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# A Catalog for Prediction-Preserving Reducibility with Membership Queries on Formal Languages

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## Abstract

In this paper, we present *prediction-preserving reducibility with membership queries* on formal languages, in particular, simple CFGs and finite unions of regular pattern languages.

## 1 Introduction

The task of predicting the classification of a new example is frequently discussed from the viewpoints of both *passive* and *active* settings. In a passive setting, the examples are all chosen independently according to a fixed but unknown probability distribution, and the learner has no control over selection of examples [7, 9]. In an active setting, on the other hand, the learner is allowed to ask about particular examples, that is, the learner makes *membership queries*, before the new example to predict is given to the learner [1, 4].

Pitt and Warmuth [9] have been formalized the model of prediction and a reduction between two prediction problems that preserves polynomial-time predictability called a *prediction-preserving reduction* in a passive setting. Angluin and Kharitonov [4] have extended to the model and the reduction in an active setting. The reduction is called a *prediction-preserving reduction with membership queries* or *pwm-reduction* for short.

Concerned with language learning, we can design a polynomial-time algorithm to predict deterministic finite automata (DFAs) in an active setting [1], while predicting DFAs is as hard as computing certain apparently hard cryptographic predicates in a passive setting [7]. Furthermore, predicting nondeterministic finite automaton (NFAs) and unrestricted context-free grammars (CFGs) is also hard under the same cryptographic assumptions in an active setting [4]. Here, the *cryptographic assumptions* denote the intractability of inverting RSA encryption, recognizing quadratic residues and factoring Blum integers.

In this paper, we present the prediction-preserving reducibility with membership queries on formal languages. First, we deal with the following simple CFGs: *linear grammars* ( $\mathcal{L}_{\text{linear}}$ ), *right-linear grammars* ( $\mathcal{L}_{\text{right-linear}}$ ), and *left-linear grammars* ( $\mathcal{L}_{\text{left-linear}}$ ), *k-bounded CFGs* [2] ( $\mathcal{L}_{k\text{-bounded-CFG}}$ ), the *sequential CFGs* [5] ( $\mathcal{L}_{\text{sqCFG}}$ ), the *properly sequential CFGs* ( $\mathcal{L}_{\text{psqCFG}}$ ), and the *k-CFGs* ( $\mathcal{L}_{k\text{-CFG}}$ ). Next, we introduce a *regular pattern* [10] that is a string of variables and constants of which each variable occurs at most once. A *regular pattern language* is a language consisting of strings as instances of a given pattern. Then, we deal with the *bounded finite union* of regular pattern languages by some constant  $m$  ( $\mathcal{L}_{\cup m\text{RP}}$ ) and the *unbounded finite union* of regular pattern languages ( $\mathcal{L}_{\text{URP}}$ ) [11].

By using pwm-reduction, we present the following results:  $\mathcal{L}_{\text{NFA}} \cong_{\text{pwm}} \mathcal{L}_{\text{right-linear}}$ ,  $\mathcal{L}_{\text{NFA}} \cong_{\text{pwm}} \mathcal{L}_{\text{left-linear}}$ ,  $\mathcal{L}_{\text{NFA}} \trianglelefteq_{\text{pwm}} \mathcal{L}_{\text{linear}}$ ,  $\mathcal{L}_{\text{NFA}} \trianglelefteq_{\text{pwm}} \mathcal{L}_{k\text{-bounded-CFG}}$  for each  $k \geq 1$ ,  $\mathcal{L}_{\text{DNF}} \trianglelefteq_{\text{pwm}} \mathcal{L}_{\text{psqCFG}}$ ,

$\mathcal{L}_{\text{DNF}} \trianglelefteq_{\text{pwm}} \mathcal{L}_{k\text{-CFG}}$  for each  $k \geq 1$ ,  $\mathcal{L}_{\text{DNF}} \trianglelefteq_{\text{pwm}} \mathcal{L}_{\text{sqCFG}}$ ,  $\mathcal{L}_{\cup_m \text{RP}} \trianglelefteq_{\text{pwm}} \mathcal{L}_{\text{DFA}}$  for each  $m \geq 0$ , and  $\mathcal{L}_{\text{DNF}} \trianglelefteq_{\text{pwm}} \mathcal{L}_{\text{URP}}$ . Hence, we obtain the following predictability with membership queries.

1.  $\mathcal{L}_{\text{linear}}$ ,  $\mathcal{L}_{\text{right-linear}}$ ,  $\mathcal{L}_{\text{left-linear}}$  and  $\mathcal{L}_{k\text{-bounded-CFG}}$  ( $k \geq 1$ ) are not polynomial-time predictable with membership queries under the cryptographic assumptions.
2. If  $\mathcal{L}_{\text{sqCFG}}$ ,  $\mathcal{L}_{\text{psqCFG}}$ ,  $\mathcal{L}_{k\text{-CFG}}$  and  $\mathcal{L}_{\text{URP}}$  are polynomial-time predictable with membership queries, then so are DNF formulas.
3.  $\mathcal{L}_{\cup_m \text{RP}}$  ( $m \geq 0$ ) is polynomial-time predictable with membership queries.

## 2 Preliminaries

### 2.1 Simple CFGs and finite unions of regular pattern languages

Let  $\Sigma$  and  $N$  be two non-empty finite sets of symbols such that  $\Sigma \cap N = \emptyset$ . A *production*  $A \rightarrow \alpha$  on  $\Sigma$  and  $N$  is an association from a nonterminal  $A \in N$  to a string  $\alpha \in (N \cup \Sigma)^*$ . A *context-free grammar* (CFG, for short) is a 4-tuple  $(N, \Sigma, P, S)$ , where  $S \in N$  is the distinguished *start symbol* and  $P$  is a finite set of productions on  $\Sigma$  and  $N$ . Symbols in  $N$  are said to be *nonterminals*, while symbols in  $\Sigma$  *terminals*. Then:

- A *linear grammar* is a CFG  $G = (N, \Sigma, P, S)$  such that each production in  $P$  is of the forms  $T \rightarrow wUv$  or  $T \rightarrow w$  for  $T, U \in N$  and  $w, v \in \Sigma^*$ . In particular, a *right-linear* (resp., *left-linear*) *grammar* if it is a linear grammar such that each production is of the forms either  $T \rightarrow wU$  (resp.,  $T \rightarrow Uw$ ) or  $T \rightarrow w$  for  $T, U \in N$  and  $w \in \Sigma^*$ .
- A CFG  $G = (N, \Sigma, P, S)$  is called *k-bounded* [2] if the right-hand side of each production in  $P$  has at most  $k$  nonterminals.
- A CFG  $G = (N, \Sigma, P, S)$  is called *sequential* [5] if the nonterminals in  $N$  are labeled  $S = T_1, \dots, T_n$  such that, for each production  $T_i \rightarrow w$ ,  $w \in (\Sigma \cup \{T_j \mid i \leq j \leq n\})^*$ . In particular, A sequential CFG satisfying that, for each production  $T_i \rightarrow w$ ,  $w \in (\Sigma \cup \{T_j \mid i < j \leq n\})^*$  is called *properly sequential*.
- A CFG  $G = (N, \Sigma, P, S)$  is called a *k-CFG* if  $|N| \leq k$ .

Let  $G$  be a CFG  $(N, \Sigma, S, P)$  and  $\alpha$  and  $\beta$  be strings in  $(\Sigma \cup N)^*$ . We denote  $\alpha \Rightarrow_G \beta$  if there exist  $\alpha_1, \alpha_2 \in (\Sigma \cup N)^*$  such that  $\alpha = \alpha_1 X \alpha_2$ ,  $\beta = \alpha_1 \gamma \alpha_2$  and  $X \rightarrow \gamma \in P$ . We extend the relation  $\Rightarrow_G$  to the reflexive and transitive closure  $\Rightarrow_G^*$ . For a nonterminal  $A \in N$ , the *language*  $L_G(A)$  of  $A$  is the set  $\{w \in \Sigma^* \mid A \Rightarrow_G^* w\}$ . The *language*  $L(G)$  of  $G$  just refers to  $L_G(S)$ .

Let  $X$  be a countable set of *variables* such that  $\Sigma \cap X = \emptyset$ . A *pattern* is an element of  $(\Sigma \cup X)^+$ . A pattern  $\pi$  is called *regular* [10] if each variable in  $\pi$  occurs at most once. A *substitution* is a homomorphism from patterns to patterns that maps each symbol  $a \in \Sigma$  to itself. A substitution that maps some variables to empty string  $\varepsilon$  is called an  $\varepsilon$ -*substitution*. In this paper, we do not deal with  $\varepsilon$ -substitution. By  $\pi\theta$ , we denote the image of a pattern by a substitution  $\theta$ . For a pattern  $\pi$ , the *pattern language*  $L(\pi)$  is the set  $\{w \in \Sigma^+ \mid w = \pi\theta \text{ for some substitution } \theta\}$ .

### 2.2 Prediction-preserving reduction with membership queries

Let  $U$  denote  $\Sigma^*$ . If  $w$  is a string,  $|w|$  denotes its length. For each  $n > 0$ ,  $U^{[n]} = \{w \in U \mid |w| \leq n\}$ . A *representation of concepts*  $\mathcal{L}$  is any subset of  $U \times U$ . We interpret an element  $\langle u, w \rangle$  of  $U \times U$  as consisting a *concept representation*  $u$  and an *example*  $w$ . The example  $w$

is a member of a concept  $u$  if  $\langle u, w \rangle \in \mathcal{L}$ . Furthermore, define the *concept represented by  $u$*  as  $\kappa_{\mathcal{L}}(u) = \{w \mid \langle u, w \rangle \in \mathcal{L}\}$ . The *set of concepts represented by  $\mathcal{L}$*  is  $\{\kappa_{\mathcal{L}}(u) \mid u \in U\}$ .

To represent CFGs, we define the class  $\mathcal{L}_{\text{CFG}}$  as the set of pairs  $\langle u, w \rangle$  such that  $u$  encodes a CFG  $G$  and  $w \in L(G)$ . Also we define the classes  $\mathcal{L}_{\text{linear}}$ ,  $\mathcal{L}_{\text{right-linear}}$ ,  $\mathcal{L}_{\text{left-linear}}$ ,  $\mathcal{L}_{k\text{-bounded-CFG}}$ ,  $\mathcal{L}_{\text{seqCFG}}$ ,  $\mathcal{L}_{\text{psqCFG}}$ , and  $\mathcal{L}_{k\text{-CFG}}$ , corresponding to a linear grammar, right-linear grammar, left-linear grammar,  $k$ -bounded CFG, sequential CFG, properly sequential CFG, and  $k$ -CFG, respectively, as similar.

To represents regular pattern languages, the class  $\mathcal{L}_{\text{RP}}$  denotes the set of pairs  $\langle u, w \rangle$  such that  $u$  encodes a regular pattern  $\pi$  and  $w$  is in the concept represented by  $c$  iff  $w \in L(\pi)$ . Furthermore, the class  $\mathcal{L}_{\cup m \text{RP}}$  of a *bounded* finite union of regular pattern languages [11] denotes the set of pairs  $\langle u, w \rangle$  such that  $u$  encodes  $m$  and a finite set  $\pi_1, \dots, \pi_m$  of  $m$  regular patterns and  $w$  is in the concept represented by  $c$  iff  $w \in L(\pi_i)$  for at least one  $\pi_i$ . Similarly, the class  $\mathcal{L}_{\cup r \text{RP}}$  of an *unbounded* finite union of regular pattern languages [11] denotes the set of pairs  $\langle u, w \rangle$  such that  $u$  encodes a finite set  $\pi_1, \dots, \pi_r$  of regular patterns and  $w$  is in the concept represented by  $c$  iff  $w \in L(\pi_i)$  for at least one  $\pi_i$ .

Additionally, we introduce the following classes. The class  $\mathcal{L}_{\text{DFA}}$  (*resp.*,  $\mathcal{L}_{\text{NFA}}$ ) denotes the set of pairs  $\langle u, w \rangle$  such that  $u$  encodes a DFA (*resp.*, NFA)  $M$  and  $M$  accepts  $w$ . The class  $\mathcal{L}_{\text{DNF}}$  denotes the set of pairs  $\langle u, w \rangle$  such that  $u$  encodes a positive integer  $n$  and a DNF formula  $d$  over  $n$  Boolean variables  $x_1, \dots, x_n$  such that  $|w| = n$  ( $w = w_1 \dots w_n$ ) and the assignment  $x_i = w_i$  ( $1 \leq i \leq n$ ) satisfies  $d$ .

In order to obtain the results of this paper, it is sufficient to introduce the following concept of *prediction-preserving reducibility* [4, 9]. Hence, we omit the formal definitions of the prediction algorithm and the predictability. See the papers [4, 7, 9] for more detail.

Angluin and Kharitonov [4] have extended the prediction-preserving reduction by Pitt and Warmuth [9] with membership queries. It also a tool for showing hardness results of predicting some classes of representations with membership queries.

**Definition 1 (Angluin & Kharitonov [4])** Let  $\mathcal{L}_i$  be a representation of concepts over domain  $U_i$  ( $i = 1, 2$ ). We say that *predicting  $\mathcal{L}_1$  reduces to predicting  $\mathcal{L}_2$  with membership queries* (*pwm-reduces*, for short), denoted by  $\mathcal{L}_1 \trianglelefteq_{\text{pwm}} \mathcal{L}_2$ , if there exist an *instance mapping*  $f : \mathbf{N} \times \mathbf{N} \times U_1 \rightarrow U_2$ , a *concept mapping*  $g : \mathbf{N} \times \mathbf{N} \times \mathcal{L}_1 \rightarrow \mathcal{L}_2$ , and a *query mapping*  $h : \mathbf{N} \times \mathbf{N} \times U_2 \rightarrow U_1 \cup \{\top, \perp\}$  satisfying the following conditions.

1. For each  $x \in U_1^{[n]}$  and  $u \in \mathcal{L}_1^{[s]}$ ,  $x \in \kappa_{\mathcal{L}_1}(u)$  iff  $f(n, s, x) \in \kappa_{\mathcal{L}_2}(g(n, s, u))$ .
2.  $f$  is computable in time bounded by a polynomial in  $n$ ,  $s$  and  $|x|$ .
3. The size of  $g(n, s, u)$  is bounded by a polynomial in  $n$ ,  $s$  and  $|u|$ .
4. For each  $x' \in U_2$  and  $u \in \mathcal{L}_1^{[s]}$ , if  $h(n, s, x') = \top$  then  $x' \in \kappa_{\mathcal{L}_2}(g(n, s, u))$ ; if  $h(n, s, x') = \perp$  then  $x' \notin \kappa_{\mathcal{L}_2}(g(n, s, u))$ ; if  $h(n, s, x') = x \in U_1$ , then it holds that  $x' \in \kappa_{\mathcal{L}_2}(g(n, s, u))$  iff  $x \in \kappa_{\mathcal{L}_1}(u)$ .
5.  $h$  is computable in time bounded by a polynomial in  $n$ ,  $s$  and  $|x'|$ .

If  $\mathcal{L}_1 \trianglelefteq_{\text{pwm}} \mathcal{L}_2$  and  $\mathcal{L}_2 \trianglelefteq_{\text{pwm}} \mathcal{L}_1$ , we denote  $\mathcal{L}_1 \cong_{\text{pwm}} \mathcal{L}_2$ .

The following theorem is useful for showing the predictability or the hardness of predictability of the class of representations.

**Theorem 1 (Angluin & Kharitonov [4])** *Let  $\mathcal{L}_1$  and  $\mathcal{L}_2$  be representations of concepts and suppose that  $\mathcal{L}_1 \trianglelefteq_{\text{pwm}} \mathcal{L}_2$ . If  $\mathcal{L}_2$  is polynomial-time predictable with membership queries, then so is  $\mathcal{L}_1$ . If  $\mathcal{L}_1$  is not polynomial-time predictable with membership queries, then neither is  $\mathcal{L}_2$ .*

It is well known the following statements:

1.  $\mathcal{L}_{\text{DFA}}$  is polynomial-time predictable with membership queries [1].
2.  $\mathcal{L}_{\text{NFA}}$  and  $\mathcal{L}_{\text{CFG}}$  are not polynomial-time predictable with membership queries under the cryptographic assumptions [4].
3.  $\mathcal{L}_{\text{DNF}}$  is either polynomial-time predictable or not polynomial-time predictable with membership queries, if there exist one-way functions that cannot be inverted by polynomial-sized circuits [4].

### 3 PWM-Reducibility

In this section, we fix  $f$ ,  $g$  and  $h$  to an instance mapping, a concept mapping, and a query mapping. Furthermore, the parameters  $n$  and  $s$  denote the bounds of examples and representations, respectively.

#### 3.1 Simple CFGs

First note that, by using the equivalent transformation between a NFA and a right-linear grammar [6] as a concept mapping, we observe that  $\mathcal{L}_{\text{NFA}} \cong_{\text{pwm}} \mathcal{L}_{\text{right-linear}}$ . Furthermore, for a CFG  $G = (N, \Sigma, P, S)$ , let  $G^R$  be a CFG  $(N, \Sigma, P', S)$  such that  $T \rightarrow w^R \in P'$  for each  $T \rightarrow w \in P$ . Here,  $R$  denotes the reversal of a word. For a right-linear (*resp.*, left-linear) grammar  $G$ , construct  $f$ ,  $g$  and  $h$  as  $f(n, s, e) = e^R$ ,  $g(n, s, G) = G^R$  and  $h(n, s, e') = e'^R$ . Then, it is obvious that  $\mathcal{L}_{\text{right-linear}} \trianglelefteq_{\text{pwm}} \mathcal{L}_{\text{left-linear}}$  (*resp.*,  $\mathcal{L}_{\text{left-linear}} \trianglelefteq_{\text{pwm}} \mathcal{L}_{\text{right-linear}}$ ), so it holds that  $\mathcal{L}_{\text{right-linear}} \cong_{\text{pwm}} \mathcal{L}_{\text{left-linear}}$ . Summary:

**Theorem 2**  $\mathcal{L}_{\text{NFA}} \cong_{\text{pwm}} \mathcal{L}$  for  $\mathcal{L} \in \{\mathcal{L}_{\text{right-linear}}, \mathcal{L}_{\text{left-linear}}\}$ . Also,  $\mathcal{L}_{\text{NFA}} \trianglelefteq_{\text{pwm}} \mathcal{L}_{\text{linear}}$  and  $\mathcal{L}_{\text{NFA}} \trianglelefteq_{\text{pwm}} \mathcal{L}_{k\text{-bounded-CFG}}$  for each  $k \geq 1$ .

**Theorem 3**  $\mathcal{L}_{\text{DNF}} \trianglelefteq_{\text{pwm}} \mathcal{L}$  for  $\mathcal{L} \in \{\mathcal{L}_{\text{psqCFG}}, \mathcal{L}_{\text{sqCFG}}\}$ .

*Proof.* Let  $d$  be a DNF formula  $t_1 \vee \dots \vee t_m$  over  $n$  Boolean variables  $x_1, \dots, x_n$ . First, we define  $w_i^j$  ( $1 \leq i \leq n, 1 \leq j \leq m$ ) as  $w_i^j = 1$  if  $t_j$  contains  $x_i$ ;  $w_i^j = 0$  if  $t_j$  contains  $\bar{x}_i$ ;  $w_i^j = T$  otherwise. Then, construct  $f$ ,  $g$  and  $h$  as follows:

$$\begin{aligned} f(n, s, e) &= e, \\ g(n, s, d) &= (\{S, T\}, \{0, 1\}, S, \{S \rightarrow w_1^1 \dots w_n^1 \mid \dots \mid w_1^m \dots w_n^m, T \rightarrow 0 \mid 1\}), \\ h(n, s, e') &= e'. \end{aligned}$$

It is obvious that the above  $f$ ,  $g$  and  $h$  satisfy the conditions of Definition 1.  $\square$

**Theorem 4** For each  $k \geq 1$ ,  $\mathcal{L}_{\text{DNF}} \trianglelefteq_{\text{pwm}} \mathcal{L}_{k\text{-CFG}}$ .

*Proof.* Theorem 3 implies that  $\mathcal{L}_{\text{DNF}} \trianglelefteq_{\text{pwm}} \mathcal{L}_{k\text{-CFG}}$  for each  $k \geq 2$ . Then, it is sufficient to show that  $\mathcal{L}_{\text{DNF}} \trianglelefteq_{\text{pwm}} \mathcal{L}_{1\text{-CFG}}$ . Let  $d = t_1 \vee \dots \vee t_m$  be a DNF formula over  $n$  Boolean variables  $x_1, \dots, x_n$ . Then, define  $w_i^j$  ( $1 \leq i \leq n, 1 \leq j \leq m$ ) as  $w_i^j = 1$  if  $t_j$  contains  $x_i$ ;  $w_i^j = 0$  if  $t_j$  contains  $\bar{x}_i$ ;  $w_i^j = S$  otherwise. Then, construct  $f$ ,  $g$  and  $h$  as follows:

$$\begin{aligned} f(n, s, e) &= e, \\ g(n, s, d) &= (\{S\}, \{0, 1\}, S, \{S \rightarrow 0 \mid 1 \mid w_1^1 \dots w_n^1 \mid \dots \mid w_1^m \dots w_n^m \mid \underbrace{S \dots S}_{n+1} \mid \dots \mid \underbrace{S \dots S}_{2n}\}), \\ h(n, s, e') &= \begin{cases} e' & \text{if } |e'| = n, \\ \perp & \text{if } 1 < |e'| < n, \\ \top & \text{if } |e'| = 1 \text{ or } |e'| > n. \end{cases} \end{aligned}$$

For each  $e \in \{0, 1\}^n$ , it holds that  $e$  satisfies  $d$  iff  $S \Rightarrow_{g(n,s,d)}^* f(n, s, e)$ . Furthermore, for each  $e' \in \{0, 1\}^*$ , if  $h(n, s, e') = \perp$ , then  $S \not\Rightarrow_{g(n,s,d)}^* e'$ , because  $g(n, s, d)$  generates no strings of length more than 1 and less than  $n$ ; If  $h(n, s, e') = e'$ , then it holds that  $S \Rightarrow_{g(n,s,d)}^* e'$  iff  $h(n, s, e')$  satisfies  $d$ .

Finally, consider the case that  $h(n, s, e') = \top$ . It is sufficient to show that, for each  $k \geq 1$ , it holds that  $S \Rightarrow_{g(n,s,d)}^* \underbrace{S \cdots S}_{kn+m}$  for each  $m$  ( $1 \leq m \leq n-1$ ). If  $k = 1$ , then, by the definition, it holds that  $S \Rightarrow_{g(n,s,d)}^* \underbrace{S \cdots S}_{n+m}$  for each  $m$  ( $1 \leq m \leq n-1$ ). Suppose that it holds that, for some  $k \geq 1$ ,  $S \Rightarrow_{g(n,s,d)}^* \underbrace{S \cdots S}_{n+m}$  for each  $m$  ( $1 \leq m \leq n-1$ ). Then, it holds that  $S \Rightarrow_{g(n,s,d)}^* \underbrace{S \cdots S}_{kn+m} S \Rightarrow_{g(n,s,d)}^* \underbrace{S \cdots S}_{kn+(m-1)} \underbrace{S \cdots S}_{n+1} = \underbrace{S \cdots S}_{(k+1)n+m}$  for each  $m$  ( $1 \leq m \leq n-1$ ). Hence,  $g(n, s, d)$  generates all strings of length more than  $n$ , so if  $h(n, s, e') = \top$ , then  $S \Rightarrow_{g(n,s,d)}^* e'$ .  $\square$

### 3.2 Finite union of regular pattern languages

Since each regular pattern language is regular [10], we can construct a DFA  $M_\pi$  such that  $L(M_\pi) = L(\pi)$  for each regular pattern  $\pi$  as follows: Suppose that  $\pi$  is a regular pattern of the form  $\pi = x_0 \alpha_1 x_1 \alpha_2 \cdots x_{n-1} \alpha_n x_n$ , where  $x_i \in X$  and  $\alpha_i = a_1^i a_2^i \cdots a_{m_i}^i \in \Sigma^+$ . Then, the corresponding DFA  $M_\pi$  of  $\pi$  is a DFA  $(\Sigma, Q, \delta, q_0, F)$  such that:

1.  $Q = \{q_0, p_1^1, \dots, p_{m_1}^1, q_1, p_1^2, \dots, p_{m_2}^2, q_2, \dots, q_{n-1}, p_1^n, \dots, p_{m_n}^n, q_n\}$  and  $F = \{q_n\}$ ,
2.  $\delta(q_i, a) = p_1^{i+1}$  and  $\delta(q_n, a) = q_n$  for each  $a \in \Sigma$  and  $0 \leq i \leq n-1$ ,
3.  $\delta(p_j^i, a_j^i) = p_{j+1}^i$  and  $\delta(p_{m_i}^i, a_{m_i}^i) = q_i$  for each  $1 \leq i \leq n$  and  $1 \leq j \leq m_i - 1$ ,
4.  $\delta(p_j^i, a) = p_1^i$  for each  $a \in \Sigma$  such that  $a \neq a_j^i$ .

It is obvious that  $|M_\pi|$  is bounded by a polynomial in  $|\pi|$ . We can easily shown that  $\mathcal{L}_{\text{RP}} \leq_{\text{pwm}} \mathcal{L}_{\text{DFA}}$  by constructing  $f$ ,  $g$  and  $h$  for each regular pattern  $\pi$  as  $f(n, s, e) = e$ ,  $g(n, s, \pi) = M_\pi$  and  $h(n, s, e') = e'$ . Then,  $\mathcal{L}_{\text{RP}}$  is polynomial-time predictable with membership queries [8].

**Theorem 5** For each  $m \geq 0$ ,  $\mathcal{L}_{\cup_m \text{RP}} \leq_{\text{pwm}} \mathcal{L}_{\text{DFA}}$ .

*Proof.* Let  $\pi_1, \dots, \pi_m$  be  $m$  regular patterns. Also let  $M_{\pi_i} = (Q_i, \Sigma, \delta_i, q_0^i, F_i)$  be the corresponding DFA of  $\pi_i$ . First, construct a DFA  $M_{\pi_1, \dots, \pi_m} = (Q_1 \times \cdots \times Q_m, \Sigma, \delta, (q_0^1, \dots, q_0^m), F_1 \times \cdots \times F_m)$  such that  $\delta((q_1, \dots, q_m), a) = (p_1, \dots, p_m)$  iff  $\delta_i(q_i, a) = p_i$  for each  $i$  ( $1 \leq i \leq m$ ). Then, construct  $f$ ,  $g$  and  $h$  as  $f(n, s, e) = e$ ,  $g(n, s, \{\pi_1, \dots, \pi_m\}) = M_{\pi_1, \dots, \pi_m}$  and  $h(n, s, e') = e'$ . Note that the size of  $g(n, s, \{\pi_1, \dots, \pi_m\})$  is bounded by a polynomial in  $s$ , i.e.,  $O(s^m)$ . It is obvious that  $L(\pi_1) \cup \cdots \cup L(\pi_m) = L(M_{\pi_1, \dots, \pi_m})$ , which implies that  $\mathcal{L}_{\cup_m \text{RP}} \leq_{\text{pwm}} \mathcal{L}_{\text{DFA}}$ .  $\square$

**Theorem 6**  $\mathcal{L}_{\text{DNF}} \leq_{\text{pwm}} \mathcal{L}_{\text{URP}}$ .

*Proof.* Let  $d = t_1 \vee \cdots \vee t_m$  be a DNF formula over  $n$  Boolean variables  $x_1, \dots, x_n$ . First, for each term  $t_j$  ( $1 \leq j \leq m$ ), construct a regular pattern  $\pi_j = \pi_1^j \cdots \pi_n^j$  as  $\pi_i^j = 1$  if  $t_j$  contains  $x_i$ ;  $\pi_i^j = 0$  if  $t_j$  contains  $\bar{x}_i$ ;  $\pi_i^j = x_i^j$  otherwise. Furthermore, let  $\pi$  be a regular pattern  $x_1 \cdots x_n x_{n+1}$ . Then, construct  $f$ ,  $g$  and  $h$  as follows:

$$\begin{aligned} f(n, s, e) &= e, \\ g(n, s, d) &= \{\pi_1, \dots, \pi_m, \pi\}, \\ h(n, s, e') &= \begin{cases} e' & \text{if } |e'| = n, \\ \top & \text{if } |e'| > n, \\ \perp & \text{if } |e'| < n. \end{cases} \end{aligned}$$

For each  $e' \in \{0,1\}^*$ , we can check the properties of  $h$  in Definition 1 as follows. Since  $L(\pi) = \{w \in \{0,1\}^* \mid |w| \geq n+1\}$ , if  $h(n,s,e') = \top$ , then  $e' \in \kappa_{\mathcal{L}_{\text{URP}}}(g(n,s,d))$ . On the other hand, since  $|\pi_j| = n$  ( $1 \leq j \leq m$ ) and  $|\pi| = n+1$ ,  $\kappa_{\mathcal{L}_{\text{URP}}}(g(n,s,d))$  contains no strings of length  $< n$ . So, if  $h(n,s,e') = \perp$ , then  $e' \notin \kappa_{\mathcal{L}_{\text{URP}}}(g(n,s,d))$ . Otherwise, i.e. if  $h(n,s,e') = e'$ , note that  $|e'| = n$ , so  $e' \notin L(\pi)$ . Then,  $e' \in L(\pi_1) \cup \dots \cup L(\pi_m)$ . Thus, there exists an index  $i$  ( $1 \leq i \leq m$ ) such that  $e' \in L(\pi_i)$  iff  $e'$  is obtained by replacing the variables in  $\pi_i$  with 0 or 1, which is corresponding to a truth assignment satisfying  $t_i$ . Hence,  $e' \in \kappa_{\mathcal{L}_{\text{URP}}}(g(n,s,d))$  iff  $e' \in \kappa_{\mathcal{L}_{\text{DNF}}}(d)$ .

Furthermore, for each  $e \in \{0,1\}^n$ ,  $e \in \kappa_{\mathcal{L}_{\text{DNF}}}(d)$  iff  $f(n,s,e) \in \kappa_{\mathcal{L}_{\text{URP}}}(g(n,s,d))$ . Hence, it holds that  $\mathcal{L}_{\text{DNF}} \leq_{\text{pwm}} \mathcal{L}_{\text{URP}}$ .  $\square$

Shinohara and Arimura [11] have discussed the inferability of  $\mathcal{L}_{\text{URP}}$  and  $\mathcal{L}_{\text{URP}}$  in the framework of inductive inference. They have shown that  $\mathcal{L}_{\text{URP}}$  is inferable from positive data, whereas  $\mathcal{L}_{\text{URP}}$  is not. In contrast, by Theorem 5 and 6,  $\mathcal{L}_{\text{URP}}$  is polynomial-time predictable with membership queries, whereas  $\mathcal{L}_{\text{URP}}$  is not polynomial-time predictable with membership queries if neither are DNF formulas.

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