}(\{1,2,\ldots,n\}) \\ \pi=\{B_1,B_2,\ldots,B_k\}}} q^{rc(\pi)} \prod_{i=1}^{k} \alpha_{|B_i|} \quad (n \geq 1), \qquad (A)$$

where the difference can be found only on the set partition statistics, that is, the number of restricted crossings $rc(\pi)$ is adopted instead of $c_o(\pi)$. Of course, it also interpolates between formulae for the usual case (U) at $q = 1$ and for the free case (F) at $q = 0$, exactly.

Recently, Anshelevich has defined in [An] a q-convolution related to the above formula (A) for a large class of probability measures (q-infinitely divisible families). He also introduced the combinatorial cumulants as the canonical self-adjoint operators for the q-deformed moments-cumulants formula (A).

In this talk, we are going to give another canonical random variable for the q-deformed moments-cumulants formula (A). Our canonical operator is not self-adjoint but it is a relative of ones for Voiculescu's R-transform and for Nica's R_q-series. Furthermore, it can be regarded as an extension of the q-deformed Poisson random variable on the q-Fock space and more straightforward one for the combinatorial structure of restricted crossings.

2. Set partition statistics

Let S be an ordered set. Then $\pi = \{B_1, B_2, \ldots, B_k\}$ is a *partition* of S, if $B_i \neq \phi$ are ordered and disjoint sets, of which union is S. We shall call $B_i \in \pi$ a *block* of the partition π.

For $n \geq 1$, we denote by $\mathcal{P}(\{1,2,\ldots,n\})$ the set of partitions of the ordered set $\{1,2,\ldots,n\}$. For $\pi \in \mathcal{P}(\{1,2,\ldots,n\})$ and $1 \leq m_1, m_2 \leq n$, we will write $m_1 \overset{\pi}{\sim} m_2$ for the fact that m_1 and m_2 are in the same block of π.

A partition π of $\{1,2,\ldots,n\}$ is said to be *noncrossing* if there is no 4-tuple (m_1, m_2, m_3, m_4) such that $1 \leq m_1 < m_2 < m_3 < m_4 \leq n$ and $m_1 \overset{\pi}{\sim} m_3 \overset{\pi}{\not\sim} m_2 \overset{\pi}{\sim} m_4$. This notion of noncrossing partition was first introduced in [Kr]. We denote the set of noncrossing partitions of the ordered set $\{1,2,\ldots,n\}$ by $\mathcal{NC}(\{1,2,\ldots,n\})$.

The various kinds of set partition statistics have been introduced related to inversions or crossings of partition. Here, we shall recall *the* *number of restricted crossings*, which was investigated by Biane in [Bi] related to the combinatorial theory of continued fractions.

Let $\pi = \{B_1, B_2, ..., B_k\}$ be a partition in $\mathcal{P}(\{1, ..., n\})$. If the block B_j has more than one elements (i.e. $|B_j| = m_j \geq 2$), put $B_j = \{b_{j,1}, b_{j,2}, \ ..., b_{j,m_j}\}$ where $b_{j,1} < b_{j,2} < ... < b_{j,m_j}$, then we make $(m_j - 1)$ connections like bridges $(b_{j,1}, b_{j,2})$, $(b_{j,2}, b_{j,3})$, ..., (b_{j,m_j-1}, b_{j,m_j}), successively. We have, of course, totally $\sum_{j=1}^{k}(|B_j| - 1)$ connections and we shall call them *arcs* of the partition π.

The number of restricted crossings for a partition $\pi \in \mathcal{P}(\{1, ..., n\})$ is the number:

$$rc(\pi) = \# \left\{ (m_1, m_2, m_3, m_4) \,\middle|\, \begin{array}{l} 1 \leq m_1 < m_2 < m_3 < m_4 \leq n, \\ (m_1, m_3) \text{ and } (m_2, m_4) \text{ are arcs} \\ \text{of } \pi \end{array} \right\}.$$

For the partition of the ordered set $\{1, 2, ..., n\}$, we shall introduce the notion of the parenthesis number and the depth of the block of size i, which play an important role in the construction of our canonical random variable.

Let π be a partition in $\mathcal{P}(\{1, 2, ..., n\})$. We shall concentrate our attention upon the blocks of the same size except singletons (blocks of size 1). Suppose π has m blocks of size $i \geq 2$ and we denote them by $\{f_k, ..., e_k\}_{k=1,2,...,m}$ with the first element f_k and the last element e_k of each block of size i because we are interested only with the first and the last elements in each block. Here we have numbered blocks in increasing order of the first elements, that is, $f_1 < f_2 < \cdots < f_m$.

We shall renumber all the first and the last elements of the blocks of size i, $\{f_1, e_1, f_2, e_2, ..., f_m, e_m\}$, in increasing order as $\{p_j\}_{j=1,2,...,2m}$.

The subscript j in the above renumbering p_j is called *the parenthesis number* for the blocks of size i.

For $k \in \{1, 2, ..., n\}$, we shall count the number of the blocks of size i, in which k is contained as an intermediate element or the last element. We call such a number *the depth of the blocks of size i at k* and denote $\mathrm{depth}_i(k)$, that is,

$$\mathrm{depth}_i(k) = \# \left\{ j \,\middle|\, \begin{array}{l} \{f_j, ..., e_j\} \text{ is a block of size } i \text{ such that} \\ k \text{ is in the open-closed interval } (f_j, e_j] \end{array} \right\}.$$

3. A canonical random variable

Let $\mathcal{K}$ be an infinite dimensional separable Hilbert space. We take a complete orthonormal basis $\{\eta_j\}_{j\geq 0}$ in $\mathcal{K}$. We put the index set I as

$$I = \{(i,j) \in \mathbb{N} \times \mathbb{N} \,|\, i \geq 2, j \geq 1\}$$

and consider the infinite tensor product

$$\widetilde{\mathcal{K}} = \bigotimes_{(i,j)\in I} \mathcal{K}_{(i,j)},$$

where each tensor factor $\mathcal{K}_{(i,j)}$ is a copy of $\mathcal{K}$.

For an operator $x \in B(\mathcal{K})$, we denote the operator

$$1_{\mathcal{K}} \otimes \cdots \otimes 1_{\mathcal{K}} \otimes \underbrace{x}_{(i,j)\text{th}} \otimes 1_{\mathcal{K}} \otimes \cdots$$

on the infinite tensor product $\widetilde{\mathcal{K}}$ as $\Gamma_{(i,j)}(x)$ where x acts only on the (i,j)th factor. Let ϕ be the vector state given by $\phi(x) = \langle x\eta_0 | \eta_0 \rangle$. Then we can endow the infinite product state

$$\widetilde{\phi} = \bigotimes_{(i,j)\in I} \phi_{(i,j)}$$

on the infinite tensor product space $\widetilde{\mathcal{K}}$ where $\phi_{(i,j)}$ is a copy of ϕ. We define the shift operator ℓ on $\mathcal{K}$ by

$$\ell\eta_j = \eta_{j+1} \quad (j \geq 0),$$

of which adjoint operator ℓ^* is given by

$$\ell^*\eta_j = \begin{cases} \eta_{j-1} & \text{if } j \geq 1, \\ 0 & \text{if } j = 0. \end{cases}$$

Let $\mathcal{L}$ be an infinite dimensional separable Hilbert space and take a doubly indexed orthonormal system (not necessary complete) $\{\zeta_{j,k}\}$ in $\mathcal{L}$ (i.e. $\langle \zeta_{j_1,k_1} | \zeta_{j_2,k_2} \rangle = \delta_{j_1,j_2}\delta_{k_1,k_2}$). We also consider the infinite tensor product

$$\widetilde{\mathcal{L}} = \bigotimes_{i=2}^{\infty} \mathcal{L}_i,$$

where each tensor factor $\mathcal{L}_i$ is a copy of $\mathcal{L}$.

For an operator $x \in B(\mathcal{L})$, we denote the operator

$$1_{\mathcal{L}} \otimes \cdots \otimes 1_{\mathcal{L}} \otimes \underbrace{x}_{i\text{th}} \otimes 1_{\mathcal{L}} \otimes \cdots$$

on the infinite tensor product $\widetilde{\mathcal{L}}$ as $\Lambda_i(x)$ where x acts only on the ith factor. Let ψ be the unital linear functional given by $\psi(x) = \langle x\zeta_{0,0}|\zeta_a\rangle$ where $\zeta_a = \sum_{j\geq 0} \zeta_{j,0}$.

We shall consider the product linear functional

$$\widetilde{\psi} = \bigotimes_{i=2}^{\infty} \psi_i$$

on the infinite tensor product space $\widetilde{\mathcal{L}}$ where ψ_i is a copy of ψ.

Given vectors $\xi, \eta \in \mathcal{L}$, we denote by $t_{\xi,\eta}$ the rank one operator on $\mathcal{L}$ defined by

$$t_{\xi,\eta}\zeta = \langle\zeta|\eta\rangle\xi, \qquad \zeta \in \mathcal{L}.$$

Here we shall make special operators on $\mathcal{L}$ using the rank one operators.

We put the index sets J_0 and J_1 as

$$J_0 = \{(j,k) \in \mathbb{N} \times \mathbb{N} \,|\, 0 \leq k \leq j\}$$

and

$$J_1 = \{(j,k) \in \mathbb{N} \times \mathbb{N} \,|\, 1 \leq k \leq j\},$$

respectively. For each $(j,k) \in J_0$, we define the rank one operator $r_{j,k}$ on $\mathcal{L}$ as

$$r_{j,k} = t_{\zeta_{j+1,k+1},\zeta_{j,k}},$$

and make the operators

$$r_j = \sum_{k=0}^{j} r_{j,k}, \quad \text{for } j \geq 0.$$

Of course, all of the operators r_j $(j \geq 0)$ are of finite rank.

For each $(j,k) \in J_1$, we define the rank one operator $s_{j,k}$ as

$$s_{j,k} = t_{\zeta_{j+1,k-1},\zeta_{j,k}}.$$

Then we put the operator s by

$$s = \sum_{(j,k)\in J_1} s_{j,k}.$$

Let $\mathcal{H}$ be an infinite dimensional Hilbert space and we take an orthonormal system $\{\xi_{i,j}\}_{(i,j)\in I}$ in $\mathcal{H}$. We make the q-Fock space $\mathcal{F}_q(\mathcal{H})$ and consider the q-annihilation operator $a(\xi_{i,j})$ and the q-creation operator $a^*(\xi_{i,j})$. The vacuum state ω on $\mathcal{F}_q(\mathcal{H})$ is given by $\omega(x) = \langle x\Omega|\Omega\rangle_q$, where Ω is the vacuum vector. For the definition of the q-Fock space, see, for instance, [BKS].

Now we adopt the Hilbert space $\mathcal{F}_q(\mathcal{H}) \otimes \widetilde{\mathcal{K}} \otimes \widetilde{\mathcal{L}}$ as the base space on which our canonical random variable will act, together with the expectation $\varepsilon = \omega \otimes \widetilde{\phi} \otimes \widetilde{\psi}$.

For each $(i,j) \in I$, we define the operators $C_{i,j}$, $N_{i,j}$, and $A_{i,j}$ on the Hilbert space $\mathcal{F}_q \otimes \widetilde{\mathcal{K}} \otimes \widetilde{\mathcal{L}}$ as

$$\begin{aligned} C_{i,j} &= a^*(\xi_{i,j}) \otimes \Gamma_{(i,j)}(\ell^{i-1}) \otimes \Lambda_i(r_{j-1}), \\ N_{i,j} &= a^*(\xi_{i,j}) a(\xi_{i,j}) \otimes \Gamma_{(i,j)}(\ell^*) \otimes \Lambda_i(1_{\mathcal{L}}), \\ A_{i,j} &= a(\xi_{i,j}) \otimes \Gamma_{(i,j)}(\ell^*) \otimes \Lambda_i(s), \end{aligned}$$

and call *the* (i,j)*-creation*, *the* (i,j)*-number*, and *the* (i,j)*-annihilation* operator, respectively.

Consequently, we obtain the operator

$$T = \alpha_1 \mathbf{1} + \sum_{i=2}^{\infty} \sum_{j=1}^{\infty} \left(\alpha_i C_{i,j} + N_{i,j} + A_{i,j} \right),$$

where $\mathbf{1}$ is the identity operator on the Hilbert space $\mathcal{F}_q \otimes \widetilde{\mathcal{K}} \otimes \widetilde{\mathcal{L}}$.

The operators $N_{2,j}$ $(j \geq 1)$ are not essential in evaluating the moments.

The operator T is our desired canonical random variable, of which moments are given by the q-deformed moments-cumulants formula (A) as follows:

THEOREM 3.1. *The nth moment of the operator T with respect to ε can be given as*

$$\varepsilon(T^n) = \sum_{\substack{\pi \in \mathcal{P}(\{1,\dots,n\}) \\ \pi = \{B_1, B_2, \dots, B_k\}}} q^{rc(\pi)} \prod_{i=1}^{k} \alpha_{|B_i|},$$

where $rc(\pi)$ is the number of restricted crossings of a partition π.

4. The proof of the Theorem

We shall start this section with seeing the role of the operators $\Lambda_i(r_{j-1})$ and $\Lambda_i(s)$ on the infinite tensor product space $\widetilde{\mathcal{L}}$, which can be used as the counter for the parenthesis number and the depth of blocks of size i.

On the Hilbert space $\mathcal{L}_i$, we consider the product of the operators $r_{j,k}$ and $s_{j,k}$,

$$z = y_n y_{n-1} \cdots y_2 y_1,$$

where $y_m \in \{r_{j_0,k_0}, s_{j_1,k_1}\}_{(j_0,k_0) \in J_0, (j_1,k_1) \in J_1}$ for $m = 1, 2, \dots, n$. Then it is easy to see that $\psi_i(z) = \langle z\zeta_{0,0} | \zeta_a \rangle$, will vanish if $z\zeta_{0,0} \neq \zeta_{n,0}$ because the product of the rank one operators r_{j_0,k_0} and s_{j_1,k_1} will induce the transitions on an orthonormal family of vectors $\{\zeta_{j,k}\}_{(j,k)\in J_0}$.

We assume the equality

$$(y_n y_{n-1} \cdots y_2 y_1)\zeta_{0,0} = \zeta_{n,0}$$

holds, which derives the path of steps n on the square lattice started from the origin $(0,0)$ and ended at $(n,0)$ by tracing the subscripts of the vectors $\zeta_{0,0}$, $y_1\zeta_{0,0}$, $(y_2y_1)\zeta_{0,0}, \ldots$, $(y_n \cdots y_2y_1)\zeta_{0,0}$ as the coordinates of the through points. It is obvious that the length of the product, n is even, automatically.

Furthermore, from the definition of the operators r_j and s, it can be said that the $\phi_i(z)$ would not be changed even if we replace the factors $r_{j,k}$ and $s_{j,k}$ in the product z by r_j and s, respectively.

Such a path is nothing but the Catalan path. This fact allows us to use the subscripts of the orthogonal vectors $\zeta_{j,k}$ on the ith tensor factor $\mathcal{L}_i$ of the infinite tensor product $\widetilde{\mathcal{L}}$ as the indicators of the parenthesis number and the depth of the blocks of size i.

Indeed, we can use the first subscript of the vector $\zeta_{j,k}$ for the counter of the parenthesis number and the second one for the indicator the depth of the blocks because the operator $\Lambda_i(r_{j-1})$ makes 1-increments both on the first and the second subscripts, and the operator $\Lambda_i(s)$ makes 1-increment on the first subscript and 1-decrement on the second subscript.

Next we shall see the role of the shift operator ℓ and its adjoint ℓ^* on each factor of the infinite tensor product space $\widetilde{\mathcal{K}}$. On the Hilbert space $\mathcal{K}$, we consider a product of ℓ and ℓ^*,

$$P = \ell^{\varepsilon_m}\ell^{\varepsilon_{m-1}} \cdots \ell^{\varepsilon_2}\ell^{\varepsilon_1}, \qquad (\varepsilon_j = \pm 1),$$

where we use the convention that $\ell^{-1} = \ell^*$. It is rather well-known that if the product P has non-zero expectation with respect to the vector state ϕ, that is, $\langle P\eta_0|\eta_0\rangle \neq 0$, then the sequence $\{\varepsilon_j\}_{j=1}^m$ should satisfy the condition for the Catalan path that

$$\sum_{j=1}^{k} \varepsilon_j \geq 0, \ (k = 1, 2, \ldots, m) \quad \text{and} \quad \sum_{j=1}^{m} \varepsilon_j = 0$$

(see, for instance, [Ni1], [VDN]). This fact allows us to use the operators $\Gamma_{(i,j)}(\ell^{i-1})$ and $\Gamma_{(i,j)}(\ell^*)$ on the infinite tensor product space $\widetilde{\mathcal{K}}$ as the counter for the elements of a block of size i, of which first element has the parenthesis number j.

In order to evaluate the moments of the operator T, we expand

$$T^n = \left(\alpha_1 \mathbf{1} + \sum_{i=2}^{\infty}\sum_{j=1}^{\infty}(\alpha_i C_{i,j} + N_{i,j} + A_{i,j})\right)^n$$

and consider the expectation in a term wise.

A product of operators $(\alpha_i C_{i,j})$, $(N_{i,j})$, $(A_{i,j})$, and $(\alpha_1 \mathbf{1})$ is called *admissible* if it has non-trivial expectation with respect to ε. The word 'trivial' means, of course, that it has zero expectation for any sequence $\{\alpha_i\}_{i=1}^{\infty}$. Here we will treat $(\alpha_i C_{i,j})$, $(N_{i,j})$, $(A_{i,j})$, and $(\alpha_1 \mathbf{1})$ as non-commutative operators and, moreover, a multiplication of the scalar operator $(\alpha_1 \mathbf{1})$ should not be reduced any more.

First we shall make the partition of the ordered set of n elements $\{1, 2, ..., n\}$ from given an admissible product of length n. It will be required to control the several counters for an admissible product. As we mentioned above, the counter Λ_i on the ith factor in the infinite tensor product Hilbert space $\widetilde{\mathcal{L}}$ will control the parenthesis number and the depth of a block of size i and the counter $\Gamma_{(i,j)}$ on the (i,j)th factor in the infinite tensor product Hilbert space $\widetilde{\mathcal{K}}$ will count the elements in the block of size i, of which first element has the parenthesis number j.

Now we assume that the product of length n,

$$Y = Z_n Z_{n-1} \cdots Z_2 Z_1$$

where

$$Z_m \in \{(\alpha_i C_{i,j}), (N_{i,j}), (A_{i,j})\}_{(i,j)\in I} \cup \{(\alpha_1 \mathbf{1})\} \quad (m = 1, 2, ..., n)$$

is given as an admissible product. In scanning the factors from right side of the admissible product, if we encounter the (i_0, j_0)-creation operator $(\alpha_{i_0} C_{i_0,j_0})$ for some $(i_0, j_0) \in I$ at the m_1th factor, that is,

$$^{\exists}m_1 \text{ s.t. } Z_{m_1} = (\alpha_{i_0} C_{i_0,j_0}) \text{ for some } (i_0, j_0) \in I,$$

then it can be ensured by the counter $\Gamma_{(i_0,j_0)}$ and definitions of the q-creation and the q-annihilation operators that there exist $(i_0 - 2)$'s (N_{i_0,j_0}) operators in the subsequent factors in case of $i_0 \geq 3$, that is,

$$^{\exists}m_2 < {}^{\exists}m_3 < \cdots < {}^{\exists}m_{i_0-1} \text{ s.t. } Z_{m_2} = Z_{m_3} = \cdots = Z_{m_{i_0-1}} = (N_{i_0,j_0}),$$

and we can find one (A_{i_0,j_0}) operator after them, that is,

$$^{\exists}m_{i_0} \text{ s.t. } Z_{m_{i_0}} = (A_{i_0,j_0}) \text{ with } m_{i_0-1} < m_{i_0}.$$

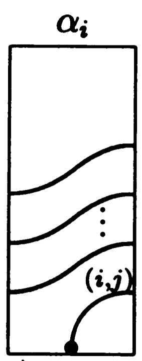
Here we can regard that the set $\{m_1, m_2, ..., m_{i_0}\}$ makes a block of size i_0. As we remarked at the beginning of this section, the second subscript j_0 of the operator $(\alpha_{i_0} C_{i_0,j_0})$ corresponds to the parenthesis number of the first element of the block $\{m_1, m_2, ..., m_{i_0}\}$ because, in general, the operators $(\alpha_i C_{i,j})$ and $(A_{i,j})$ have $\Lambda_i(r_{j-1})$ and $\Lambda_i(s)$ as the third tensor factor, respectively. Thus, the subscript j will be increased at every $(\alpha_i C_{i,j})$ and $(A_{i,j})$ that we will encounter. Of course, any (i,j)-annihilation operator, $(A_{i,j})$ or (i,j)-number operator, $(N_{i,j})$ would not appear without the corresponding (i,j)-creation operator $(\alpha_i C_{i,j})$ before their appearance.

Furthermore, if we encounter the scalar operator $(\alpha_1 \mathbf{1})$ then we should consider it makes a singleton.

In order to evaluate the expectation of an admissible product with respect to ε, we introduce the cards arrangement technique which is similar as in [ER] for juggling patterns but we will use considerably different kinds of cards. Depending on the factors in an admissible product, we will arrange the cards in reverse order, that is, the position number of cards should be counted from left side, and concatenate the flow lines drawn on the cards.

The (i,j)-creation card.

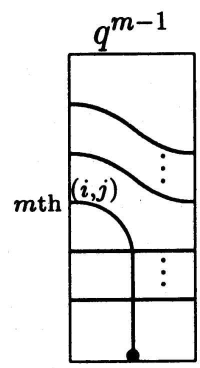
If we encounter the operator $(\alpha_i C_{i,j})$ in an admissible product then we put the following (i,j)-creation card: The (i,j)-creation card has 1 more many outflow lines than inflow ones. Hence, a new line will be created, which is started from the middle point on the ground and flows out at the first lowest level. We shall give the label (i,j) to this newly created line. If there are some inflow lines then they will flow out at the 1-increased level without any crossing, respectively, that is, the line inflowed at the ℓth level flows out at the $(\ell+1)$st level, and none of their labels will be changed. Moreover, we shall give the weight to the card by the coefficient α_i.



The (i,j)-creation card

The (i,j)-annihilation card.

If we encounter the operator $(A_{i,j})$ in an admissible product then we put the following (i,j)-annihilation card: It has 1 less many outflow lines than inflow ones, thus one line will be deleted. In this case, we can find the unique (i,j)-labelled inflow line because if there is no (i,j)-labelled line then the operator $(A_{i,j})$ will not be allowed to use there in an admissible product. Now we assume the (i,j)-labelled line has been inflowed at the mth level then we make it go down to the middle point on the ground and it will be deleted. The lines inflowed at lower than the mth level go in horizontally parallel and keep their levels. Hence $(m-1)$ crossings will occur. The lines inflowed higher than the mth level will flow out at the 1-decreased level without any crossing, respectively, that is, the line inflowed at the $\ell(> m)$th level flows out at the $(\ell-1)$st level. Any labels of lines on the card will not be changed. We shall give the weight to the card by q to the number of the crossings, hence this card has the weight q^{m-1}.



The (i,j)-annihilation card

REMARK 4.1. The (i,j)-creation and the (i,j)-annihilation cards represent the relations of the definition for the q-creation and the q-annihilation operators, respectively. Indeed, on the q-creation operator, we have

$$a^*(\xi_{i_0,j_0})\Omega = \xi_{i_0,j_0},$$
$$a^*(\xi_{i_0,j_0})\xi_{i_1,j_1} \otimes \cdots \otimes \xi_{i_n,j_n} = \xi_{i_0,j_0} \otimes \xi_{i_1,j_1} \otimes \cdots \otimes \xi_{i_n,j_n}$$

Each flow line corresponds to the vector ξ_{i_ℓ,j_ℓ} and its label indicates the subscripts of the vector. The set of the inflow lines and one of the outflow lines represent the tensor product vector of the operand and the result for the creation operator $a^*(\xi_{i_0,j_0})$, respectively. The order of piled lines corresponds to one of factors in the tensor product vector. The vacuum vector can be expressed as no flow line.

On the q-annihilation operator, we have

$$a(\xi_{i_0,j_0})\Omega = 0,$$

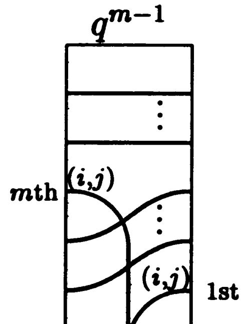
$$a(\xi_{i_0,j_0})\xi_{i_1,j_1} = \begin{cases} 0, & \text{if } (i_0,j_0) \neq (i_1,j_1), \\ \Omega, & \text{if } (i_0,j_0) = (i_1,j_1), \end{cases}$$

$$a(\xi_{i_0,j_0})\xi_{i_1,j_1} \otimes \cdots \otimes \xi_{i_n,j_n}$$
$$= \begin{cases} 0, & \text{if } (i_0,j_0) \neq (i_\ell,j_\ell) \text{ for } \ell = 1,2,...,n, \\ q^{m-1}\xi_{i_1,j_i} \otimes \cdots \otimes \overset{\curlyvee}{\xi}_{i_m,j_m} \otimes \cdots \otimes \xi_{i_n,j_n}, & \text{if } (i_0,j_0) = (i_m,j_m). \end{cases}$$

where the symbol $\overset{\curlyvee}{\xi}_{i_m,j_m}$ means that ξ_{i_m,j_m} has to be deleted in the tensor product and, of course, the number m for $(i_0,j_0) = (i_m,j_m)$ is unique if it exists. The right hand side to be 0 means that we can not use the operator (A_{i_0,j_0}) there for an admissible product.

The (i,j)-number card.

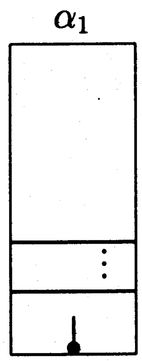
If we encounter the operator $(N_{i,j})$ in an admissible product then we put the following (i,j)-number card: Similarly as for the (i,j)-annihilation card, we can find unique (i,j)-labelled inflow line. Assume that the (i,j)-labelled line has been inflowed at the mth level then we make it go down to the middle point on the ground and its flow will be continued as the first lowest line. The inflow lines of lower than the mth level will flow out at the 1-increased level, respectively, that is, the line inflowed at the ℓth level flows out at the $(\ell+1)$st level, and ones of higher than the mth level will keep their levels. Hence we have $(m-1)$ crossings. Any labels of lines on the card will not be changed. We shall also give the weight to the card by q to the number of the crossings, thus this card has also the weight q^{m-1}.



The (i,j)-number card

The scalar card.

If we encounter the operator $(\alpha_1 \mathbf{1})$ in an admissible product then we put the following scalar card: The scalar cards has the short pole-like segment of line at the middle point on the ground. If there are some inflow lines then they will go in horizontally parallel and keep their levels, respectively. No label, of course, will be changed. The height of the pole is smaller than the 1st level, thus we have no crossing on the card. We shall give the weight to the card by α_1.



The scalar card

It is clear that given an admissible cards arrangement determines the partition of the ordered set $\{1, 2, ..., n\}$, of which blocks constituted from the points connected by flow lines in the pattern of the arrangement. Here we regard that the short poles at the middle points on the scalar card will make singletons.

From the construction of the cards, it is also obvious that the crossings which will appear in the cards arrangement are nothing else but restricted crossings for the partition determined by the arrangement because the flow line which makes a connection between two elements becomes an arc of the partition. Here we remind how to give the weights to the cards then it follows that the expectation of an admissible product can be evaluated by the product of all the weights of the cards used in the arrangement.

Now we have reached that the expectation of the admissible product Y of length n can be evaluated as

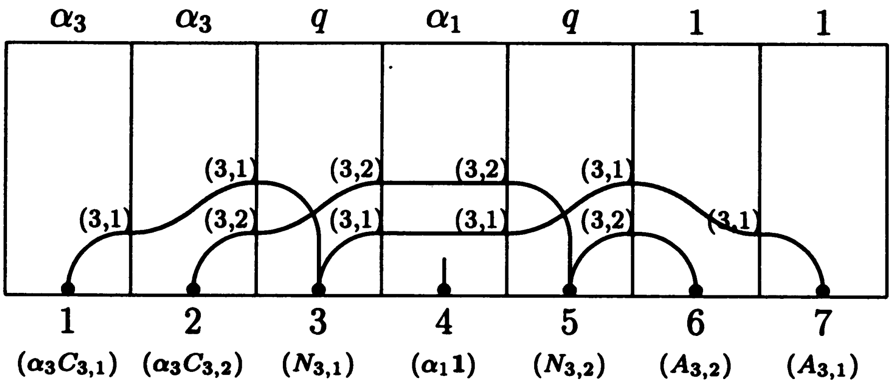
$$\varepsilon(Y) = q^{rc(\pi_Y)} \prod_{i=1}^{k} \alpha_{|B_i|},$$

where $\pi_Y = \{B_1, B_2, ..., B_k\} \in \mathcal{P}(\{1, 2, ..., n\})$ is the partition arisen from the admissible product Y as we mentioned above.

Example 4.2. For the admissible product

$$Y = (A_{3,1})\,(A_{3,2})\,(N_{3,2})\,(\alpha_1\mathbf{1})\,(N_{3,1})\,(\alpha_3 C_{3,2})\,(\alpha_3 C_{3,1}),$$

we have the following cards arrangement:



The above cards arrangement yields the partition

$$\{\{1,3,7\},\{2,5,6\},\{4\}\},$$

and, on the expectation, we have $\varepsilon(Y) = q^2\alpha_1\alpha_3^2$.

Conversely, given a partition $\pi \in \mathcal{P}(\{1,2,...,n\})$, we can make the admissible product of the operators $(\alpha_i C_{i,j})$, $(N_{i,j})$, $(A_{i,j})$, and $(\alpha_1\mathbf{1})$ of the length n as the following manner: For $k \in \{1,2,...,n\}$, we first take the size i of the block in which k is contained. If $i = 1$, that is, $\{k\}$ is a singleton in the partition π, then we put the scalar operator $(\alpha_1\mathbf{1})$ as the kth factor in our product. Now we assume that $i \geq 2$. Then we seek the parenthesis number of the first element of the block in which k is contained, say j. If k is the first (resp. last) element of the block then we use the operator $(\alpha_i C_{i,j})$ (resp. $(A_{i,j})$) as the kth factor in our product. For the rest of the above cases, that is, k is an intermediate element of a block, then we adopt the operator $(N_{i,j})$ as the kth factor in our product. It should be noted that the position of the factors is counted from right side.

Using the card arrangement again, it is easy to see that such a product has non-trivial expectation with respect to ε, which can be obtained as the product of the weights of the cards used in the arrangement. □

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