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Author(s)	Murakami, Masahiko
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NONSTANDARD UNIVERSE

MASAHIKO MURAKAMI
DEPARTMENT OF MATHEMATICS
HOSEI UNIVERSITY

ABSTRACT. The nonstandard universes are frameworks of nonstandard analysis. We find sheaf representation for a nonstandard universe. in Theorem 3.7.

1. NONSTANDARD UNIVERSE

Definitions 1.1 (superstructure, base set). Given a set X, we define the iterated power set $V_n(X)$ over X recursively by

$$V_0(X) = X, \quad \text{and} \quad V_{n+1}(X) = V_n(X) \cup \mathcal{P}(V_n(X)).$$

The *superstructure* $V(X)$ is the union $\bigcup_{n<\omega} V_n(X)$. The set X is said to be a *base set* if $\varnothing \notin X$ and each element of X is disjoint from $V(X)$.

Definition 1.2 (nonstandard universe). A *nonstandard universe* is a triple $\langle V(X), V(Y), \star\rangle$ such that:

(1) X and Y are infinite base sets.
(2) (Transfer Principle) The symbol $\star$ is a map from $V(X)$ into $V(Y)$ such that

$$V(X) \models \varphi(a_1, \ldots, a_n) \quad \text{if and only if} \quad V(Y) \models \varphi({}^\star a_1, \ldots, {}^\star a_n)$$

holds for any bounded formula $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in V(X)$.
(3) ${}^\star X = Y$.
(4) For every infinite subset of A of X, $\{{}^\star a \mid a \in A\}$ is a proper subset of ${}^\star A$.

Definitions 1.3 (standard, internal). For $a \in V({}^\star X)$, we call a *standard* if there is an $x \in V(X)$ such that $a = {}^\star x$.

For $a \in V({}^\star X)$, we call a *internal* if there is an $x \in V(X)$ such that $a \in {}^\star x$. We denote by ${}^\star V(X)$ the set of all internal elements in $V({}^\star X)$.

From now on, we denote a nonstandard universe by single ${}^\star V(X)$.

Definitions 1.4 (norm, radius). The *norm (of standardness)* of an internal element a is a cardinal defined by

$$\text{nos}(a) = \min \left\{ |x| \;\middle|\; a \in {}^\star x \right\}.$$

The *radius* of ${}^\star V(X)$ is a cardinal defined by

$$\text{rad}({}^\star V(X)) = \min \left\{ \kappa \;\middle|\; \forall y \in {}^\star V(X) \;\; \text{nos}(y) < \kappa \right\}.$$

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Definition 1.5 (covering number). Let a be an internal element. The *local ultrapower* at a is defined by

$$V(X)[a] = \{ ({}^{*}w)(a) \mid w \in V(X) \text{ and } a \in {}^{*}(\mathrm{dom}(w)) \}.$$

For a subset $E \subseteq {}^{*}V(X)$, we denote

$$V(X)[E] = \bigcup \{ V(X)[s] \mid s \text{ is a finite subset of } E \}.$$

The *covering number* of ${}^{*}V(X)$ is defined by

$$\mathrm{cov}({}^{*}V(X)) = \min \{ |E| \mid E \subseteq {}^{*}V(X) \text{ and } V(X)[E] = {}^{*}V(X) \}.$$

2. Locally atomic complete algebra

Definition 2.1 (regular complete subalgebra). Let $\langle \mathcal{B}, \wedge, \vee, \neg, \mathbf{0}_{\mathcal{B}}, \mathbf{1}_{\mathcal{B}} \rangle$ be a Boolean algebra. A subset $C \subseteq \mathcal{B}$ is said be a *regular complete subalgebra of* $\mathcal{B}$ if C is a complete subalgebra of $\mathcal{B}$ and the inclusion map is also complete.

Notation. Let $\mathcal{B}$ be a Boolean algebra. For a subset $\mathcal{S} \subseteq \mathcal{P}(\mathcal{B})$, we denote

$$\mathcal{S}^{\diamond} = \{ C \in \mathcal{S} \mid C \text{ is a regular complete subalgebra of } \mathcal{B} \}.$$

Definition 2.2 (LCA). A *locally complete algebra (LCA)* is a set Λ of subsets of a Boolean algebra $\mathcal{B}$ satisfying the conditions below.

(1) $\bigcup \Lambda = \mathcal{B}$.
(2) If $S_1, S_2 \in \Lambda$ then $S_1 \cup S_2 \in \Lambda$.
(3) If $S \in \Lambda$ and $T \subseteq S$ then $T \in \Lambda$.
(4) For every $S \in \Lambda$, there is a $C \in \Lambda^{\diamond}$ containing S.

For an LCA Λ, we denote by $\mathcal{B}(\Lambda)$ the Boolean algebra $\bigcup \Lambda$. We call the Boolean algebra $\mathcal{B}(\Lambda)$ the *base Boolean algebra of* Λ.

Definition 2.3 (LACA). An LCA Λ is a *locally atomic complete algebra (LACA)* if every $C \in \Lambda^{\diamond}$ is atomic. We denote the set of atoms of $C \in \Lambda^{\diamond}$ by $\mathrm{Atom}(C)$.

Definition 2.4 (homomorphism). We introduce notation $R``` \mathcal{S} = \{ R``S \mid S \in \mathcal{S} \}$. Let Λ and Ξ be LCAs. A Boolean homomorphism $f\colon \mathcal{B}(\Lambda) \to \mathcal{B}(\Xi)$ is a *pseudo-homomorphism* of LCAs if $f``` \Lambda \subseteq \Xi$. We denotes a pseudo-homomorphism by $f :: \Lambda \to \Xi$. A pseudo-homomorphism $h :: \Lambda \to \Xi$ of LCAs is a *(complete) homomorphism* if $\bigvee h``S = h(\bigvee S)$ for all $S \in \Lambda$. An *embedding* or *monomorphism* $j :: \Lambda \to \Xi$ is an injective homomorphism.

Definition 2.5 (subLCA). A *subLCA of* an LCA Λ is a nonempty subset of Λ which is itself an LCA and the inclusion map is an embedding.

Definition 2.6 (generator). Let Λ be an LCA. A subset $\mathcal{G} \subseteq \Lambda^{\diamond}$ is a *generator of* Λ or $\mathcal{G}$ *generates* Λ if Λ is the only subLCA of Λ containing $\mathcal{G}$.

Definitions 2.7 (radius, covering number, diameter). The *radius* of an LACA Λ is a cardinal defined by

$$\mathrm{rad}(\Lambda) = \min\left\{\kappa \;\middle|\; \forall C \in \Lambda^{\diamond}\ |\mathrm{Atom}(C)| < \kappa\right\}$$

The *covering number* of an LCA Λ is a cardinal defined by

$$\mathrm{cov}(\Lambda) = \min\left\{|\mathcal{G}| \;\middle|\; \mathcal{G} \text{ is a generator of } \Lambda\right\}.$$

The *diameter* of an LACA Λ is a cardinal defined by

$$\mathrm{diam}(\Lambda) = \min\left\{\sum_{C\in\mathcal{G}} |\mathrm{Atom}(C)| \;\middle|\; \mathcal{G} \text{ is a generator of } \Lambda\right\}.$$

Definition 2.8 (direct product). Let I be an index set. The *direct product* $\Lambda^{[I]}$ of the LCA Λ is defined by:

$$\Lambda^{[I]} = \left\{S \subseteq \mathcal{B}(\Lambda)^I \;\middle|\; \bigcup_{g\in S} \mathrm{rng}\, g \in \Lambda\right\}$$

with the pointwise Boolean operations on $\mathcal{B}(\Lambda^{[I]}) = \bigcup \Lambda^{[I]} \subseteq \mathcal{B}(\Lambda)^I$. Then $\Lambda^{[I]}$ is an LCA. The LCA Λ is embedded into $\Lambda^{[I]}$ by the *canonical embedding* $b \mapsto I \times \{b\}$.

Definitions 2.9 (embedding system, direct limit). The embedding system of LCAs is a family of embeddings

$$\mathcal{E} = \{j_d^{d'} :: \Lambda_d \to \Lambda_{d'}\}_{d\le d', d, d' \in D}$$

satisfying $j_{d'}^{d''} \circ j_d^{d'} = j_d^{d''}$ for all $d \le d' \le d''$, where D is an upper direct set. The *direct limit* of $\mathcal{E}$ is $\bigcup\{j_d``\Lambda_d \mid d \in D\}$ where $\{j_d \colon \mathcal{B}(\Lambda_d) \to \mathcal{B}\}_{d\in D}$ is the direct limit of $\{j_d^{d'} : \mathcal{B}(\Lambda_d) \to \mathcal{B}(\Lambda_{d'})\}_{d\le d', d, d'\in D}$ as Boolean algebras.

Definition 2.10 (ultrafilter). Let Λ be an LCA. A subset $\mathcal{U}$ of $\mathcal{B}(\Lambda)$ is an *ultrafilter of an LCA* Λ if it is an ultrafilter of the base Boolean algebra $\mathcal{B}(\Lambda)$.

3. Ultralimit

Definition 3.1 (LACA-valued model). Let Λ be an LACA and let M be a model for a language $\mathcal{L}$. The $\mathcal{B}(\Lambda)$-valued universe of M is defined by

$$M^{\langle\!\langle\Lambda\rangle\!\rangle} = \left\{u \colon M \to \mathcal{B}(\Lambda) \;\middle|\; \begin{array}{l} u(x) \wedge u(y) = \mathbf{0} \text{ for } x \ne y, \\ \mathrm{rng}\, u \in \Lambda,\ \bigvee \mathrm{rng}\, u = \mathbf{1} \end{array}\right\}.$$

For $u \in M^{\langle\!\langle\Lambda\rangle\!\rangle}$, the *support* of u is a subset of M defined by

$$\mathrm{supp}\, u = \{x \in M \mid u(x) \ne \mathbf{0}\}.$$

To each function F of $\mathcal{L}(M)$ and each $u_1, \dots, u_n \in M^{\langle\!\langle\Lambda\rangle\!\rangle}$, we assign a $\check{F}(u_1, \dots, u_n) \in M^{\langle\!\langle\Lambda\rangle\!\rangle}$ by:

$$\check{F}(u_1, \dots, u_n)(y) = \bigvee\left\{\bigwedge_{i=1}^{n} u_i(x_i) \;\middle|\; M \models y = F(x_1, \dots, x_n)\right\} \quad \text{for } y \in M.$$

We regard a constant of $\mathcal{L}(M)$ as a function without any variables. Note that $\bigwedge_{i=1}^{n} u_i(x_i) = \mathbf{1}$ if $n = 0$. To each sentence φ of $\mathcal{L}(M^{\langle\!\langle\Lambda\rangle\!\rangle})$ we assign a truth value $[\![\varphi]\!] \in \overline{\mathcal{B}(\Lambda)}$ by following recursive rules:

$$\begin{aligned} [\![u = v]\!] &= \bigvee \{u(x) \wedge v(x) \mid x \in M\}, \\ [\![\mathrm{R}(u_1, \ldots, u_m)]\!] &= \bigvee \left\{ \bigwedge_{i=1}^{m} u_i(x_i) \,\middle|\, \mathfrak{M} \models \mathrm{R}(x_1, \ldots, x_m) \right\}, \\ [\![\neg\varphi]\!] &= \neg [\![\varphi]\!], \\ [\![\varphi_1 \vee \varphi_2]\!] &= [\![\varphi_1]\!] \vee [\![\varphi_2]\!], \\ [\![\exists x\, \varphi(x)]\!] &= \bigvee \{[\![\varphi(u)]\!] \mid u \in M^{\langle\!\langle\Lambda\rangle\!\rangle}\}, \end{aligned}$$

where R is any predicate in $\mathcal{L}$.

Definition 3.2 (LACA-valued superstructure). Let Λ be an LACA. The *Λ-valued superstructure* of $V(X)$ is defined by

$$\widehat{V}(X)^{\langle\!\langle\Lambda\rangle\!\rangle} = \{u \in V(X)^{\langle\!\langle\Lambda\rangle\!\rangle} \mid \operatorname{supp} u \in V(X)\}.$$

While the truth values range over $\overline{\mathcal{B}(\Lambda)}$ on this definition, we shall see $[\![\varphi]\!]_\Lambda \in \mathcal{B}(\Lambda)$.

Theorem 3.1. *Let $\varphi(x_1, \ldots, x_r)$ be a formula of $\mathcal{L}$ with only $x_1, \ldots, x_r$ free. For $u_1, \ldots, u_r \in M^{\langle\!\langle\Lambda\rangle\!\rangle}$,*

$$(*) \qquad [\![\varphi(u_1, \ldots, u_r)]\!]_\Lambda = \bigvee \left\{ \bigwedge_{i=1}^{r} u_i(x_i) \,\middle|\, \mathfrak{M} \models \varphi(x_1, \ldots, x_r) \right\}.$$

Proof. For φ either "$x_1 = x_2$" or $\mathbf{R}$, (*) holds by definition. If (*) holds for an atomic formula $\varphi(x)$ then, by simple calculus of Boolean algebra, (*) holds for $\varphi(\mathbf{F}(x_1, \ldots, x_n))$. Thus, by induction, (*) holds for φ atomic. Suppose (*) holds for φ, φ_1 and φ_2. Since there is an atomic $C \in \Lambda^{\diamond}$ containing all the ranges of $u_1, \ldots, u_r$ and every range of u_i is a partition of unity except for $\mathbf{0}$,

$$[\![\neg\varphi]\!]_\Lambda = \bigvee \left\{ \bigwedge_{i=1}^{r} u_i(x_i) \,\middle|\, \mathfrak{M} \models \neg\varphi(x_1, \ldots, x_r) \right\}.$$

It is easy to see:

$$[\![\varphi_1 \vee \varphi_2]\!]_\Lambda = \bigvee \left\{ \bigwedge_{i=1}^{r} u_i(x_i) \,\middle|\, \mathfrak{M} \models \varphi_1(x_1, \ldots, x_r) \vee \varphi_2(x_1, \ldots, x_r) \right\}.$$

Since $[\![\varphi(u)]\!]_\Lambda = \bigvee_{x \in M} (u(x) \wedge [\![\varphi(\check{x})]\!]_\Lambda)$, we have $[\![\exists x\, \varphi(x)]\!]_\Lambda = \bigvee_{x \in M} [\![\varphi(\check{x})]\!]_\Lambda$. Therefore (*) holds for $\exists x\, \varphi(x)$. □

Similarly, we shall obtain the superstructure version.

Corollary 3.2. *Let $\varphi(x_1, \ldots, x_r)$ be a formula of $\mathcal{L}_\in$ with only $x_1, \ldots, x_r$ free. For $u_1, \ldots, u_r \in \widehat{V}(X)^{\langle\!\langle\Lambda\rangle\!\rangle}$,*

$$[\![\varphi(u_1, \ldots, u_r)]\!]_\Lambda = \bigvee \left\{ \bigwedge_{i=1}^{r} u_i(x_i) \,\middle|\, V(X) \models \varphi(x_1, \ldots, x_r) \right\}.$$

By the theorem and the corollary above, we have a fundamental property $[\![u = v]\!] \wedge [\![\varphi(u)]\!] \leq [\![\varphi(v)]\!]$. We have just introduced $\mathcal{B}(\Lambda)$-valued model $\mathfrak{M}^{\langle\!\langle\Lambda\rangle\!\rangle} = \langle M^{\langle\!\langle\Lambda\rangle\!\rangle}, \check{R}, \check{F}, \check{c}\rangle$ and $\mathcal{B}(\Lambda)$-valued superstructure $\widehat{V}(X)^{\langle\!\langle\Lambda\rangle\!\rangle}$. We say that a sentence φ of $\mathcal{L}(M^{\langle\!\langle\Lambda\rangle\!\rangle})$ *holds* in $\mathfrak{M}^{\langle\!\langle\Lambda\rangle\!\rangle}$ if $[\![\varphi]\!]_\Lambda = \mathbf{1}$ and that a sentence ψ of $\mathcal{L}_\in(\widehat{V}(X)^{\langle\!\langle\Lambda\rangle\!\rangle})$ *holds* in $\widehat{V}(X)^{\langle\!\langle\Lambda\rangle\!\rangle}$ if $[\![\psi]\!]_\Lambda = \mathbf{1}$. Theorem 3.1 and Corollary 3.2 follow that we consider the values $u(x)$ only for $x \in \operatorname{supp} u$. For $E \subseteq M$, we may regard $E^{\langle\!\langle\Lambda\rangle\!\rangle}$ as a subset of $M^{\langle\!\langle\Lambda\rangle\!\rangle}$ by extending the domain of $u \in E^{\langle\!\langle\Lambda\rangle\!\rangle}$ to M. This means that we define for $u \in E^{\langle\!\langle\Lambda\rangle\!\rangle}$

$$u(x) = \mathbf{0} \quad \text{if } x \notin E.$$

In the superstructure version, if E is a set relative to $V(X)$ then we may assume

$$E^{\langle\!\langle\Lambda\rangle\!\rangle} = \{u \in \widehat{V}(X)^{\langle\!\langle\Lambda\rangle\!\rangle} \mid u \in \check{E} \text{ holds in } \widehat{V}(X)^{\langle\!\langle\Lambda\rangle\!\rangle}\}.$$

Theorem 3.3 (Maximum principle). *Let $\varphi(x)$ be a formula of $\mathcal{L}(M^{\langle\!\langle\Lambda\rangle\!\rangle})$ with only x free. Then there is $u \in M^{\langle\!\langle\Lambda\rangle\!\rangle}$ such that $[\![\varphi(u)]\!]_\Lambda = [\![\exists x\, \varphi(x)]\!]_\Lambda$.*

Proof. Let $\{a_\zeta\}_{\zeta<\alpha}$ be a well-ordering for M. By theorem 3.1, there is $C \in \Lambda^\Diamond$ containing $\{[\![\varphi(\check{x})]\!] \mid x \in M\}$. Putting $b_\zeta = [\![\varphi(\check{a_\zeta})]\!] \wedge \neg \bigvee_{\xi<\zeta} [\![\varphi(\check{a_\xi})]\!]$, we have $\{b_\zeta\}_{\zeta<\alpha} \subseteq C$. Since $\{b_\zeta\}_{\zeta<\alpha}$ is a pairwise disjoint family, we can pick $u \in M^{\langle C\rangle}$ with $u(a_\zeta) \geq b_\zeta$. Then $[\![\varphi(u)]\!] \geq u(a_\zeta) \wedge [\![\varphi(\check{a_\zeta})]\!] \geq b_\zeta$ for any $\zeta < \alpha$. Since $[\![\exists x\, \varphi(x)]\!] = \bigvee_{\zeta<\alpha} \varphi(\check{a_\zeta}) = \bigvee_{\zeta<\alpha} b_\zeta$, we have $[\![\varphi(u)]\!] \geq [\![\exists x\, \varphi(x)]\!]$. □

Corollary 3.4. *Let $\varphi(x)$ be a formula of $\mathcal{L}_\in(\widehat{V}(X)^{\langle\!\langle\Lambda\rangle\!\rangle})$ with only x free and let v be an element of $\widehat{V}(X)^{\langle\!\langle\Lambda\rangle\!\rangle}$. Then there is $u \in \widehat{V}(X)^{\langle\!\langle\Lambda\rangle\!\rangle}$ such that $[\![u \in v \wedge \varphi(u)]\!]_\Lambda = [\![\exists x \in v\, \varphi(x)]\!]_\Lambda$.*

Proof. Since there is n such that $\operatorname{supp} v \subseteq V_{n+1}(X)$,

$$[\![\check{x} \in v]\!] = \bigvee \{v(y) \mid x \in y \in \operatorname{supp} v\} = \mathbf{0} \quad \text{for } x \notin V_n(X).$$

Therefore we can choose u whose support is a subset of $V_n(X)$. □

Definition 3.3 (ultralimit). We denote by $u/\mathcal{U}$ the equivalence class of $u \in M^{\langle\!\langle\Lambda\rangle\!\rangle}$ by the equivalence relation

$$x \sim_\mathcal{U} y \quad \equiv \quad [\![x = y]\!]_\Lambda \in \mathcal{U}.$$

The *ultralimit* $\mathfrak{M}^{\langle\!\langle\Lambda\rangle\!\rangle}/\mathcal{U}$ *of* $\mathfrak{M}$ *modulo* $\mathcal{U}$ *of* Λ is defined by:

$$M^{\langle\!\langle A\rangle\!\rangle}/\mathcal{U} = \{u/\mathcal{U} \mid u \in M^{\langle\!\langle\Lambda\rangle\!\rangle}\}.$$

$$\check{F}/\mathcal{U}(u_1/\mathcal{U}, \ldots, u_n/\mathcal{U}) = (\check{F}(u_1, \ldots, u_n))/\mathcal{U}.$$

$$\mathfrak{M}^{\langle\!\langle\Lambda\rangle\!\rangle}/\mathcal{U} \models \mathbf{R}(u_1/\mathcal{U}, \ldots, u_m/\mathcal{U}) \quad \text{iff} \quad [\![\mathbf{R}(u_1, \ldots, u_m)]\!] \in \mathcal{U}.$$

Definition 3.4 (bounded ultralimit). We denote by $u/\mathcal{U}$ the equivalence class of $u \in \widehat{V}(X)^{\langle\!\langle\Lambda\rangle\!\rangle}$ by the equivalence relation

$$x \sim_\mathcal{U} y \quad \equiv \quad [\![x = y]\!]_\Lambda \in \mathcal{U}.$$

The *bounded ultralimit* $\widehat{V}(X)^{\langle\langle\Lambda\rangle\rangle}/\mathfrak{U}$ *of* $V(X)$ *modulo* $\mathfrak{U}$ *of* Λ is defined by:

$$\widehat{V}(X)^{\langle\langle\Lambda\rangle\rangle}/\mathfrak{U} = \{u/\mathfrak{U} \mid u \in \widehat{V}(X)^{\langle\langle\Lambda\rangle\rangle}\},$$

$$\widehat{V}(X)^{\langle\langle\Lambda\rangle\rangle}/\mathfrak{U} \models u/\mathfrak{U} \in v/\mathfrak{U} \quad \text{iff} \quad [\![u \in v]\!] \in \mathfrak{U}.$$

Theorem 3.5 (Łoś Principle of Ultralimits). *Let* $\varphi(x_1, \ldots, x_r)$ *be a formula of* $\mathfrak{L}$ *with only* $x_1, \ldots, x_r$ *free. For* $u_1, \ldots, u_r \in M^{\langle\langle\Lambda\rangle\rangle}$,

$$\mathfrak{M}^{\langle\langle\Lambda\rangle\rangle}/\mathfrak{U} \models \varphi(u_1/\mathfrak{U}, \ldots, u_r/\mathfrak{U}) \quad \textit{iff} \quad [\![\varphi(u_1, \ldots, u_r)]\!] \in \mathfrak{U}.$$

Proof. The proof proceeds by induction on the complexity of formulae. The only nontrivial step is the case where φ is of the form $\exists x\psi(x)$. Suppose $[\![\exists x\psi(x)]\!] \in \mathfrak{U}$. By the maximal principle (Theorem 3.3), there is u satisfying $[\![\psi(u)]\!] = [\![\exists x\psi(x)]\!]$. Then $\mathfrak{M}^{\langle\langle\Lambda\rangle\rangle} \models \psi(u/\mathfrak{U})$ by the induction assumption. We have thus $\mathfrak{M}^{\langle\langle\Lambda\rangle\rangle} \models \exists x\psi(x)$. Conversely, suppose $\mathfrak{M}^{\langle\langle\Lambda\rangle\rangle} \models \exists x\psi(x)$. Then there is some u such that $\mathfrak{M}^{\langle\langle\Lambda\rangle\rangle} \models \psi(u/\mathfrak{U})$. By the induction assumption, $[\![\exists x\psi(x)]\!] \geq [\![\psi(u/\mathfrak{U})]\!] \in \mathfrak{U}$. □

Corollary 3.6 (Łoś-Mostowski Principle of Bounded Ultralimits).
Let $\varphi(x_1, \ldots, x_r)$ *be a* Δ_0*-formula of* $\mathfrak{L}_\in$ *with only* $x_1, \ldots, x_r$ *free. For* $u_1, \ldots, u_r \in \widehat{V}(X)^{\langle\langle\Lambda\rangle\rangle}$,

$$\widehat{V}(X)^{\langle\langle\Lambda\rangle\rangle}/\mathfrak{U} \models \varphi(u_1/\mathfrak{U}, \ldots, u_r/\mathfrak{U}) \quad \textit{iff} \quad [\![\varphi(u_1, \ldots, u_r)]\!] \in \mathfrak{U}.$$

Proof. The proof is similar to that of Theorem 3.5. The only different part is the if-part of the case where φ is of the form $\exists x \in y\ \psi(x)$. Suppose $[\![\exists x \in u_k\ \psi(x)]\!] \in \mathfrak{U}$. It follows from Corollary 3.4 that there is $u \in \widehat{V}(X)^{\langle\langle\Lambda\rangle\rangle}$ satisfying $[\![u \in u_k \wedge \psi(u)]\!] = [\![\exists x \in u_k\ \psi(x)]\!]$. □

A bounded ultralimit is a pre-nonstandard universe: that satisfies (1),(2) and (3) of Definition 1.2 with Mostowski collapsing.

Definition 3.5 (atlas). An *atlas* is a pair $\langle\Lambda, \mathfrak{U}\rangle$ of an LACA Λ and an ultrafilter of Λ such that $\mathrm{rad}(\widehat{V}(X)^{\langle\langle\Lambda\rangle\rangle}/\mathfrak{U}) = \mathrm{rad}(\Lambda)$ and $\mathrm{cov}(\widehat{V}(X)^{\langle\langle\Lambda\rangle\rangle}/\mathfrak{U}) = \mathrm{cov}(\Lambda)$.

Theorem 3.7 (Sheaf representation Theorem for Nonstandard Universes). *For any nonstandard universe* $^*V(X)$, *there is an atlas* $\langle\Lambda, \mathfrak{U}\rangle$ *such that* $\widehat{V}(X)^{\langle\langle\Lambda\rangle\rangle}/\mathfrak{U}$ *isomorphic to* $^*V(X)$.

We prove the theorem in the next section.

4. Local Ultralimits

We shall see that a homomorphism of LACAs induces an elementary embedding of ultralimits and a bounded elementary embedding of bounded ultralimits. Let $h :: \Lambda \to \Xi$ be a homomorphism. The *induced map* $h_*: \mathfrak{M}^{\langle\langle\Lambda\rangle\rangle} \to \mathfrak{M}^{\langle\langle\Xi\rangle\rangle}$ is defined by $h_*(u) = h \circ u$. Then we have the lemma below.

Lemma 4.1. *Let* $\varphi(x_1, \ldots, x_r)$ *be a formula of* $\mathfrak{L}$ *with only* $x_1, \ldots, x_r$ *free. For* $u_1, \ldots, u_r \in M^{\langle\langle\Lambda\rangle\rangle}$

$$[\![\varphi(h_*(u_1), \ldots, h_*(u_r))]\!]_\Xi = h([\![\varphi(u_1, \ldots, u_r)]\!]_\Lambda).$$

Proof. There is $C \in \Lambda^{\Diamond}$ containing all the ranges of u_k. Since $h \restriction C$ is complete, we have from Theorem 3.1

$$\bigvee\left\{ \bigwedge_{i=1}^{r} h(u_i(x_i)) \;\middle|\; \mathfrak{M} \models \varphi(x_1,\ldots,x_r) \right\} = h(\bigvee\left\{ \bigwedge_{i=1}^{r} u_i(x_i) \;\middle|\; \mathfrak{M} \models \varphi(x_1,\ldots,x_r) \right\}$$

$$[\![\varphi(h_*(u_1),\ldots,h_*(u_r))]\!]_{\Xi} = h([\![\varphi(u_1,\ldots,u_r)]\!]_{\Lambda}).$$

We have thus proved the lemma. □

For $u \in \widehat{V}(X)^{\langle\!\langle\Lambda\rangle\!\rangle}$, since $\operatorname{supp}(h \circ u) \subseteq \operatorname{supp} u$, we can define the *induced map* $h_*\colon \widehat{V}(X)^{\langle\!\langle\Lambda\rangle\!\rangle} \to \widehat{V}(X)^{\langle\!\langle\Lambda\rangle\!\rangle}$ similarly.

Corollary 4.2. *Let $\varphi(x_1,\ldots,x_r)$ be a formula of $\mathcal{L}_{\in}$ with only $x_1,\ldots,x_r$ free. For $u_1,\ldots,u_r \in \widehat{V}(X)^{\langle\!\langle\Lambda\rangle\!\rangle}$*

$$[\![\varphi(h_*(u_1),\ldots,h_*(u_r))]\!]_{\Xi} = h([\![\varphi(u_1,\ldots,u_r)]\!]_{\Lambda}).$$

Proof. Using Corollary 3.2, we see the proof is similar to that of Lemma 4.1. □

Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters of Λ and Ξ, respectively. Suppose $h^{-1}``\mathcal{V} = \mathcal{U}$. Then we have from Lemma 4.1 or from Corollary 4.2

$$[\![u = u']\!]_{\Lambda} \in \mathcal{U} \quad \text{iff} \quad [\![h_*(u) = h_*(u')]\!]_{\Xi} \in \mathcal{V}.$$

Therefore we can define the injection $h_*\colon M^{\langle\!\langle\Lambda\rangle\!\rangle}/\mathcal{U} \to M^{\langle\!\langle\Xi\rangle\!\rangle}/\mathcal{V}$,denoted by same h_*, by $h_*(u/\mathcal{U}) = h_*(u)/\mathcal{V}$. Since $\operatorname{supp} h_*(u) \subseteq \operatorname{supp} u$, we can define the injection $h_*\colon \widehat{V}(X)^{\langle\!\langle\Lambda\rangle\!\rangle}/\mathcal{U} \to \widehat{V}(X)^{\langle\!\langle\Xi\rangle\!\rangle}/\mathcal{V}$ similarly.

Lemma 4.3. *The injection h_* is an elementary embedding of $M^{\langle\!\langle\Lambda\rangle\!\rangle}/\mathcal{U}$ into $M^{\langle\!\langle\Xi\rangle\!\rangle}/\mathcal{V}$.*

Proof. Let $\varphi(x_1,\ldots,x_r)$ be a formula of $\mathcal{L}$ with only $x_1,\ldots,x_r$ free. From Theorem 3.5, we have for $u_1,\ldots,u_r \in M^{\langle\!\langle\Lambda\rangle\!\rangle}$

$$\begin{aligned} h([\![\varphi(u_1,\ldots,u_r)]\!]_{\Lambda}) \in \mathcal{V} \quad &\text{iff} \quad [\![\varphi(u_1,\ldots,u_r)]\!]_{\Lambda} \in h^{-1}``\mathcal{V} \\ [\![\varphi(h_*(u_1),\ldots,h_*(u_r))]\!]_{\Xi} \in \mathcal{V} \quad &\text{iff} \quad [\![\varphi(u_1,\ldots,u_r)]\!]_{\Lambda} \in \mathcal{U} \\ \mathfrak{M}^{\langle\!\langle\Xi\rangle\!\rangle}/\mathcal{V} \models \varphi(h_*(u_1/\mathcal{U}),\ldots,h_*(u_r/\mathcal{U})) \quad &\text{iff} \quad \mathfrak{M}^{\langle\!\langle\Lambda\rangle\!\rangle}/\mathcal{U} \models \varphi(u_1,\ldots,u_r). \end{aligned}$$

□

Corollary 4.4. *The injection h_* is a bounded elementary embedding of $\widehat{V}(X)^{\langle\!\langle\Lambda\rangle\!\rangle}/\mathcal{U}$ into $\widehat{V}(X)^{\langle\!\langle\Xi\rangle\!\rangle}/\mathcal{V}$.*

Proof. Using Corollary 3.6, we see the proof is similar to that of Lemma 4.3. □

Let I be a set relative to $V(X)$. We shall find a one-to-one correspondence between $\mathcal{P}(I)^{\langle\!\langle\Lambda\rangle\!\rangle}$ and $\mathcal{B}(\Lambda^{[I]})$. Note that $\mathcal{P}(I)^{\langle\!\langle\Lambda\rangle\!\rangle}$ is the set of "the subsets of $\check{I}$ in $\widehat{V}(X)^{\langle\!\langle\Lambda\rangle\!\rangle}$". For $A \in \mathcal{P}(I)^{\langle\!\langle\Lambda\rangle\!\rangle}$, there is $C \in \Lambda^{\Diamond}$ such that $\operatorname{rng} A \subseteq C$. Define $g\colon I \to \mathcal{B}(\Lambda)$ by $g(i) = [\![\check{i} \in A]\!]_{\Lambda}$. Then we have $\operatorname{rng} g \subseteq C$ and $g \in \mathcal{B}(\Lambda^{[I]})$. Conversely, for $g \in \mathcal{B}(\Lambda^{[I]})$, there is $C \in \Lambda^{\Diamond}$ such that $\operatorname{rng} g \subseteq C$. Define $A\colon \mathcal{P}(I) \to C$ by

$$A(x) = \bigwedge_{i \in I} \operatorname{sg}_x(i, g(i)), \quad \text{where } \operatorname{sg}_x(i,b) = \begin{cases} b & \text{if } i \in x, \\ \neg b & \text{if } i \in I \setminus x. \end{cases}$$

Since C is completely distributive, we have $A \in \mathcal{P}(I)^{\langle\!\langle \Lambda \rangle\!\rangle}$. Suppose $g(i) = [\![\check{i} \in A]\!]_\Lambda$ and $g'(i) = [\![\check{i} \in A']\!]_\Lambda$. Then we see $(g \wedge g')(i) = [\![\check{i} \in A \cap A']\!]_\Lambda$ and $(\neg g)(i) = [\![\check{i} \in \check{I} \setminus A]\!]_\Lambda$. In the context above, the relation $g(i) = [\![\check{i} \in A]\!]_\Lambda$ sets up a one-to-one correspondence between $\mathcal{P}(I)^{\langle\!\langle \Lambda \rangle\!\rangle}$ and $\mathcal{B}(\Lambda^{[I]})$ as Boolean algebras. From now on, we identify $\mathcal{P}(I)^{\langle\!\langle \Lambda \rangle\!\rangle}$ with $\mathcal{B}(\Lambda^{[I]})$.

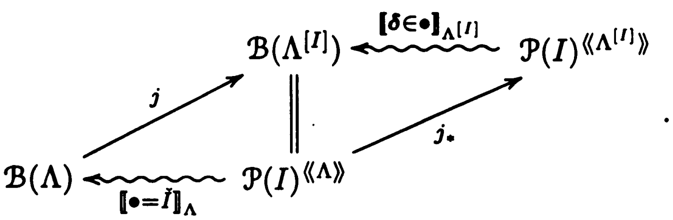
We shall define the special element $\boldsymbol{\delta} \in I^{\langle\!\langle \Lambda^{[I]} \rangle\!\rangle} \subseteq \widehat{V}(X)^{\langle\!\langle \Lambda^{[I]} \rangle\!\rangle}$ by

$$\boldsymbol{\delta}(x)(i) = \begin{cases} \mathbf{1} & \text{if } x = i, \\ \mathbf{0} & \text{if } x \neq i. \end{cases}$$

We call the $\boldsymbol{\delta}$ *diagonal element* of I on Λ. Let $j :: \Lambda \to \Lambda^{[I]}$ be the canonical embedding. Then j is also a Boolean monomorphism of $\mathcal{B}(\Lambda)$ into $\mathcal{P}(I)^{\langle\!\langle \Lambda \rangle\!\rangle}$. The diagonal element $\boldsymbol{\delta}$ has following properties.

Lemma 4.5. *The following statements hold.*

(1) $[\![j(b) = \check{I}]\!]_\Lambda = b$ *for every* $b \in \mathcal{B}(\Lambda)$.
(2) $[\![\boldsymbol{\delta} \in j_*(g)]\!]_{\Lambda^{[I]}} = g$ *for every* $g \in \mathcal{P}(I)^{\langle\!\langle \Lambda \rangle\!\rangle}$.



Proof. Since $[\![\check{i} \in j(b)]\!]_\Lambda = j(b)(i) = b$ for all $i \in I$, $[\![j(b) = \check{I}]\!]_\Lambda = [\![j(b) \supseteq \check{I}]\!]_\Lambda = \bigwedge_{i \in I} [\![\check{i} \in j(b)]\!] = b$. From the definition of $\boldsymbol{\delta}$, it is clear that $[\![\boldsymbol{\delta} = \check{i}]\!]_{\Lambda^{[I]}}(i) = \boldsymbol{\delta}(i)(i) = \mathbf{1}$. Then we have $[\![\boldsymbol{\delta} \in j_*(g)]\!]_{\Lambda^{[I]}}(i) = [\![\check{i} \in j_*(g)]\!]_{\Lambda^{[I]}}(i) = [\![\check{i} \in g]\!]_\Lambda = g(i)$. $\square$

Theorem 4.6. *For any* $v \in \widehat{V}(X)^{\langle\!\langle \Lambda^{[I]} \rangle\!\rangle}$, *there is a map* $w \colon \check{I} \to (\operatorname{supp} v)^{\checkmark}$ *in* $\widehat{V}(X)^{\langle\!\langle \Lambda \rangle\!\rangle}$ *such that* $v = j_*(w)(\boldsymbol{\delta})$ *holds in* $\widehat{V}(X)^{\langle\!\langle \Lambda \rangle\!\rangle}$.

Proof. Since $\operatorname{rng} v \in \Lambda^{[I]}$, $\bigcup_{g \in \operatorname{rng} v} \operatorname{rng} g \in \Lambda$. Therefore we can define $w \colon (\operatorname{supp} v)^I \to \mathcal{B}(\Lambda)$ by

$$w(s) = \bigwedge_{i \in I} v\big(s(i)\big)(i).$$

Then we get w as required. First, we show $w \in \widehat{V}(X)^{\langle\!\langle \Lambda \rangle\!\rangle}$. If $s \neq s'$, then there is $i_0 \in I$ such that $s(i_0) \neq s'(i_0)$. Since $\operatorname{rng} v$ is pairwise disjoint, we have

$$w(s) \wedge w(s') \leq v\big(s(i_0)\big)(i_0) \wedge v\big(s'(i_0)\big)(i_0) = \mathbf{0}.$$

There is $C \in \Lambda^\diamond$ such that $\bigcup_{g \in \operatorname{rng} v} \operatorname{rng} g \subseteq C$. Then we have $\operatorname{rng} w \subseteq C$ and then

$$\bigvee_{s \in (\operatorname{supp} v)^I} w(s) = \bigvee_{s \in (\operatorname{supp} v)^I} \bigwedge_{i \in I} v\big(s(i)\big)(i) = \bigwedge_{i \in I} \bigvee_{y \in \operatorname{supp} v} v(y)(i) = \mathbf{1}.$$

We have thus shown $w \in \widehat{V}(X)^{\langle\!\langle \Lambda \rangle\!\rangle}$. For each $i \in I$, since $w(s) \leq v(s(i))(i)$ holds for every $s \in (\operatorname{supp} v)^I$, we have

$$\begin{aligned}
[\![v = j_*(w)(\boldsymbol{\delta})]\!]_{\Lambda^{[I]}}(i) &= [\![v = j_*(w)(\check{i})]\!]_{\Lambda^{[I]}}(i) \\
&= \left(\bigvee \{v(y) \wedge j(w(s)) \wedge \check{i}(x) \mid y = s(x)\}\right)(i) \\
&= \bigvee_{s \in (\operatorname{supp} v)^I} (v(s(i))(i) \wedge w(s)) \\
&= \bigvee_{s \in (\operatorname{supp} v)^I} w(s) = \mathbf{1}.
\end{aligned}$$

We have thus proved the theorem. □

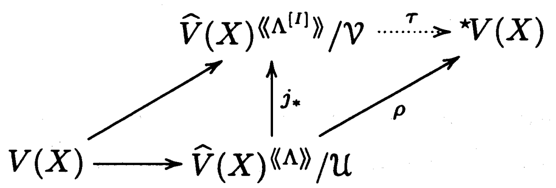
Let $\mathcal{U}$ be an ultrafilter of an LACA Λ. A *local ultralimit* $\rho\colon \widehat{V}(X)^{\langle\!\langle \Lambda \rangle\!\rangle}/\mathcal{U} \to {}^*V(X)$ is a bounded elementary embedding satisfying $\rho(\check{x}/\mathcal{U}) = {}^*x$ for every $x \in V(X)$.

Theorem 4.7 (Local Ultralimit Theorem). *Let* $\rho\colon \widehat{V}(X)^{\langle\!\langle \Lambda \rangle\!\rangle}/\mathcal{U} \to {}^*V(X)$ *be a local ultralimit and let* p *be an internal element of* ${}^*V(X)$. *Then there is a local ultralimit* $\tau\colon \widehat{V}(X)^{\langle\!\langle \Lambda^{[I]} \rangle\!\rangle}/\mathcal{V} \to {}^*V(X)$ *such that the following conditions hold.*

(i) *The index set* I *is a set relative to* $V(X)$ *and* $|I| = \operatorname{nos}(p)$.

(ii) *Let* $j :: \Lambda \to \Lambda^{[I]}$ *be the canonical embedding. Then* $\mathcal{U} = j^{-1}``\mathcal{V}$ *and* $\rho = \tau \circ j_*$.

(iii) *The submodel* $\operatorname{rng}\tau$ *of* ${}^*V(X)$ *is the minimal bounded elementary submodel of* ${}^*V(X)$ *that contains* $\{p\} \cup \operatorname{rng}\rho$.



$, p \in \operatorname{rng}\tau.$

Proof. Let I be a set relative to $V(X)$ such that $p \in {}^*I$ and $|I| = \operatorname{nos}(p)$. We have identified $\mathcal{B}(\Lambda^{[I]})$ with $\mathcal{P}(I)^{\langle\!\langle \Lambda \rangle\!\rangle}$. Define $\mathcal{V} \subseteq \mathcal{B}(\Lambda^{[I]})$ by

$$g \in \mathcal{V} \quad \text{iff} \quad p \in \rho(g/\mathcal{U}).$$

Then $\mathcal{V}$ is an ultrafilter of $\Lambda^{[I]}$. Let b be an element of $\mathcal{U}$. From (1) of Lemma 4.5, $\rho(j(b)/\mathcal{U})$ coincides ${}^*I$. Then we have $j(b) \in \mathcal{V}$ from the definition of $\mathcal{V}$. Since $\mathcal{U}$ and $\mathcal{V}$ are maximal filters, we obtain $j^{-1}``\mathcal{V} = \mathcal{U}$. Let $\varphi(x_1, \dots, x_r)$ be a Δ_0-formula of $\mathcal{L}_\in$ with only $x_1, \dots, x_r$ free. Let $v_1, \dots, v_r$ be elements of $\widehat{V}(X)^{\langle\!\langle \Lambda^{[I]} \rangle\!\rangle}$. By Theorem 4.6, there are maps $w_1, \dots, w_r$ from $\check{I}$ in $\widehat{V}(X)^{\langle\!\langle \Lambda \rangle\!\rangle}$ such that $v_k = j_*(w_k)(\boldsymbol{\delta})$ hold, where $\boldsymbol{\delta}$ is the diagonal element of I on Λ. Putting $g_0 = \{i \in \check{I} \mid \varphi(w_1(i), \dots, w_r(i))\}$ in $\widehat{V}(X)^{\langle\!\langle \Lambda \rangle\!\rangle}$, we have from (2) of Lemma 4.5

$$\begin{aligned}
g_0 &= [\![\boldsymbol{\delta} \in j_*(g_0)]\!]_{\Lambda^{[I]}} \\
&= [\![\boldsymbol{\delta} \in j_*(\{i \in \check{I} \mid \varphi(w_1(i), \dots, w_r(i))\})]\!]_{\Lambda^{[I]}} \\
&= [\![\boldsymbol{\delta} \in \{i \in \check{I} \mid \varphi(j_*(w_1)(i), \dots, j_*(w_r)(i))\}]\!]_{\Lambda^{[I]}} \\
&= [\![\varphi(j_*(w_1)(\boldsymbol{\delta}), \dots, j_*(w_r)(\boldsymbol{\delta}))]\!]_{\Lambda^{[I]}} \\
&= [\![\varphi(v_1, \dots, v_r)]\!]_{\Lambda^{[I]}}
\end{aligned}$$

and we have

$$\widehat{V}(X)^{\langle\!\langle \Lambda \rangle\!\rangle}/\mathcal{U} \models g_0/\mathcal{U} = \{i \in \check{I}/\mathcal{U} \mid \varphi((w_1/\mathcal{U})(i), \ldots, (w_r/\mathcal{U})(i))\}$$
$$\rho(g_0/\mathcal{U}) = \{i \in {}^*I \mid \varphi(\rho(w_1/\mathcal{U})(i), \ldots, \rho(w_r/\mathcal{U})(i))\}.$$

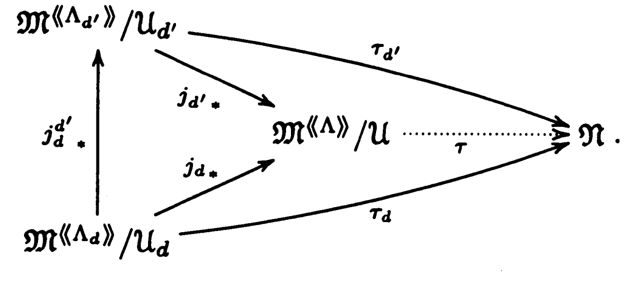
By the definition of $\mathcal{V}$, we obtain

$$g_0 \in \mathcal{V} \quad \text{iff} \quad p \in \rho(g_0/\mathcal{U})$$
$$[\![\varphi(v_1, \ldots, v_r)]\!]_{\Lambda^{[I]}} \in \mathcal{V} \quad \text{iff} \quad \varphi(\rho(w_1/\mathcal{U})(p), \ldots, \rho(w_r/\mathcal{U})(p)).$$

The case $\varphi(x_1, x_2) \equiv$ "$x_1 = x_2$" enables us to define the operation $v/\mathcal{V} \mapsto \rho(w/\mathcal{U})(p)$ where $v = j_*(w)(\boldsymbol{\delta})$ holds in $\widehat{V}(X)^{\langle\!\langle \Lambda^{[I]} \rangle\!\rangle}$. Thus, defining $\tau\colon \widehat{V}(X)^{\langle\!\langle \Lambda^{[I]} \rangle\!\rangle}/\mathcal{V} \to {}^*V(X)$ by $\tau(v/\mathcal{V}) = \rho(w/\mathcal{U})(p)$ where $v = j_*(w)(\boldsymbol{\delta})$ holds in $\widehat{V}(X)^{\langle\!\langle \Lambda^{[I]} \rangle\!\rangle}$, we get τ as required. In fact, it is clear in the preceding context that τ is an bounded elementary embedding of $\widehat{V}(X)^{\langle\!\langle \Lambda^{[I]} \rangle\!\rangle}/\mathcal{V}$ into ${}^*V(X)$. Let ι be the identity map on I, then we see $\tau(j_*(\check{\iota}/\mathcal{U})(\boldsymbol{\delta}/\mathcal{V})) = \rho(\check{\iota}/\mathcal{U})(p) = {}^*\iota(p) = p$. For $u/\mathcal{U} \in \widehat{V}(X)^{\langle\!\langle \Lambda \rangle\!\rangle}/\mathcal{U}$, let $\tilde{u}$ be the constant map from $\check{I}$ onto $\{u\}$ in $\widehat{V}(X)^{\langle\!\langle \Lambda \rangle\!\rangle}$, then we have $\tau(j_*(u/\mathcal{U})) = \rho(\tilde{u}/\mathcal{U})(p) = \rho(u/\mathcal{U})$. Suppose a bounded elementary submodel W of ${}^*V(X)$ contains $\{p\} \cup \operatorname{rng} \rho$. From the definition of τ, $\tau(v/\mathcal{V}) = \rho(w/\mathcal{U})(p) \in W$ for some $w/\mathcal{U} \in \widehat{V}(X)^{\langle\!\langle \Lambda \rangle\!\rangle}/\mathcal{U}$. Therefore $\operatorname{rng} \tau$ is the minimum. We have completed the proof of Theorem 4.7. □

Let $\{j_d^{d'} :: \Lambda_d \to \Lambda_{d'}\}_{d \le d', d, d' \in D}$ be an embedding system of LACAs with direct limit $\{j_d :: \Lambda_d \to \Lambda\}_{d \in D}$. Let $\mathcal{U}$ be an ultrafilter of Λ, then each $\mathcal{U}_d = {j_d}^{-1}{}``\mathcal{U}$ is an ultrafilter of Λ_d.

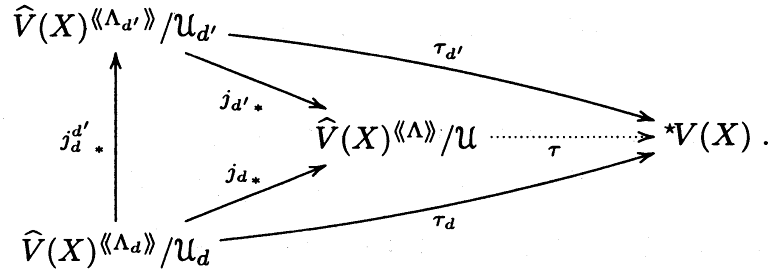
Theorem 4.8 (Elementary Net Theorem of Ultralimits). *Let $\mathfrak{M}$ and $\mathfrak{N}$ be models for $\mathcal{L}$. Suppose there are elementary embeddings $\tau_d\colon \mathfrak{M}^{\langle\!\langle \Lambda_d \rangle\!\rangle}/\mathcal{U}_d \to \mathfrak{N}$ satisfying the condition $\tau_d = \tau_{d'} \circ {j_d^{d'}}_*$ for $d \le d'$. Then there is an elementary embedding $\tau\colon \mathfrak{M}^{\langle\!\langle \Lambda \rangle\!\rangle}/\mathcal{U} \to \mathfrak{N}$ such that $\tau_d = \tau \circ j_{d*}$ for $d \in D$.*



Proof. Let v be an element of $\mathfrak{M}^{\langle\!\langle \Lambda \rangle\!\rangle}$. Since $\operatorname{rng} v \in \Lambda = \{j_d{}``S \mid d \in D \text{ and } S \in \Lambda_d\}$ from the definition of direct limits, there is $u \in \mathfrak{M}^{\langle\!\langle \Lambda_d \rangle\!\rangle}$ such that $v = j_{d*}(u)$. Therefore defining $\tau(v/\mathcal{U}) = \tau_d(u/\mathcal{U}_d)$ where $v = j_{d*}(u)$, we get τ as required. Let $\varphi(x_1, \ldots, x_r)$ be a formula of $\mathcal{L}$ with only $x_1, \ldots, x_r$ free and let $v_1, \ldots, v_r$ be elements $M^{\langle\!\langle \Lambda \rangle\!\rangle}$. Then there are $d \in D$ and $u_1, \ldots, u_r \in M^{\langle\!\langle \Lambda_d \rangle\!\rangle}$ such that $v_k = j_{d*}(u_k)$. We conclude as below.

$$\mathfrak{M}^{\langle\!\langle \Lambda_d \rangle\!\rangle}/\mathcal{U}_d \models \varphi(u_1/\mathcal{U}_d, \ldots, u_r/\mathcal{U}_d) \quad \text{iff} \quad \mathfrak{N} \models \varphi(\tau_d(u_1/\mathcal{U}), \ldots, \tau_d(u_r/\mathcal{U}))$$
$$\mathfrak{M}^{\langle\!\langle \Lambda \rangle\!\rangle}/\mathcal{U} \models \varphi(v_1/\mathcal{U}, \ldots, v_r/\mathcal{U}) \quad \text{iff} \quad \mathfrak{N} \models \varphi(\tau(v_1/\mathcal{U}), \ldots, \tau(v_r/\mathcal{U})).$$

Theorem 4.9 (Bounded Elementary Net Theorem of Bounded Ultralimits). *Suppose there are local ultralimits* $\tau_d\colon \widehat{V}(X)^{\langle\!\langle \Lambda_d \rangle\!\rangle}/\mathfrak{U}_d \to {}^*V(X)$ *satisfying the condition* $\tau_d = \tau_{d'} \circ j^{d'}_{d\,*}$ *for* $d \leq d'$. *Then there is a local ultralimit* $\tau\colon \widehat{V}(X)^{\langle\!\langle \Lambda \rangle\!\rangle}/\mathfrak{U} \to {}^*V(X)$ *such that* $\tau_d = \tau \circ j_{d*}$ *for* $d \in D$.



Proof. Similar to the proof of Theorem 4.8. □

We call the pair $\langle \Lambda, \mathfrak{U} \rangle$ in Theorem 4.8 or Theorem 4.7 the *direct limit* of $\{\langle \Lambda_d, \mathfrak{U}_d \rangle\}_{d \leq d', d, d' \in D}$.

Proof of Theorem 3.7 Let $\{p_\zeta\}_{\zeta<\kappa}$ be a sequence in ${}^*V(X)$ with $\kappa = \mathrm{cov}({}^*V(X))$. We define local ultralimits $\{\rho_\zeta\colon \widehat{V}^{\langle\!\langle \Lambda_\zeta \rangle\!\rangle}/\mathfrak{U}_\zeta \to {}^*V(X)\}_{\zeta<\kappa}$ of ${}^*V(X)$ by:

$\Lambda_0 = \mathcal{P}(\{\mathbf{0}, \mathbf{1}\})$, $\mathfrak{U}_0 = \{\mathbf{1}\}$.

$\Lambda_{\zeta+1} = \Lambda^{[I_\zeta]}$, where $|I_\zeta| = \mathrm{nos}(p_\zeta)$, $p_\zeta \in \mathrm{rng}\,\rho_{\zeta+1}$ in Theorem 4.7.

$\langle \Lambda_\lambda, \mathfrak{U}_\lambda \rangle$ is the direct limit of $\{\langle \Lambda_\zeta, \mathfrak{U}_\zeta \rangle_{\zeta<\lambda}\}$ in Theorem 4.9.

Then the direct limit $\langle \Lambda, \mathfrak{U} \rangle$ of $\{\langle \Lambda_\zeta, \mathfrak{U}_\zeta \rangle\}_{\zeta<\kappa}$ is an atlas of ${}^*V(X)$.

□

References

[1] C. Chang and J. Keisler, ***Model Theory***, 3rd ed, North-Holland, Amsterdam, (1990).

[2] R. Mansfield, *The theory of Boolean ultrapowers*, ***Ann. Math. Logic***, 2 (1971) 297–323.

[3] M. Ozawa, *Forcing in nonstandard analysis*, ***Annals of Pure and Applied Logic***, 68 (1994), 263–297.

[4] M. Murakami, *Standardization principle of Nonstandard universes*, ***Journal of Symbolic Logic***, 64, 4(1999), 1645–1655.

E-mail address: `muramasa@ms.u-tokyo.ac.jp`