

Title	GEOMETRIC ASPECTS OF LARGE DEVIATIONS FOR RANDOM WALKS ON A CRYSTAL LATTICE (Geometry of Submanifolds and Related Topics)
Author(s)	Kotani, Motoko; Sunada, Toshikazu
Citation	数理解析研究所講究録 (2001), 1236: 112-114
Issue Date	2001-11
URL	http://hdl.handle.net/2433/41548
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

GEOMETRIC ASPECTS OF LARGE DEVIATIONS FOR RANDOM WALKS ON A CRYSTAL LATTICE

東北大・理学研究科 小谷 元子 (Motoko Kotani)
 東北大・理学研究科 砂田 利一 (Toshikazu Sunada)
 Mathematical Institute, Graduate School of Sciences,
 Tohoku University

The purpose of this talk is to discuss some remarkable relations among convex polyhedra showing up in various circumstances, say Gromov-Hausdorff limits of crystal lattices, homological directions of infinite paths in finite graphs, and the large deviation property (LDP) of random walks on crystal lattices.

Let us start with a simple example. Consider the square lattice \mathbb{Z}^2 as a metric space with the graph-distance d . Given a positive constant ϵ , we have the metric space $(\mathbb{Z}^2, \epsilon d)$ homothetic to (\mathbb{Z}^2, d) . We then ask what the limit $\lim_{\epsilon \downarrow 0} (\mathbb{Z}^2, \epsilon d)$ is as ϵ tends to zero? The answer is, as we may anticipate, the Euclidean 2-space \mathbb{R}^2 with the taxi-cab distance. In this view, it is natural to ask what happens for a more general infinite graph with periodicity. Graphs we would like to consider are *crystal lattices* which are defined to be abelian covering graphs of finite graphs.

Theorem 1. (1) *(a special case of Gromov's result [2]) Let (X, d) be a crystal lattice with the graph-distance. There exists a normed linear space $(L, \|\cdot\|)$ of finite dimension such that*

$$\lim_{\epsilon \downarrow 0} (X, \epsilon d) = (L, d_1),$$

where $d_1(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$.

(2) *The unit ball $\overline{D} = \{\mathbf{x} \in L \mid \|\mathbf{x}\| \leq 1\}$ is a polyhedron.*

Let X_0 be a finite connected graph. We denote the set of all oriented edges by E_0 . Let $c = (e_1, e_2, \dots)$ be an infinite path in X_0 . If the limit

$$\gamma(c) = \lim_{n \rightarrow \infty} \frac{1}{n} (e_1 + \dots + e_n)$$

exists in the 1-chain group $C_1(X_0, \mathbb{R})$, then $\gamma(c)$ is said to be the *homological direction* of c . It is easy to see that $\gamma(c)$ is a 1-cycle so that $\gamma(c) \in H_1(X_0, \mathbb{R})$. To describe the range of homological directions, define the ℓ^1 -norm on $C_1(X_0, \mathbb{R})$ by

$$\left\| \sum_{e \in E_0^+} a_e e \right\|_1 = \sum_{e \in E_0^+} |a_e|,$$

where E_0^+ is an orientation of X_0 .

Theorem 2. *The range of homological directions coincides with*

$$\mathcal{D}_0 = \{\alpha \in H_1(X_0, \mathbb{R}) \mid \|\alpha\|_1 \leq 1\}.$$

Note that \mathcal{D}_0 is a convex polyhedron in $H_1(X_0, \mathbb{R})$, symmetric around the origin.

The convex polyhedron \mathcal{D}_0 is related to the combinatorics of the finite graph X_0 in the following way.

Theorem 3. 1. \mathcal{D}_0 is “rational” in the sense that all extreme points of \mathcal{D}_0 are in $H_1(X_0, \mathbb{Q})$.

2. $\alpha \in H_1(X_0, \mathbb{Q})$ is a vertex of \mathcal{D}_0 if and only if $\alpha = c/\|c\|_1$ for a circuit (simple closed path) c in X_0 .

We shall go back to crystal lattices. To be exact, a crystal lattice X is a connected infinite graph X on which a free abelian group Γ acts as an automorphism group with a finite quotient $X_0 = \Gamma \backslash X$.

A piecewise linear map Φ of X into $\Gamma \otimes \mathbb{R} \cong \mathbb{R}^k$ ($k = \text{rank } \Gamma$) is said to be a *periodic realization* if it satisfies $\Phi(\sigma x) = \Phi(x) + \sigma$. We consider a random walk on X given by a Γ -invariant transition probability p . Given a periodic realization Φ , we put $\xi_n(c) = \Phi(x_n(c))$ for an infinite path c . We thus obtain a $\Gamma \otimes \mathbb{R}$ -valued process $\{\xi_n\}_{n=0}^\infty$.

Now comes a discussion about large deviations principle for the process $\{\xi_n\}$.

Theorem 4. *A large deviation property holds for $\{\xi_n\}$. Namely, there exists $I : \Gamma \otimes \mathbb{R} \rightarrow [0, \infty]$, (which is called entropy function) and satisfies, for $A \subset \Gamma \otimes \mathbb{R}$,*

$$\begin{aligned} -I(\text{int}A) &\leq \liminf_{n \rightarrow \infty} \frac{1}{n} \log P_x\left(\frac{1}{n}\xi_n \in \text{int}A\right) \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log P_x\left(\frac{1}{n}\xi_n \in \overline{A}\right) \leq -I(\overline{A}), \end{aligned}$$

where $I(K) = \inf\{I(\mathbf{z}) \mid \mathbf{z} \in K\}$ for $K \subset \Gamma \otimes \mathbb{R}$.

To give more details, we let

$$\langle \cdot, \cdot \rangle : (\Gamma \otimes \mathbb{R}) \times \text{Hom}(\Gamma, \mathbb{R}) \rightarrow \mathbb{R}$$

be the pairing map between $\Gamma \otimes \mathbb{R}$ and its dual $(\Gamma \otimes \mathbb{R})^* = \text{Hom}(\Gamma, \mathbb{R})$, and let $\rho : H_1(X_0, \mathbb{Z}) \rightarrow \Gamma$ be the surjective homomorphism coming from the covering map $X \rightarrow X_0$.

Lemma 5. *Let $\chi \in \text{Hom}(\Gamma, \mathbb{R})$.*

1. The limit $\lim_{n \rightarrow \infty} \frac{1}{n} \log E(e^{\langle \xi_n, \chi \rangle}) = c(\chi)$ exists. Here $e^{c(\chi)}$ is the maximal positive eigenvalue of the “twisted” transition operator associated with χ .
2. The function c is real analytic, and the hessian of c is strictly positive definite everywhere. Thus the correspondence $\chi \mapsto (\nabla c)(\chi)$ is a diffeomorphism of $\text{Hom}(\Gamma, \mathbb{R})$ onto an open subset U in $\Gamma \otimes \mathbb{R}$.

By using a general recipe in the theory of large deviation (see [1]), with the entropy function $I : \Gamma \otimes \mathbb{R} \rightarrow [0, \infty]$ defined by

$$I(\mathbf{z}) = \sup_{\chi} (\langle \mathbf{z}, \chi \rangle - c(\chi)),$$

we have the LDP for our R.W. It should be noted that the function I assumes finite values on U . We also see

Proposition 6. $\bar{U} = \rho_{\mathbb{R}}(\mathcal{D}_0)$, and hence is independent of p . Moreover

$$\bar{U} = \{\mathbf{x} \in \Gamma \otimes \mathbb{R} \mid \|\mathbf{x}\|_1 \leq 1\},$$

where

$$\|\mathbf{x}\|_1 = \inf \{\|\alpha\|_1 \mid \alpha \in H_1(X_0, \mathbb{R}), \rho_{\mathbb{R}}(\alpha) = \mathbf{x}\}.$$

Therefore \bar{U} is a convex polyhedron, symmetric around the origin, and rational in the sense that the vertices of \bar{U} are in $\Gamma \otimes \mathbb{Q}$.

Finally, we come back to the theorem mentioned in the beginning. As an application of the LDP, we have

Theorem 7.

$$\lim_{\epsilon \downarrow 0} (X, \epsilon d) = (\Gamma \otimes \mathbb{R}, d_1),$$

where $d_1(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_1$.

REFERENCES

- [1] R. S. Ellis, *Large deviations for a general class of random vectors*, Ann. Prob. **12**(1984), 1-12.
- [2] M. Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces*, Birkhäuser, 1999.
- [3] M. Kotani and T. Sunada, *Albanese maps and off diagonal long time asymptotics for the heat kernels*, Comm.Math.Phys. **209**(2000), 633-670.
- [4] M. Kotani and T. Sunada, *Standard realizations of crystal lattices via harmonic maps*, Trans. Amer. Math. Soc. **353**(2000), 1-20

E-mail address: kotani@math.tohoku.ac.jp

E-mail address: sunada@math.tohoku.ac.jp

MATHEMATICAL INSTITUTE, GRADUATE SCHOOL OF SCIENCES, TOHOKU UNIVERSITY, AOBA, SENDAI 980-77, JAPAN