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On a Limiting System in the Lotka–Volterra Competition with Cross-Diffusion

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1 Introduction

This is a joint work with Yuan Lou (The Ohio State State Univ.) and Wei-Ming Ni (Univ. of Minnesota). For details, see Lou–Ni–Yotsutani [4].

In an attempt to model segregation phenomena in population dynamics, Shigesada, Kawasaki and Teramoto [7] in 1979 incorporated the inter– and intra–specific population pressures into the classical Lotka–Volterra competition system.

In particular, the following system was proposed:

$$\begin{cases} u_t = \Delta[(d_1+\rho_{11}u+\rho_{12}v)u]+u\ (a_1-b_1u-c_1v) & \text{in } \Omega_T, \\ v_t = \Delta[(d_2+\rho_{21}u+\rho_{22}v)v]+v\ (a_2-b_2u-c_2v) & \text{in } \Omega_T, \\ \frac{\partial u}{\partial \nu}=\frac{\partial v}{\partial \nu} \quad \text{on } \partial\Omega_T, \\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x) \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Delta = \Sigma_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, Ω is a bounded domain in $\mathbf{R}^n$, $n \geq 1$, with smooth boundary $\partial\Omega$ and its unit outer normal ν, $\Omega_T = \Omega \times [0,T)$ and $\partial\Omega_T = \partial\Omega \times [0,T)$. The system (1.1) was proposed to model segregation of interacting species, where u and v represent the densities of two competing species, hence only nonnegative u and v are of interest. The constants a_j, b_j, c_j and d_j $(j=1,2)$ are all positive, where a_1, a_2 denote the intrinsic growth rates of these two species, b_1 and c_2 account for intra–specific competitions while b_2, c_1 account for inter–specific competitions, and d_1, d_2 are their diffusion rates. The constants ρ_{11}, ρ_{22} represent intra-specific population pressures, also known as *self–diffusion* rates, and ρ_{12}, ρ_{21} are the coefficients of inter–specific population pressures, also known as *cross-diffusion* rates.

We should remark that it is well known that the important quantities involving the constants a_j, b_j, c_j, $j=1,2$, are only

$$A=\frac{a_1}{a_2}, \qquad B=\frac{b_1}{b_2} \qquad \text{and} \qquad C=\frac{c_1}{c_2}. \tag{1.2}$$

We refer to [7] and [6] for further details of the model (1.1).

In [2, 3] steady states of (1.1) are studied, especially in the case when one of the cross-diffusion rates, say, ρ_{12}, is large. In particular, the limiting characterization as ρ_{12} tends to infinity was established. (See Theorems 1.4 and 4.1 in [3].) For simplicity, here we only state the special case $\rho_{11} = \rho_{21} = \rho_{22} = 0$.

Theorem 1.1 *Suppose that $n \leq 3$, $B \neq A \neq C$ and $a_2/d_2 \neq \lambda_k$ for any $k \geq 1$, where $0 = \lambda_0 < \lambda_1 \leq \cdots$ denote the eigenvalues of $-\Delta$ on Ω under the homogeneous Neumann boundary condition. Then, as $\rho_{12,k} \to \infty$, by passing to a subsequence if necessary, a positive steady state (u_k, v_k) of (1.1) must satisfy either* (i) *or* (ii) *below.*

(i) $(u_k, \rho_{12,k} v_k)$ converges uniformly to (u, v) where (u, v) is a positive solution of

$$\begin{cases} \Delta[(d_1 + v)u] + u\,(a_1 - b_1 u) = 0 & \text{in } \Omega, \\ d_2\, \Delta v + v\,(a_2 - b_2 u) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} \quad \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

(ii) (u_k, v_k) converges to $\left(\frac{\tau}{v}, v\right)$ where $\tau > 0$ is a constant, and (v, τ) satisfies

$$\begin{cases} d_2\, \Delta v + v\,(a_2 - c_2 v) - b_2 \tau = 0 \quad \text{in } \Omega, \\ v > 0 \quad \text{in } \Omega, \qquad \dfrac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \\ \displaystyle\int_\Omega \frac{1}{v}\left(a_1 - b_1 \frac{\tau}{v} - c_1 v\right) = 0. \end{cases} \tag{1.4}$$

The main purpose of this talk is to understand *all possible* solutions of the "shadow" system (1.4) as thoroughly as possible in the one dimensional case $n = 1$, $\Omega = (0, 1)$. Roughly speaking, our results can be categorized into two classes: Existence and qualitative behavior of solutions, depending on various cases involving the coefficients A, B and C.

In studying non-constant solutions of the 1–dimensional version of the shadow system (1.4), i.e.

$$\begin{cases} d_2\, v'' + v\,(a_2 - c_2 v) - b_2 \tau = 0 \quad \text{in } (0,1), \\ v > 0 \ \text{ and } \ v' \not\equiv 0 \quad \text{in } (0,1), \\ v'(0) = v'(1) = 0, \\ \displaystyle\int_0^1 \frac{1}{v}\left(a_1 - b_1 \frac{\tau}{v} - c_1 v\right) = 0 \ \text{ and } \ \tau > 0, \end{cases} \tag{1.5}$$

it seems natural to consider the following two cases separately: the "strong competition" case $B < C$ and the "weak competition" case $B > C$.

In the strong competition $B < C$, our results in this paper will include

(I) *If* $d_2 \geq a_2/\pi^2$, *then* (1.4) *does not have any non-constant solutions.*

(II) *For* $d_2 < a_2/\pi^2$, *we have that*

- *(i)* (1.5) *has no solution for any* $d_2 \in (0, a_2/\pi^2)$ *if* $A \leq B$*;*
- *(ii) for every* $d_2 \in (0, a_2/\pi^2)$, (1.5) *has a solution* (v, τ) *with* v *being strictly increasing in* $(0, 1)$ *if* $A \geq (B + C)/2$*;*
- *(iii) the following two cases hold if* $B < A < (B + C)/2$*:*
  - *(1) for every* $d_2 \in \left(\dfrac{B+C-2A}{C-B} \cdot \dfrac{a_2}{\pi^2}, \dfrac{a_2}{\pi^2}\right)$, (1.5) *has a solution* (v, τ) *with* v *being strictly increasing in* $(0, 1)$*;*
  - *(2) for every integer* $k > 0$, *there exists* $D_k > 0$ *such that* (1.5) *has no solution of mode* k *for* $d_2 \in (0, D_k)$.

We refer to Section 5 for the definition of *solutions of mode k*.

The above results show a general tendency that for (1.5) to have solutions the diffusion rate d_2 needs to be small while the parameter A needs to be large. Comparing this to the well-known non-existence for the "semi-trivial" case in the classical diffusion (only) case, i.e. $\rho_{11} = \rho_{12} = \rho_{21} = \rho_{22} = 0$ in (1.1), for which the only possible steady states are $(a_1/b_1, 0)$ and $(0, a_2/c_2)$, we see a striking difference.

For the "weak competition" $B > C$, a similar list of our results includes:

(III) *If* $d_2 \geq a_2/\pi^2$, *then* (1.4) *does not have any non-constant solutions.*

(IV) *For* $d_2 < a_2/\pi^2$, *we have*

- *(i)* (1.5) *has no solution for any* $d_2 \in (0, a_2/\pi^2)$ *if* $A \leq (B + 3C)/4$*;*
- *(ii) for every* $d_2 \in (0, a_2/\pi^2)$, (1.5) *has a solution* (v, τ) *with* v *being strictly increasing in* $(0, 1)$ *if* $A \geq B$*;*
- *(iii) the following two cases hold if* $B > A > (B + 3C)/4$*:*
  - *(1) for every* $d_2 \in \left(0, \dfrac{2A-(B+C)}{B-C} \cdot \dfrac{a_2}{\pi^2}\right)$, (1.5) *has a solution* (v, τ) *with* v *being strictly increasing in* $(0, 1)$*;*
  - *(2) there exists* $D \in (0, a_2/\pi^2)$ *such that* (1.5) *has no solution for* $d_2 \in (D, a_2/\pi^2)$.

Results (I) – (IV) above are nearly optimal, and our understanding of (1.5), as far as existence and non-existence are concerned, is nearly complete, except for the ranges that $B < A < (B+C)/2$ in the strong competition case and $B > A > (B+3C)/4$ in the weak competition case.

We can also obtain the asymptotic behaviors of solutions to (1.5) as $d_2 \to 0$ or $d_2 \to a_2/\pi^2$, whenever the existence is guaranteed. It is interesting to note that as $d_2 \to a_2/\pi^2$, while both v and τ tend to 0, the ratio τ/v remains a large amplitude solution. On the other hand, as $d_2 \to 0$, both v and τ/v exhibit "spike-layer" phenomena. To illustrate our results in this direction we use the following example for (1.5) (i.e. $a_1 = 4$, $b_1 = 1$, $c_1 = 6$, and $a_2 = b_2 = c_2 = 1$):

$$\begin{cases} d_2\, v'' + v\,(1-v) - \tau = 0 \quad \text{in } (0,1), \\ v > 0 \quad \text{and} \quad v' \not\equiv 0 \quad \text{in } (0,1), \\ v'(0) = v'(1) = 0, \\ \displaystyle\int_0^1 \frac{1}{v}\left(4 - \frac{\tau}{v} - 6v\right) = 0 \quad \text{and} \quad \tau > 0. \end{cases} \tag{1.6}$$

Note that by (II) (ii) above the existence of a solution (v, τ) of (1.6) with v being strictly increasing in $(0,1)$ is guaranteed for every $d_2 \in (0, 1/\pi^2)$.

Theorem 1.2 *Let* $(v(\cdot; d_2), \tau(d_2))$ *be a solution of* (1.6) *with* v *being strictly increasing in* $(0,1)$. *Then*

(1) as $d_2 \to 0$, $\tau(d_2) \to 0.24$, $v(0; d_2) \to 0.3$ *and* $v(x; d_2) \to 0.6$ *for all* $x \in (0,1]$; *and*

(2) as $d_2 \to 1/\pi^2$, $\tau(d_2) \to 0$, $v(\cdot; d_2) \to 0$ *uniformly, and*

$$\frac{\tau(d_2)}{v(x; d_2)} \to \frac{2}{2 - \sqrt{3}\cos(\pi x)}.$$

Observe that if we drop the requirement that $v' \not\equiv 0$ in $(0,1)$ in (1.6), then (1.6) allows a constant solution $(v, \tau) = (0.6, 0.24)$, and the spike-layer solution guaranteed by Theorem 1.2 (1) above stays "close" to this constant solution except near the "spike". This is typical if $(B+C)/2 < A < (B+3C)/4$. *If* $A > (B+3C)/4$, *a strikingly different spike-layer solution will appear.* For instance, consider the following variant of (1.6)

$$\begin{cases} d_2\, v'' + v\,(1-v) - \tau = 0 \quad \text{in } (0,1), \\ v > 0 \quad \text{and} \quad v'' \not\equiv 0 \quad \text{in } (0,1), \\ v'(0) = v'(1) = 0, \\ \displaystyle\int_0^1 \frac{1}{v}\left(a_1 - b_1\frac{\tau}{v} - c_1 v\right) = 0 \quad \text{and} \quad \tau > 0, \end{cases} \tag{1.7}$$

where a_1, b_1 and c_1 satisfy

$$\frac{1}{4}b_1 + \frac{3}{4}c_1 < a_1 < c_1.$$

In this case (II) (ii) still applies and we have

Theorem 1.3 *For* $0 < d_2 < 1/\pi^2$, *let* $(v(\cdot; d_2), \tau(d_2))$ *be a solution of* (1.7) *with* v *being strictly increasing in* $(0,1)$. *Then, as* $d_2 \to 0$, $\tau(d_2) \to 3/16$, $v(0; d_2) \to 0$ *and* $v(x; d_2) \to 3/4$ *for all* $x \in (0,1]$.

Although (1.7) still has a constant solution

$$\tau = \frac{(c_1 - a_1)(a_1 - b_1)}{(c_1 - b_1)^2}, \quad v = \frac{a_1 - b_1}{c_1 - b_1},$$

if we drop the requirement that $v' \not\equiv 0$ in (1.7), we see that the spike-layer solution in Theorem 1.3 is nowhere close to the constant solution. Furthermore, the solution in Theorem 1.3 does not depend on a_1, b_1 and c_1 in the limit!

Our basic strategy of proofs is to convert the problem of solving (v, τ) of the shadow system (1.5) to a problem of solving its "representation" (δ, α) in a different parameter space. This is done first *without* the integral constraint in (1.5). Then we use the integral constraint to find the "solution curve" in the space (δ, α) as the diffusion rate d_2 varies. This turns out to be a powerful method as it gives fairly precise information about the solutions.

We ought to mention that earlier work on (1.1) include [5], in which under suitable hypothesis steady states of (1.1) with internal transition-layers are constructed, and their stability properties are studied in [1]. Moreover, "spike-layer" steady states of (1.1), as well as (1.5), are constructed in [3], which are directly related to the work presented here.

2 Existence

In one-dimension, i.e. $n = 1$ and $\Omega = (0,1)$, a non-constant solution (v, τ) of the shadow system (1.4) satisfies (1.5).

Due to the scaling and reflection properties of solutions to autonomous ordinary differential equations, all solutions to the above system may be obtained, by several reflections and a suitable re-scaling, from solutions of the following more stringent system (with perhaps a different value of d_2):

$$\begin{cases} d_2\, v'' + v\,(a_2 - c_2 v) - b_2 \tau = 0 \quad \text{in } (0,1), \\ v > 0 \quad \text{and} \quad v' > 0 \quad \text{in } (0,1), \\ v'(0) = v'(1) = 0, \\ \displaystyle\int_0^1 \frac{1}{v}\left(a_1 - b_1 \frac{\tau}{v} - c_1 v\right) = 0 \quad \text{and} \quad \tau > 0. \end{cases} \tag{2.1}$$

Thus we may focus on (2.1). Note that the strict monotonicity of v is required in (2.1) now.

We can obtain the following existence result.

Theorem 2.1 *For every $d_2 \in (0, a_2/\pi^2)$, the shadow system (2.1) always possesses a solution if $A \geq max\{B, (B+C)/2\}$, except that in the case $A = B = C$ the system (2.1) has no solution.*

In many ways Theorem 2.1 is optimal. For instance, if $d_2 > a_2/\pi^2$, then (2.1) does not have any solution. On the other hand, if A is sufficiently small, then (2.1) has no solution. (See Section 4 for details.) For the case that A is comparable to $(B+C)/2$ (or, more generally, A lies in between B and C), the situation becomes somewhat delicate and complicated. Nevertheless, we still have the following a "dual" result.

Proposition 2.2 *If $B < A < (B+C)/2$, then for every*

$$\frac{(B+C)-2A}{C-B} \cdot \frac{a_2}{\pi^2} < d_2 < \frac{a_2}{\pi^2}, \tag{2.2}$$

the system (2.1) possesses a solution.

3 Asymptotic Behavior as $d_2 \to a_2/\pi^2$

For a sequence $d_{2,i} \to a_2/\pi^2$ in $(0, a_2/\pi^2)$, if (v_i, τ_i) is a solution of (2.1) corresponding to $d_2 = d_{2,i}$, we are interested in the limiting behavior of (v_i, τ_i). In this section, we do not need the existence of such solutions (v_i, τ_i), although they are guaranteed to exist in the case $A > B$ by Theorem 2.1 and Proposition 2.2. In fact, some non-existence results for $d_2 < a_2/\pi^2$ but close to a_2/π^2 will be derived as consequences of the limiting behavior of (v_i, τ_i).

Theorem 3.1 *Let $(v(\cdot; d_{2,i}), \tau(d_{2,i}))$ be a solution of (2.1), where $d_{2,i} \in (0, a_2/\pi^2)$ and $d_{2,i} \to a_2/\pi^2$ as $i \to \infty$. Then, as $i \to \infty$, $(v(\cdot; d_{2,i}), \tau(d_{2,i})) \to (0,0)$ uniformly, with*

$$\frac{\tau(d_{2,i})}{v(x; d_{2,i})} \to \frac{a_2}{b_2} \cdot \frac{1}{1 - \sqrt{1 - \frac{B}{A}\cos(\pi x)}} \tag{3.1}$$

uniformly on $[0,1]$. In particular, it must hold that $A \geq B$ for (2.1) to have solutions for d_2 close to a_2/π^2.

4 Asymptotic Behavior as $d_2 \to 0$

In this section, again, we let (v, τ) denote a solution of (2.1), i.e. $\tau > 0$ and v is positive and strictly increasing in $(0,1)$, and we will continue to use the notation and results in the previous sections. Our main results here gives a *complete* description of the behavior of all solutions of (2.1) for d_2 small. In particular, we shall show that all solutions of (2.1) exhibit various kinds of "spike-layer" patterns as $d_2 \to 0$.

Theorem 4.1 *Let $(v(\cdot; d_2), \tau(d_2))$ be a solution of* (2.1)*. Suppose that*

$$A < \frac{1}{4}B + \frac{3}{4}C. \tag{4.1}$$

Then, as $d_2 \to 0$ we have

$$v(0; d_2) \to 2\frac{a_2}{c_2} \cdot \frac{A - (\frac{1}{4}B + \frac{3}{4}C)}{B - C}, \tag{4.2}$$

$$v(x; d_2) \to \frac{a_2}{c_2} \cdot \frac{B - A}{B - C}, \quad \text{for all } 0 < x \leq 1, \tag{4.3}$$

$$\tau(d_2) \to \frac{a_2^2}{b_2 c_2} \cdot \frac{(B - A)(A - C)}{(B - C)^2}. \tag{4.4}$$

It turns out that Theorem 4.1 is also useful in establishing non-existence results concerning (2.1) for d_2 small. We shall continue our discussions in this direction in Section 5.

We obtain the following result which complements Theorem 4.1.

Theorem 4.2 *Let $(v(\cdot; d_2), \tau(d_2))$ be a solution of* (2.1)*. Suppose that*

$$A \geq \frac{1}{4}B + \frac{3}{4}C \quad \text{and} \quad B < C. \tag{4.5}$$

Then, as $d_2 \to 0$ we have

$$v(0; d_2) \to 0, \tag{4.6}$$

$$v(x; d_2) \to \frac{3}{4}\frac{a_2}{c_2}, \quad \text{for all} \quad x \in (0, 1], \tag{4.7}$$

$$\tau(d_2) \to \frac{3}{16}\frac{a_2^2}{b_2 c_2}. \tag{4.8}$$

Combining Theorems 4.1 and 4.2 we see that in the "strong competition" case, $B < C$, the asymptotic behavior of solutions (v, τ) to (2.1) for d_2 small is well understood. (Here, we ought to remark that for $A < (B + C)/2$ and

$B < C$, Theorem 4.1 actually implies that (2.1) has no solution for d_2 small. See Proposition 5.7 for details.)

For the "weak competition" case, namely, $B > C$, similar results can be obtained.

Theorem 4.3 *Let $(v(\cdot; d_2), \tau(d_2))$ be a solution of* (2.1). *Suppose that*

$$A > \frac{B+C}{2} \quad \text{and} \quad B > C. \tag{4.9}$$

Then, as $d_2 \to 0$ we have

$$v(0; d_2) \to 0, \tag{4.10}$$

$$v(x; d_2) \to \frac{3}{4}\frac{a_2}{c_2}, \quad \text{for all} \quad x \in (0, 1], \tag{4.11}$$

$$\tau(d_2) \to \frac{3}{16}\frac{a_2^2}{b_2 c_2}. \tag{4.12}$$

Noting that (2.1) does not possess any solution if $A < (B + 3C)/4$ and $B > C$ (see, e.g. Theorem 5.5), we conclude this section by the following result which deals with the remaining case for the "weak competition" $B > C$.

Theorem 4.4 *Let $(v(\cdot; d_2), \tau(d_2))$ be a solution of* (2.1). *Suppose that*

$$\frac{B+C}{2} \geq A > \frac{B+3C}{4} \quad \text{and} \quad B > C. \tag{4.13}$$

Then, as $d_2 \to 0$, by passing to a subsequence if necessary we have either

$$v(0; d_2) \to 0, \tag{4.14}$$

$$v(x; d_2) \to \frac{3}{4}\frac{a_2}{c_2} \quad \text{for all} \quad x \in (0, 1], \tag{4.15}$$

$$\tau(d_2) \to \frac{3}{16}\frac{a_2^2}{b_2 c_2}, \tag{4.16}$$

or

$$v(0; d_2) \to 2\frac{a_2}{c_2} \cdot \frac{A - (\frac{1}{4}B + \frac{3}{4}C)}{B - C},. \tag{4.17}$$

$$v(x; d_2) \to \frac{a_2}{c_2} \cdot \frac{B - A}{B - C} \quad \text{for all } 0 < x \leq 1,. \tag{4.18}$$

$$\tau(d_2) \to \frac{a_2^2}{b_2 c_2} \cdot \frac{(B - A)(A - C)}{(B - C)^2}. \tag{4.19}$$

5 Non-existence results

Our first non-existence result for the shadow system (1.4) is a very general one – it applies to any bounded smooth domain in any dimension, and it does not even need to assume the integral constraint in (1.4).

Consider the following boundary value problem

$$\begin{cases} d_2\,\Delta v + v\,(a_2 - c_2 v) - b_2\tau = 0 \quad \text{in } \Omega, \\ v > 0 \quad \text{in } \Omega, \quad \text{and} \quad \dfrac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \\ \tau > 0. \end{cases} \tag{5.1}$$

Note that while both v and τ are kept, the integral constraint in (1.4) is dropped.

Theorem 5.1 *If $d_2 \geq a_2/\lambda_1$, then* (5.1) *only has constant solutions.*

Noting that $\lambda_1 = \pi^2$ if $\Omega = (0,1)$ in $\mathbf{R}$, we have

Corollary 5.2 *The boundary value problem*

$$\begin{cases} d_2\,v'' + v\,(a_2 - c_2 v) - b_2\tau = 0 \quad \text{in } (0,1), \\ v > 0 \;\text{ and }\; v' \not\equiv 0 \;\text{ in } (0,1), \\ v'(0) = v'(1) = 0, \;\text{ and }\; \tau > 0, \end{cases} \tag{5.2}$$

does not have any solution if $d_2 \geq a_2/\pi^2$.

Our next non-existence result is also quite general. It deals with the "strong competition" case of the shadow system:

$$\begin{cases} d_2\,\Delta v + v\,(a_2 - c_2 v) - b_2\tau = 0 \quad \text{in } \Omega, \\ v > 0 \;\; in \;\; \Omega \;\text{ and }\; \dfrac{\partial v}{\partial \nu} = 0 \;\text{ on }\; \partial\Omega, \\ \displaystyle\int_\Omega \frac{1}{v}\left(a_1 - b_1\frac{\tau}{v} - c_1 v\right) = 0, \;\text{ and }\; \tau > 0. \end{cases} \tag{5.3}$$

Theorem 5.3 *If $A \leq B < C$, then* (5.3) *does not possess any solution for any $d_2 > 0$.*

Remark 5.1. In fact, Theorem 5.3 is sharp, as is readily seen from the existence results – Theorem 2.1 and Proposition 2.2.

Non-existence for the "weak competition" case is more delicate to obtain, and our result only covers the case $n = 1$.

The rest of this section is devoted to the discussion of non-existence of the 1-dimensional shadow system

$$\begin{cases} d_2\, v'' + v\,(a_2 - c_2 v) - b_2\tau = 0 \quad \text{in } (0,1), \\ v > 0 \ \text{ and } \ v' \not\equiv 0 \ \text{ in } (0,1), \\ v'(0) = v'(1) = 0, \\ \displaystyle\int_0^1 \frac{1}{v}\left(a_1 - b_1\frac{\tau}{v} - c_1 v\right) = 0 \ \text{ and } \ \tau > 0. \end{cases} \tag{5.4}$$

First, note that any solution $(v(x), \tau)$ of (5.4) remains a solution of (5.4) after a reflection; i.e. $(v(1-x), \tau)$ is still a solution of (5.4). To relate solutions of (5.4) to that of (2.1), we introduce the following notion. If v' has $k-1$ zeros in $(0,1)$, then we call (v, τ) a solution of (5.4) *of mode k*. (Thus the solutions investigated in Sections 2 – 4, which are solutions to (2.1), are all of mode 1.) Moreover, if $(v(x), \tau)$ is a mode ℓ solution of (5.4), then $(v_k(x), \tau)$, with v_k defined by

$$v_k(x) = \begin{cases} v\left(k\left(x - \dfrac{i}{k}\right)\right) & \text{if } x \in \left[\dfrac{i}{k}, \dfrac{i+1}{k}\right] \text{ and } i \text{ is even,} \\ v\left(k\left(\dfrac{i+1}{k} - x\right)\right) & \text{if } x \in \left[\dfrac{i}{k}, \dfrac{i+1}{k}\right] \text{ and } i \text{ is odd,} \end{cases} \tag{5.5}$$

where $i = 0, 1, 2, \cdots, k-1$, is a solution of mode $k\ell$ of (5.4) except that the constant d_2 must now be replaced by d_2/k^2. Moreover, all solutions of (5.4) can be obtained this way, by the uniqueness of ordinary differential equations. Summing up our discussion above, we have

Proposition 5.4 (i) *If* (2.1) *does not have any solution for* $0 < d_2 < M$, *then* (5.4) *does not have any solutions of mode k for* $0 < d_2 < M/k^2$.
(ii) *If* (2.1) *does not have any solution for* $d_2 > 0$, *then* (5.4) *does not have any solution for* $d_2 > 0$.

We are now ready for our non-existence result for (5.4) in the "weak competition" case.

Theorem 5.5 *Suppose that* $B > C$. *If* $A < B/4 + 3C/4$, *then* (5.4) *does not possess any solution.*

A key ingredient for the proof of Theorem 5.5 is the following *a priori* estimates of solutions.

Proposition 5.6 *Let* (v, τ) *be a solution of* (5.4). *Then* $\tau \leq a_2^2/(4b_2c_2)$. *Furthermore, v must satisfy the following estimates:*

$$|v|_{L^\infty} \leq \frac{3}{4c_2}\left[a_2 - \sqrt{a_2^2 - \frac{16}{3}b_2c_2\tau}\right], \quad \text{if } \tau \leq \frac{3a_2^2}{16b_2c_2}, \tag{5.6}$$

$$|v|_{L^\infty} \leq \frac{1}{2c_2}\left[a_2 + \sqrt{a_2^2 - 4b_2c_2\tau}\right], \quad \text{if } \frac{3a_2^2}{16b_2c_2} \leq \tau \leq \frac{a_2^2}{4b_2c_2}. \tag{5.7}$$

To conclude this section, we shall prove the non-existence results stated in (II) (iii) (2) and (IV) (iii) (2). First, we consider the case $B < C$. Suppose for d_2 small, $(v(\cdot; d_2), \tau(d_2))$ is a solution of (2.1). If $B < A < (B+C)/2$, then (4.1) holds and Theorem 4.1 implies that

$$2\frac{a_2}{c_2} \cdot \frac{A - (\frac{1}{4}B + \frac{3}{4}C)}{B - C} \leq \frac{a_2}{c_2}\frac{B - A}{B - C} \tag{5.8}$$

since $v(0; d_2) \leq v(1; d_2)$. It is easy to see that (5.8) is equivalent to $A \geq (B+C)/2$, a contradiction, and we have therefore established the following result.

Proposition 5.7 *If* $B < A < (B+C)/2$, *then (2.1) does not have solution for* d_2 *sufficiently small.*

Now, we see easily that (II)(iii)(2) in Section 1 follows from Propositions 5.4 (i) and 5.7. Furthermore, observe that for $B < A < (B+C)/2$ there exists $d_2^* > 0$, satisfying (2.2), such that for each integer $k > 0$, (5.4) has a solution of mode k with $d_2 = d_2^*/k^2$, which tends to 0 as $k \to \infty$. (This follows from Propositions 2.2 and 5.4 (i).) In this sense we say that (III)(iii)(2) can not be improved.

For the weak case, the situation is simpler. Theorem 3.1 implies that for d_2 close to a_2/π^2 if (2.1) is to have a solution, then $A \geq B$. Therefore, if $B > A$ then there exists $D \in (0, a_2/\pi^2)$ such that (2.1) has no solution for $D < d_2 < a_2/\pi^2$; i.e. (5.1) does not have solution of mode 1 for $d_2 > D$, in view of Corollary 5.2. From Proposition 5.4 (i) our last result follows.

Proposition 5.8 *If* $A < B$, *then there exists* $D < a_2/\pi^2$ *such that* (5.4) *does not have any solution for* $d_2 > D$.

This settles (IV) (iii) (2) in Section 1.

References

[1] Y. Kan-on, *Stability of singularly perturbed solutions to nonlinear diffusion system arising in population dynamics*, Hiroshima Math. J. **23** (1993), 193-221.

[2] Y. Lou and W.M. Ni, *Diffusion, self-diffusion and cross-diffusion*, J. Differential Equations **131** (1996), 79-131.

[3] Y. Lou and W.M. Ni, *Diffusion vs. cross-diffusion: An elliptic approach*, J. Differential Equations **154** (1999), 157-190.

[4] Y. Lou, W.M. Ni and S. Yotsutani, *On a Limiting System in the Lotka–Volterra Competition with Cross-Diffusion*, Discrete and Continuous Dynamical Systems (2002), to appear.

[5] M. Mimura, Y. Nishiura, A. Tesei and T. Tsujikawa, *Coexistence problem for two competing species models with density-dependent diffusion*, Hiroshima Math. J. **14** (1984), 425-449.

[6] A. Okubo, "Diffusion and Ecological Problems: Mathematical Models," Springer-Verlag, Berlin/New York, 1980.

[7] N. Shigesada, K. Kawasaki and E. Teramoto, *Spatial segregation of interacting species*, J. Theoretical Biology **79** (1979), 83-99.