

Title	The Jorgensen number of the Whitehead link (Hyperbolic Spaces and Discrete Groups II)
Author(s)	Sato, Hiroki
Citation	数理解析研究所講究録 (2002), 1270: 77-83
Issue Date	2002-06
URL	<a href="http://hdl.handle.net/2433/42173">http://hdl.handle.net/2433/42173</a>
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

# The Jørgensen number of the Whitehead link

Hiroki Sato

佐藤 宏樹 (静岡大学理学部) \*

**ABSTRACT.** In this paper we will sketch out the result obtained recently: the Jørgensen number of the Whitehead link is two. Furthermore we will represent points corresponding to the Whitehead link by using the coordinates introduced in Sato [7]. The details will appear in Sato [9].

1. In 1976 Jørgensen obtained the following important theorem called Jørgensen's inequality, which gives a necessary condition for a non-elementary Möbius transformation group  $G = \langle A, B \rangle$  to be discrete.

**THEOREM A** (Jørgensen [1]). *Suppose that the Möbius transformations  $A$  and  $B$  generate a non-elementary discrete group. Then*

$$J(A, B) := |\operatorname{tr}^2(A) - 4| + |\operatorname{tr}(ABA^{-1}B^{-1}) - 2| \geq 1.$$

*The lower bound 1 is best possible.*

**DEFINITION 1.** Let  $A$  and  $B$  be Möbius transformations. The *Jørgensen number*

---

\*Partly supported by the Grants-in-Aid for Co-operative Research as well as Scientific Research, the Ministry of Education, Science, Sports and Culture, Japan.

2000 *Mathematics Subject Classification.* Primary 32G15; Secondary 20H10, 30F40.

*Key Words and Phrases.* Jørgensen's inequality, Jørgensen number, Jørgensen groups, the Whitehead link.

$J(A, B)$  of the ordered pair  $(A, B)$  is defined as

$$J(A, B) := |\operatorname{tr}^2(A) - 4| + |\operatorname{tr}(ABA^{-1}B^{-1}) - 2|.$$

We denote by  $\text{Möb}$  the set of all Möbius transformations. Throughout this paper we will always write elements of  $\text{Möb}$  as matrices with determinant 1. We recall that  $\text{Möb}$  ( $= PSL(2, \mathbb{C})$ ) acts on the upper half space  $H^3$  of  $\mathbb{R}^3$  as the group of conformal isometries of hyperbolic 3-space. A subgroup  $G$  of  $\text{Möb}$  is said to be *elementary* if there exists a finite  $G$ -orbit in  $\mathbb{R}^3$ .

**DEFINITION 2.** Let  $G$  be a non-elementary two-generator subgroup of  $\text{Möb}$ . The *Jørgensen number*  $J(G)$  for  $G$  is defined as

$$J(G) := \inf\{J(A, B) \mid A \text{ and } B \text{ generate } G\}.$$

**DEFINITION 3.** A non-elementary two-generator subgroup  $G$  of  $\text{Möb}$  is a *Jørgensen group* if  $G$  is a discrete group with  $J(G) = 1$ .

**THEOREM B (Jørgensen-Kiikka [2]).** Let  $\langle A, B \rangle$  be a non-elementary discrete group with  $J(A, B) = 1$ . Then  $A$  is elliptic of order at least seven or  $A$  is parabolic.

If  $\langle A, B \rangle$  is a Jørgensen group such that  $A$  is parabolic, then we call it a *Jørgensen group of parabolic type*. Here we only consider Jørgensen groups of parabolic type.

2. Let  $\langle A, B \rangle$  be a marked two-generator group such that  $A$  is parabolic. Then we can normalize  $A$  and  $B$  as follows:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B := B_{\sigma, \mu} = \begin{pmatrix} \mu\sigma & \mu^2\sigma - 1/\sigma \\ \sigma & \mu\sigma \end{pmatrix},$$

where  $\sigma \in \mathbb{C} \setminus \{0\}$  and  $\mu \in \mathbb{C}$ . We denote by  $G_{\sigma, \mu}$  the marked group generated by  $A$  and  $B_{\sigma, \mu}$ :  $G_{\sigma, \mu} = \langle A, B_{\sigma, \mu} \rangle$ . We say that  $(\sigma, \mu)$  is the point representing a marked group  $G_{\sigma, \mu}$  and that  $G_{\sigma, \mu}$  is the marked group corresponding to a point  $(\sigma, \mu)$ .

In particular, we consider the case of  $\mu = ik$  ( $k \in \mathbf{R}$ ). Namely, we consider marked two-generator group  $G_{\sigma, ik} = \langle A, B_{\sigma, ik} \rangle$  generated by

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B := B_{\sigma, ik} = \begin{pmatrix} ik\sigma & -k^2\sigma - 1/\sigma \\ \sigma & ik\sigma \end{pmatrix},$$

where  $\sigma \in \mathbf{C} \setminus \{0\}$  and  $k \in \mathbf{R}$ .

3. Let  $C$  be the following cylinder:  $C = \{(\sigma, ik) \mid |\sigma| = 1, k \in \mathbf{R}\}$ .

**THEOREM C** (Sato [7]). *If a marked two-generator group  $G_{\sigma, ik}$  is a Jørgensen group, then the point  $(\sigma, ik)$  representing  $G_{\sigma, ik}$  lies on the cylinder  $C$ .*

By Theorem C we consider marked two-generator groups  $G_{\sigma, \mu} = \langle A, B_{\mu, \sigma} \rangle$  with  $\sigma = -ie^{i\theta}$  ( $0 \leq \theta < 2\pi$ ) and  $\mu = ik$  ( $k \in \mathbf{R}$ ). For simplicity we set  $B_{\theta, k} := B_{\sigma, ik}$  and  $G_{\theta, k} = \langle A, B_{\sigma, ik} \rangle$  for  $\sigma = -ie^{i\theta}$ .

4. There are infinite number of Jørgensen groups (see Jørgensen-Lascurain-Pignataro [3], Sato [7]). The following families of groups are all Jørgensen groups: The modular group, the Picard group (Jørgensen-Lascurain-Pignataro [3], Sato [8, 9], Sato-Yamada [10]), the figure-eight knot group (Sato [7]), "the Gehring-Maskit group" (Sato [7]), where "the Gehring-Maskit group" is the group studied in Maskit [5]. Namely, we have the following theorem:

**THEOREM D** (Jørgensen-Lascurain-Pignataro [3], Sato [7, 8], Sato-Yamada [10]).

*Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_{\theta, k} = \begin{pmatrix} ke^{i\theta} & ie^{-i\theta}(k^2e^{2i\theta} - 1) \\ -ie^{i\theta} & ke^{i\theta} \end{pmatrix}$$

*and let  $G_{\theta, k} = \langle A, B_{\theta, k} \rangle$  be the group generated by  $A$  and  $B_{\theta, k}$ , where  $0 \leq \theta < 2\pi$  and  $k \in \mathbf{R}$ . Then*

- (i)  $G_{\pi/2,0}$  is a Jørgensen group.
- (ii)  $G_{\pi/2,1/2}$  is a Jørgensen group.
- (iii)  $G_{\pi/6,\sqrt{3}/2}$  is a Jørgensen group.
- (iv)  $G_{0,\sqrt{3}/2}$  is a Jørgensen group.

REMARK (1) The groups  $G_{\pi/2,0}$ ,  $G_{\pi/2,1/2}$ ,  $G_{\pi/6,\sqrt{3}/2}$  and  $G_{0,\sqrt{3}/2}$  are conjugate to the modular group, the Picard group, the figure-eight knot group and "the Gehring-Maskit group", respectively.

- (2) See Sato [7] for other Jørgensen groups of parabolic type.

5. Now it gives rise to the following problem.

PROBLEM. Is the Whitehead link a Jørgensen group ?

Here we can give the answer to the problem, that is, we have the following theorems.

THEOREM 1 (Sato [9]). *The Jørgensen number of the Whitehead link is two.*

COROLLARY (Sato [9]). The Whitehead link is not a Jørgensen group.

THEOREM 2 (Sato [9]). *The Whitehead link is conjugate to the marked two-generator group  $G_{\sigma,\mu}$  where  $\sigma = \sqrt{2}e^{3\pi i/4}$  and  $\mu = -1/2$ .*

6. The proofs of the theorems will appear elsewhere. Here we only give sketches of the proofs.

THEOREM E (cf. Wielenberg [11], Krushkal', Apanasov and Gusevskii,[4]). *The Whitehead link  $G_W$  has the following presentation:*

$$G_W = \langle A, B \mid (A^{-1}BAB^{-1})(ABA^{-1}B^{-1})(AB^{-1}A^{-1}B)(A^{-1}B^{-1}AB) = 1 \rangle,$$

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1-i & 1 \end{pmatrix}.$$

PROPOSITION 1 *Let  $G_W$  be the Whitehead link defined in Theorem E. Then an element  $X$  of  $G_W$  has the following form:*

$$X = \begin{pmatrix} 1 + (1-i)a & b_1 + (1-i)b_2 \\ (1-i)c & 1 + (1-i)d \end{pmatrix}.$$

where  $a, b_1, b_2, c, d \in \mathbf{Z} + i\mathbf{Z}$ ,  $a + d - b_1c + (1-i)(ad - b_2c) = 0$ .

PROPOSITION 2. *Let  $G_W$  be the Whitehead link defined in Theorem E and let  $\langle X, Y \rangle$  be a non-elementary subgroup generated by  $X$  and  $Y$ , where  $X, Y \in G_W$ . Then the Jørgensen number of  $(X, Y)$  is greater than or equal to two:  $J(X, Y) \geq 2$ .*

PROPOSITION 3. *Let  $A, B$  be the matrices in Theorem E. Set  $C = AB$ . Then  $A$  and  $C$  generate the Whitehead link  $G_W$  and  $J(A, C) = 2$ .*

Theorem 1 follows from Propositions 2 and 3.

6. Next we will give a sketch of the proof of Theorem 2.

Let  $P$  be the regular ideal octahedron in Ratcliffe [6, p.454]. Let the sides  $S_A, S_B, S_C, S_D, S_{A'}, S_{B'}, S_{C'}$  and  $S_{D'}$  be the sides of  $P$ . Let  $f_A, f_B, f_C$  and  $f_D$  be the side pairing transformations of  $S_A$  to  $S_{A'}$ , of  $S_B$  to  $S_{B'}$ , of  $S_C$  to  $S_{C'}$ , and of  $S_D$  to  $S_{D'}$ , respectively.

PROPOSITION 4. *Let  $f_A, f_B, f_C$  and  $f_D$  be the side pairing transformations defined in the above. Then  $f_A, f_B, f_C$  and  $f_D$  generate the Whitehead link  $G_{W,R}$  in the sense of Ratcliffe.*

PROPOSITION 5. *Let*

$$G_{W,R}^* = \langle A^*, B^* \mid A^*(B^*)^{-2}A^*B^*(A^*)^{-1}(B^*)^{-1} \\ (A^*)^{-1}(B^*)^2(A^*)^{-1}(B^*)^{-1}A^*B^* = 1 \rangle,$$

where

$$A^* = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B^* = \begin{pmatrix} 1/2 + i/2 & 3/4 + i/4 \\ -1 + i & 1/2 + i/2 \end{pmatrix}.$$

Then  $G_{W,R}^*$  is conjugate to the Whitehead link  $G_{W,R}$  in the sense of Ratcliffe.

(ii)  $J(A^*, B^*) = 2$ .

PROPOSITION 6. *The marked group  $G_{W,R}^* = \langle A^*, B^* \rangle$  in Proposition 5 corresponds to the point  $(-1 + i, -1/2)$ .*

Theorem 2 follows from Propositions 5 and 6,

## References

- [1] T. Jørgensen, *On discrete groups of Möbius transformations*, Amer. J. Math. **98** (1976) 739-749.
- [2] T. Jørgensen and M. Kiiikka, *Some extreme discrete groups*, Ann. Acad. Sci. Fenn. **1** (1975), 245-248.
- [3] T. Jørgensen, A. Lascurain and T. Pignataro, *Translation extentions of the classical modular group*, Complex Variable **19** (1992), 205-209.
- [4] S. L. Krushkal', B. N. Apanasov and N. A. Gusevskii, *Kleinian groups and uniformization in examples and problems*, Trans. math. Monographs **62**, Amer. Math. Soc., Providence, Rhode Island, 1986.

- [5] B. Maskit, *Some special 2-generator Kleinian groups*, Proc. Amer. Math. Soc., **106** (1989), 175-186.
- [6] J. R. Ratcliffe, *Foundations of hyperbolic manifolds*, GTM **149**, Springer-Verlag, New York, Berlin, Heidelberg, 1994.
- [7] H. Sato, *One-parameter families of extreme groups for Jørgensen's inequality*, Contemporary Math. **256** (The First Ahlfors - Bers Colloquium) edited by I. Kra and B. Maskit, 2000, 271-287.
- [8] H. Sato, *Jørgensen groups and the Picard group*, to appear in the Proc. of The Third ISAAC International Conference, Academic Scientific Publ., 2002.
- [9] H. Sato, *The Jørgensen number of the Whitehead link*, in preparation.
- [10] H. Sato and R. Yamada, *Some extreme Kleinian groups for Jørgensen's inequality*, Rep. Fac. Sci. Shizuoka Univ. **27** (1993), 1-8.
- [11] N. J. Wielenberg, *The structure of certain subgroups of the Picard group*, Math. Proc. Cambridge Philos. Soc. **84** (1978), 427-438.

Department of Mathematics

Faculty of Science

Shizuoka University

Ohya Shizuoka 422-8529

Japan

e-mail: smhsato@ipc.shizuoka.ac.jp