Title	Properties on relative normality, their absolute embeddings and related problems (Set Theoretic and Geometric Topology and Its Applications)
Author(s)	Kawaguchi, Shinji; Sokei, Ryoken
Citation	数理解析研究所講究録 (2005), 1419: 18-31
Issue Date	2005-02
URL	http://hdl.handle.net/2433/26298
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録 1419 巻 2005 年 18-31

Properties on relative normality, their absolute embeddings and related problems

筑波大学大学院数理物質科学研究科　川口　慎二 (Shinji Kawaguchi)
Graduate School of Pure and Applied Sciences, University of Tsukuba

東京学芸大学附属高等学校　祖慶　良謙 (Ryoken Sokei)
Tokyo Gakugei University Senior High School

1. Introduction

This note is a summary of [20]. Throughout this paper all spaces are assumed to be T_1 topological spaces and the symbol γ denotes an infinite cardinal.

The notions of relative normality and relative paracompactness are central in the study of relative topological properties which has been posed by Arhangel'skiĭ and Genedi [4], and also in the subsequent articles [2] and [3] by Arhangel'skiĭ.

Let X be a space and Y a subspace of X. A subspace Y is said to be *normal* (respectively, *strongly normal*) *in* X if for each disjoint closed subsets F_0, F_1 of X (respectively, of Y), there exist disjoint open subsets G_0, G_1 of X such that $F_i \cap Y \subset G_i$ for each $i = 0, 1$. A subspace Y is said to be 1- (respectively, 2-) *paracompact in* X if for every open cover $\mathcal{U}$ of X, there exists a collection $\mathcal{V}$ of open subsets of X with $X = \bigcup \mathcal{V}$ (respectively, $Y \subset \bigcup \mathcal{V}$) such that $\mathcal{V}$ is a partial refinement of $\mathcal{U}$ and $\mathcal{V}$ is locally finite at each point of Y. Here, $\mathcal{V}$ is said to be a *partial refinement* of $\mathcal{U}$ if for each $V \in \mathcal{V}$, there exists a $U \in \mathcal{U}$ containing V. The term "2-paracompact" is often simply said "paracompact". In the definition of 2-paracompactness of Y in X above, when we replace "open cover of X" by "collection of open subsets of X with $Y \subset \bigcup \mathcal{U}$", Y is said to be *Aull-paracompact in* X ([3], [5]). Each of 1-paracompactness and Aull-paracompactness of Y in X clearly implies 2-paracompactness of Y in X. Note that 1-paracompactness coincides with α-paracompactness defined by Aull [6] for a closed subset of a regular space [23]. See also Theorem 3.11.

On the other hand, it is natural to define the following two relative notions; a subspace Y of a space X is said to be γ-*collectionwise normal* (respectively, *strongly* γ-*collectionwise normal*) *in* X if for every discrete collection $\{E_\alpha \mid \alpha < \gamma\}$ of closed subsets of X (respectively, of Y), there is a pairwise disjoint collection $\{U_\alpha \mid \alpha < \gamma\}$ of open subsets of X such that $E_\alpha \cap Y \subset U_\alpha$ (respectively, $E_\alpha \subset U_\alpha$) for every $\alpha < \gamma$ ([18]). Clearly, Y being ω-collectionwise normal (respectively, strongly ω-collectionwise normal) in X is equivalent to that Y is normal (respectively, strongly normal) in X. When Y is γ-collectionwise normal (respectively, strongly γ-collectionwise normal) in X for every γ, we say Y is *collectionwise normal* (respectively, *strongly collectionwise normal*) *in* X; we see that collectionwise normality (respectively, strongly collectionwise normality) of Y in X is equal to being $\alpha - CN$ (respectively, $\gamma - CN$) of Y in the sense of Aull [7].

2. Preliminaries and 1- or 2- (collectionwise) normality of a subspace in a space

At first, we recall some preliminary notions and facts.

Let Y be a subspace of a space X. As is known, Y is said to be C^*- (respectively, C-) *embedded in* X if every bounded real-valued (respectively, real-valued) continuous function on Y is continuously extended over X. A subspace Y is said to be P^γ- (respectively, P-) embedded in X if every continuous γ-separable (respectively, continuous) pseudo-metric on Y is continuously extended over X ([1]); a pseudo-metric d on Y is γ-*separable* if the pseudo-metric space (Y, d) has weight $\leq \gamma$. It is known that P^ω-embedding is equal to C-embedding ([1]).

By [2], Y is said to be *weakly* C-*embedded* in X if for every real-valued continuous function f on Y there exists a real-valued function on X which is an extension of f and continuous at each point of Y. By [18], Y is said to be *weakly* P^γ- (respectively, *weakly* P-) embedded in X if every continuous γ-separable (respectively, continuous) pseudo-metric on Y is extended to a pseudo-metric on X which is continuous at each point of $Y \times Y$. Weak P^ω-embedding is equal to weak C-embedding ([18]). A space X is γ-*collectionwise normal* if for every discrete collection $\{E_\alpha \mid \alpha < \gamma\}$ of closed subsets there exists a pairwise disjoint collection $\{G_\alpha \mid \alpha < \gamma\}$ of open subsets such that $E_\alpha \subset G_\alpha$ for each $\alpha < \gamma$. Clearly, X is collectionwise normal if X is γ-collectionwise normal for every γ.

A subspace Y is said to be *Hausdorff in* X if for every two distinct points y_1, y_2 of Y, there are disjoint open subsets U_1, U_2 of X such that $y_i \in U_i$ for each $i = 0, 1$. A subspace Y is said to be *strongly regular in* X if for each $x \in X$ and each closed subset F of X with $x \notin F$, there exist disjoint open subsets U, V of X such that $x \in U$ and $F \cap Y \subset V$.

Let X_Y denote the space obtained from the space X, with the topology generated by a subbase $\{U \mid U \text{ is open in } X \text{ or } U \subset X \setminus Y\}$. Hence, points in $X \setminus Y$ are isolated and Y is closed in X_Y. Moreover, X and X_Y generate the same topology on Y ([12]). As is seen in [2], the space X_Y is often useful in discussing several relative topological properties. It is easy to see that Y is Hausdorff in X if and only if X_Y is Hausdorff. The following results given in [2], [18] are fundamental in the present paper.

Lemma 2.1 ([2],[18]). *For a subspace Y of a space X the following statements are equivalent.*

(*a*) *Y is strongly normal in X.*

(*b*) *Y is normal in G for every open subset G of X with $Y \subset G$.*

(*c*) *X_Y is normal.*

(*d*) *Y is normal in X_Y.*

(*e*) *Y is normal itself and weakly C-embedded in X.*

Lemma 2.2 ([18]). *For a subspace Y of a space X the following statements are equivalent.*

(*a*) *Y is strongly γ-collectionwise normal in X.*

(*b*) *Y is γ-collectionwise normal in G for every open subset G of X with $Y \subset G$.*

(*c*) *X_Y is γ-collectionwise normal.*

(*d*) *Y is γ-collectionwise normal in X_Y.*

(*e*) *Y is γ-collectionwise normal itself and weakly P^γ-embedded in X.*

Corresponding to Lemmas 2.1 and 2.2 we have the following lemma; $(a) \Leftrightarrow (c)$ was recently obtained in [30], and $(c) \Leftrightarrow (e)$ for Y being Hausdorff in X was proved in [18, Lemma 4.6]. Other equivalences are easily proved.

Lemma 2.3. *For a subspace Y of a space X, the following statements from (a) to (d) are equivalent. If Y is Hausdorff in X, these are equivalent to (e).*

(*a*) *Y is Aull-paracompact in X.*

(*b*) *Y is 2-paracompact in G for every open subset G of X with $Y \subset G$.*

(*c*) *X_Y is paracompact.*

(*d*) *Y is 2-paracompact in X_Y.*

(*e*) *Y is paracompact itself and weakly P-embedded in X.*

We now introduce notions of 1- or 2- (collectionwise) normality of Y in X. We say that a subspace Y of a space X is 1- (respectively, 2-) *normal in X* if for each disjoint closed subsets F_0, F_1 of X there exist open subsets G_0, G_1 of X such that $F_i \cap Y \subset G_i$ for each $i = 0, 1$ and $\{G_0, G_1\}$ is discrete in X (i.e. $\overline{G_0} \cap \overline{G_1} = \emptyset$) (respectively, discrete at each point of Y in X (i.e. $\overline{G_0} \cap \overline{G_1} \cap Y = \emptyset$)).

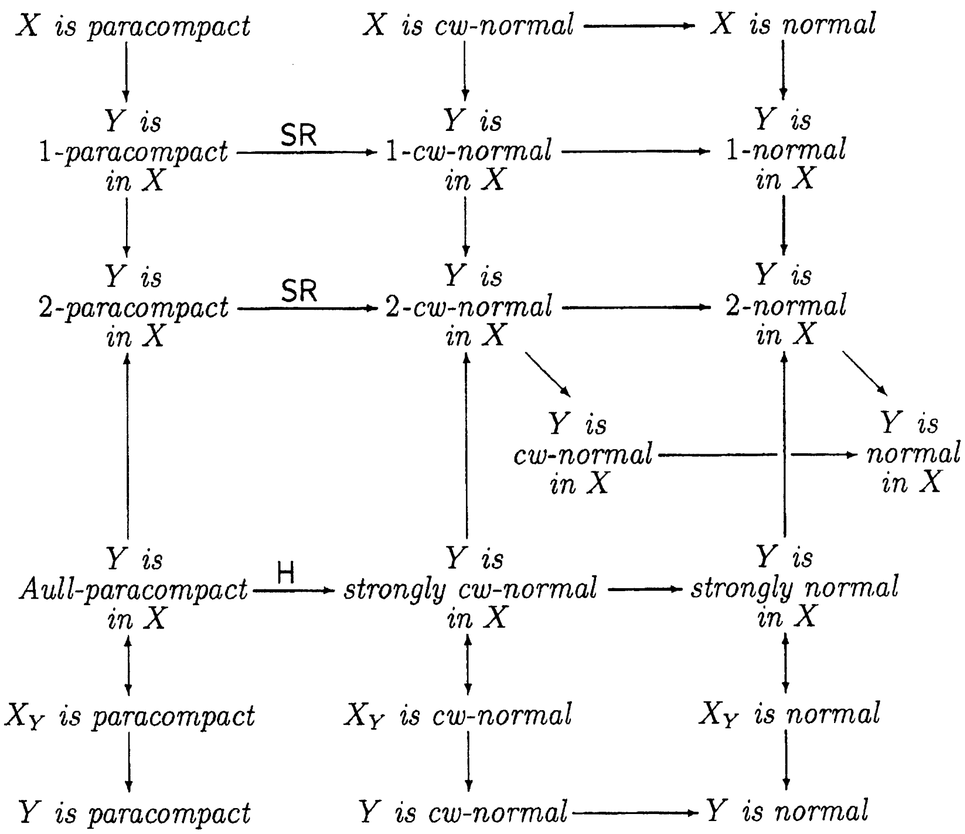
A subspace Y of a space X is 1-γ- (respectively, 2-γ-) *collectionwise normal in X* if for each discrete collection $\{F_\alpha \mid \alpha < \gamma\}$ of closed subsets of X there exists a collection $\{G_\alpha \mid \alpha < \gamma\}$ of open subsets of X such that $F_\alpha \cap Y \subset G_\alpha$ for each $\alpha < \gamma$ and $\{G_\alpha \mid \alpha < \gamma\}$ is discrete in X (respectively, discrete at each point of

Y in X). If Y is 1- (respectively, 2-) γ-collectionwise normal in X for every γ, Y is said to be 1- (respectively, 2-) *collectionwise normal in* X[†].

In the above definitions of 2-normality and 2-γ-collectionwise normality of Y in X, it is easy to see that both $\{G_1, G_2\}$ and $\{G_\alpha \mid \alpha < \gamma\}$ can be taken to be disjoint. Therefore, 2- (collectionwise) normality of Y in X implies (collectionwise) normality of Y in X.

These definitions above admit the following result; for brevity "cw-normal" means collectionwise normal. Moreover, the symbols "H" and "SR" mean the assumptions that "Y is Hausdorff in X" and "Y is strongly regular in X", respectively.

Proposition 2.4. *For a subspace Y of a space X the following implications hold.*



[†]2-collectionwise normality of Y in X is called collectionwise normality of Y in X in a recent paper of E. Grabner, G. Grabner, Miyazaki and Tartir, "*Relative collectionwise normality*" to appear in Appl. Gen. Top. Moreover, they also independently proved the implication "Y is 2-paracompact in $X \xrightarrow{\text{SR}} Y$ is 2-cw-normal in X" in Proposition 2.4 assuming further that X is Hausdorff.

Bella and Yaschenko [8] proved the following theorem. A space X is said to be *almost compact* if for every pair of disjoint zero-sets Z_0, Z_1 in X, either Z_0 or Z_1 is compact. Note that a Tychonoff space X is almost compact if and only if $|\beta X \setminus X| \leq 1$, where βX is the Stone-Čech compactification of X.

Theorem 2.5 ([8]). *For a Tychonoff space Y, the following statemants are equivalent.*

(*a*) *Y is weakly C-embedded in every larger Tychonoff (or equivalently, regular) space.*

(*b*) *Y is either Lindelöf or almost compact.*

Theorem 2.5 was improved to the following.

Theorem 2.6 ([18]). *For a Tychonoff space Y, the following statemants are equivalent.*

(*a*) *Y is weakly P^{γ}-embedded in every larger Tychonoff (or equivalently, regular) space.*

(*b*) *Y is either Lindelöf or almost compact.*

With Theorem 2.5, Bella and Yaschenko [8] further proved the following theorem, which was independently proved by Matveev et al. [25].

Theorem 2.7 ([8],[25]). *For a Tychonoff (respectively, regular) space Y, the following statemants are equivalent.*

(*a*) *Y is strongly normal in every larger Tychonoff (respectively, regular) space.*

(*b*) *Y is normal in every larger Tychonoff (respectively, regular) space.*

(*c*) *Y is either Lindelöf or normal and almost compact.*

Similarly, Theorem 2.6 and Lemma 2.2 provide the following theorem.

Theorem 2.8 ([18]). *For a Tychonoff (respectively, regular) space Y, the following statemants are equivalent.*

(*a*) *Y is strongly collectionwise normal in every larger Tychonoff (respectively, regular) space.*

(*b*) *Y is collectionwise normal in every larger Tychonoff (respectively, regular) space.*

(*c*) *Y is either Lindelöf or normal and almost compact.*

Remark 2.9. Combining Proposition 2.4 and Theorems 2.7, 2.8, we have that "strongly normal" (respectively, "strongly collectionwise normal") can be replaced by "2-normal" (respectively, "2-collectionwise normal") in Theorem 2.7 (respectively, Theorem 2.8).

Moreover, the following theorem follows from Theorem 2.6 and Lemma 2.3.

Theorem 2.10 ([4], [15], [30]). *For a Tychonoff space Y, the following statemants are equivalent.*

(*a*) *Y is Aull-paracompact in every larger Tychonoff (or equivalently, regular) space.*

(*b*) *Y is 2-paracompact in every larger Tychonoff (or equivalently, regular) space.*

(*c*) *Y is Lindelöf.*

Remark 2.11. In Theorems 2.5, 2.6, 2.7, 2.8 and 2.10, all "larger Tychonoff (respectively, regular) space" can be replaced by "larger Tychonoff (respectively, regular) space containing Y as a closed subspace".

Remark 2.12. Yamazaki [29] showed that the following are equivalent for a Hausdoff space Y:

(*a*) Y is weakly C-embedded (or equivalently, weakly P-embedded) in every larger Hausdorff space.

(*b*) Y is either compact or every continuous real-valued function on Y is constant.

In the condition (*a*), "larger Hausdorff space" can be replaced by "larger Hausdorff space containing Y as a closed subspace".

Hence, if we replace all "Tychonoff" in Theorems 2.7, 2.8 and 2.10 by "Hausdorff", the conditions (*c*) of each theorems are replaced by "Y is compact" (see also [29], [30]).

Remark 2.13. Yamazaki [31] constructed a T_1-space X and a subspace Y such that Y is normal in X, but not 2-normal in X. We do not know similar examples under higher separation axioms. Furthermore, it is unknown whether if 2-normality implies 2-ω-collectionwise normality, or collectionwise normality implies 2-collectionwise normality.

3. Quasi-C*-, quasi-C- and quasi-P$^\gamma$-embeddings

In this section, we introduce new extension properties called quasi-C^*-, quasi-C- and quasi-P-embeddings, which will play basic roles on the study of 1- (collectionwise) normality.

Let X be a space and $\mathcal{E} = \{E_\alpha \mid \alpha \in \Omega\}$ a collection of subsets of X. Then $\mathcal{E}$ is said to be *uniformly discrete* in X if there exist a collection $\{Z_\alpha \mid \alpha \in \Omega\}$ of zero-sets of X and a discrete collection $\{G_\alpha \mid \alpha \in \Omega\}$ of cozero-sets of X such that $E_\alpha \subset Z_\alpha \subset G_\alpha$ for each $\alpha \in \Omega$ ([9]).

Let us now define that a subspace Y of a space X is *quasi-C^*-embedded in X* if for each pair Z_0, Z_1 of disjoint zero-sets of Y, there exist open subsets G_0, G_1 of X such that $\{G_0, G_1\}$ is discrete in X and $Z_i \subset G_i$ for each $i = 0, 1$.

A subspace Y of a space X is said to be *quasi-P^γ-embedded in X* if for each uniformly discrete collection $\{Z_\alpha \mid \alpha < \gamma\}$ of zero-sets of Y, there exists a discrete collection $\{G_\alpha \mid \alpha < \gamma\}$ of open subsets of X such that $Z_\alpha \subset G_\alpha$ for each $\alpha < \gamma$. A subspace Y is *quasi-P-embedded in X* if Y is quasi-P^γ-embedded in X for every γ. Furthermore, quasi-P^ω-embedding is called *quasi-C-embedding.*

Definitions of quasi-C^*-embedding and quasi-P^γ-embedding should be compared with the following results in [9], [18] and [19].

Lemma 3.1 ([9]). *A subspace Y of a space X is P^γ-embedded in X if and only if if for every uniformly discrete collection of subsets of Y of cardinality $\leq \gamma$ is also uniformly discrete in X.*

Lemma 3.2 ([18]). *A subspace Y of a space X is weakly C-embedded in X if and only if if for each pair Z_0, Z_1 of disjoint zero-sets of Y, there exist disjoint open subsets G_0, G_1 of X such that $Z_i \subset G_i$ for each $i = 0, 1$.*

Lemma 3.3 ([19]). *A subspace Y of a space X is weakly P^γ-embedded in X if and only if for each uniformly discrete collection $\{E_\alpha \mid \alpha < \gamma\}$ of zero-sets of Y there exists a pairwise disjoint collection $\{G_\alpha \mid \alpha < \gamma\}$ of open subsets of X such that $E_\alpha \subset G_\alpha$ for each $\alpha < \gamma$.*

By Lemmas 3.1, 3.2 and 3.3, we have the following implications.

$$\begin{array}{ccccc} P & \longrightarrow & C & \longrightarrow & C^* \\ \downarrow & & \downarrow & & \downarrow \\ \text{quasi-}P & \longrightarrow & \text{quasi-}C & \longrightarrow & \text{quasi-}C^* \\ \downarrow & & & & \downarrow \\ \text{weak } P & & \longrightarrow & & \text{weak } C \end{array}$$

We note that none of reverse implications above is true.

Proposition 3.4. *For a subspace Y of a space X, the following statements hold.*

(*a*) *If Y is itself γ-collectionwise normal and quasi-P^γ-embedded in X, then Y is 1-γ-collectionwise normal in X.*

(*b*) *If Y is itself normal and quasi-C^*-embedded in X, then Y is 1-normal in X.*

Moreover, if Y is closed in X, each of (a) and (b) reverses.

In [6], Aull defined that a subspace Y of a space X is *α-paracompact in X* if for every collection $\mathcal{U}$ of open subsets of X with $Y \subset \bigcup \mathcal{U}$, there exists a collection $\mathcal{V}$ of open subsets of X such that $Y \subset \bigcup \mathcal{V}$, $\mathcal{V}$ is a partial refinement of $\mathcal{U}$ and $\mathcal{V}$ is locally finite in X. Note that α-paracompactness of Y in X implies Aull-paracompactness of Y in X ([3], [4]).

Related to α-paracompactness, let us recall the following results in [22] and [23, Theorem 1.3].

Theorem 3.5 ([**22**]). *A Hausdorff (respectively, regular, Tychonoff) space Y is α-paracompact in every Hausdorff (respectively, regular, Tychonoff) space containing Y as a closed subspace if and only if Y is compact.*

Theorem 3.6 ([**23**]). *For a closed subspace Y of a regular space X, Y is 1-paracompact in X if and only if Y is α-paracompact in X.*

Theorems 3.5 and 3.6 immediately induce a characterization of absolute 1-paracompactness as follows.

Corollary 3.7. *For a Tychonoff (respectively, regular) space Y, the following statemants are equivalent.*

(*a*) *Y is 1-paracompact in every larger Tychonoff (respectively, regular) space.*
(*b*) *Y is α-paracompact in every larger Tychonoff (respectively, regular) space.*
(*c*) *Y is compact.*

The following is one of our main theorems characterizing absolute quasi-P-, quasi-C- and quasi-C^*-embeddings.

Theorem 3.8. *For a Tychonoff space Y, the following statements are equivalent.*

(*a*) *Y is quasi-P-embedded in every larger Tychonoff space.*
(*b*) *Y is quasi-C-embedded in every larger Tychonoff space.*
(*c*) *Y is quasi-C^*-embedded in every larger Tychonoff space.*
(*d*) *Y is almost compact.*

In the conditions from (a) to (c), "Tychonoff" can be replaced by "regular".

By Proposition 3.4 and Theorem 3.8, we have

Corollary 3.9. *For a Tychonoff (respectively, regular) space Y, the following statements are equivalent.*

(*a*) *Y is 1-collectionwise normal in every larger Tychonoff (respectively, regular) space.*
(*b*) *Y is 1-normal in every larger Tychonoff (respectively, regular) space.*
(*c*) *Y is normal and almost compact.*

In Corollary 3.9, $(b) \Leftrightarrow (c)$ also follows from [25, Theorem 2.6]. For the Hausdorff case, we have the following.

Theorem 3.10. *For a Hausdorff space Y, the following statements are equivalent.*

(*a*) *Y is quasi-C^*-embedded in every larger Hausdorff space.*

(*b*) *Every continuous real-valued function on Y is constant.*

In (a), "quasi-C^*-embedded" can be replaced by "quasi-P-embedded" or "quasi-C-embedded" and "larger Hausdorff space" can be replaced by "larger Hausdorff space containing Y as a closed subspace".

By Theorem 3.10 and Proposition 3.4, we have the following; *a Hausdorff space Y is 1-collectionwise normal (or equivalently, 1-normal) in every larger Hausdorff space if and only if $|Y| \leq 1$. Moreover, "larger Hausdorff space" can be replaced by "larger Hausdorff space containing Y as a closed subspace".*

Finally we consider the condition under which 2-paracompactness implies 1-paracompactness. We say a subspace Y of a space X is T_4- (respectively, T_3-) *embedded in X* if for every closed subset F of X disjoint from Y (respectively, $z \in X \setminus Y$), F (respectively, z) and Y are separated by disjoint open subsets of X. The idea of these notions already appeared in Aull [6]. It is easy to see that if Y is T_3-embedded in X, then Y is closed in X.

The following is a finer result of Theorem 3.6; to show "$(b) \Rightarrow (c)$", the implication "$(b) \Rightarrow Y$ is T_4-embedded in X" is due to Aull [6, Theorem 6]. By using this fact, Lupiañez and Outerelo [23, Lemma 1.2 and Theorem1.3] proved $(a) \Rightarrow (c) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$.

Theorem 3.11 ([23]). *For a closed subspace Y of a regular space X the following statements are equivalent.*

(*a*) *Y is 1-paracompact in X.*

(*b*) *Y is α-paracompact in X.*

(*c*) *Y is 2-paracompact in X and T_4-embedded in X.*

The proof of Theorem 3.11 essentially shows the following.

Theorem 3.12. *For a subspace Y of a space X the following statements are equivalent.*

(*a*) *Y is 1-paracompact in X and T_3-embedded in X.*

(*b*) *Y is α-paracompact in X and for every $y \in Y$ and every closed subset F of X with $F \cap Y = \emptyset$, there exists an open subset U of X such that $y \in U \subset \overline{U}^X \subset X \setminus F$.*

(*c*) *Y is 2-paracompact in X and T_4-embedded in X.*

Proposition 3.13. *For a Tychonoff space Y the following statements are equivalent.*

(a) *Y is T_4-embedded in every larger Tychonoff space.*

(b) *Y is compact.*

Remark 3.14. In Theorem 3.8, Corollaries 3.7 and 3.9, Proposition 3.13, all "larger Tychonoff (respectively, regular) space" can be replaced by "larger Tychonoff (respectively, regular) space containing Y as a closed subspace".

Theorem 2.10, Theorem 3.11 and Proposition 3.13 give an alternative proof to Corollary 3.7.

In case Y is Hausdorff, we have the following; *a Hausdoff space Y is T_4-embedded in every larger Hausdorff space if and only if $Y = \emptyset$.* The similar proof provides the following; *a Hausdorff space Y is 1-paracompact in every larger Hausdorff space if and only if $Y = \emptyset$.* Moreover, in both statements, "larger Hausdorff space" can be replaced by "larger Hausdorff space containing Y as a closed subspace". This should be compared with Theorem 3.5 and Corollary 3.7.

4. On 1-metacompactness of a subspace in a space

In this section, we describe absolute case of 1-metacompactness. A subspace Y of a space X is said to be *1-metacompact in X* if for every open cover $\mathcal{U}$ of X, there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\mathcal{V}$ is point-finite at every $y \in Y$ ([21]). In [16], 1-metacompactness of Y in X is called strongly metacompactness of Y in X.

A space X satisfies the *discrete finite chain condition* (DFCC, for short) if every discrete collection of non-empty open subsets of X is finite (see [24], for example). Recall that a Tychonoff space X is pseudocompact if and only if X satisfies the DFCC. It is also known that a Tychonoff space X is compact if and only if X is pseudocompact and metacompact ([27], [28]). Furthermore, a regular space X is compact if and only if X satisfies the DFCC and is metacompact ([27]).

According to [2], in [4], Arhangel'skiĭ and Genedi remarked the following fact; *let Y be a countable dense subset of a regular space X. Then Y is 1-metacompact (or equivalently, 1-paracompact) in X if and only if X is Lindelöf.* The proof of this fact is applied to show the following lemma.

Lemma 4.1. *Take a separable space Z and a non-DFCC space Y, arbitrarily.*

Let $\{d_n \mid n \in \mathbb{N}\}$ be a countable dense subset of Z, $\{U_n \mid n \in \mathbb{N}\}$ a countable discrete collection of non-empty open subsets of Y and $\{y_n \mid n \in \mathbb{N}\}$ a countable *closed discrete subset of* Y *such that* $y_n \in U_n$ *for each* $n \in \mathbb{N}$. *Let* X *be the quotient space obtained from* $Y \oplus Z$ *by identifying* y_n *with* d_n *for each* $n \in \mathbb{N}$.

If Y *is* 1*-metacompact in* X, *then* Z *is Lindelöf.*

Moreover, if Y *and* Z *are Tychonoff (respectively, regular), then* X *is also Tychonoff (respectively, regular).*

Theorem 4.2. *A Tychonoff (respectively, regular, Hausdorff) space* Y *is* 1*-metacompact in every larger Tychonoff (respectively, regular, Hausdorff) space if and only if* Y *is compact.*

Theorem 4.2 extends the following result due to E. Grabner et al. [16]; *a normal space* Y *is* 1*-metacompact in every larger regular space if and only if* Y *is compact.*

5. On 1-subparacompactness of a subspace in a space

It was defined in [26] that a subspace Y of a space X is 1-*subparacompact in* X if for every open cover $\mathcal{U}$ of X, there exists a σ-discrete collection $\mathcal{P}$ of closed subsets of X with $Y \subset \bigcup \mathcal{P}$ such that $\mathcal{P}$ is a partial refinement of $\mathcal{U}$.

In [26], Qu and Yasui asked a question as follows; *let* X *be a regular space and* Y *a subspace of* X. *Is it true that if* Y *is* 1*-paracompact in* X, *then* Y *is* 1*-subparacompact in* X? The following theorem gives a negative answer to this question.

Theorem 5.1. *There exists a Tychonoff space* X *and a subspace* Y *of* X *such that* Y *is* 1*-paracompact but not* 1*-subparacompact in* X.

Construction. Let X be the set $(\omega_2 + 1) \times (\omega_1 + 1) \setminus \{\langle \omega_2, \omega_1 \rangle\}$. For $\alpha \in \omega_1$ and $\beta \in \omega_2$, define $G_\alpha = (\omega_2 + 1) \times \{\alpha\}$ and $H_\beta = \{\beta\} \times (\omega_1 + 1)$, respectively. Define a topology on X as follows. For $\alpha \in \omega_1$, a neighborhood base at $\langle \omega_2, \alpha \rangle$ is the family of all sets of the form $G_\alpha \setminus E$, where E is a finite subset of $\omega_2 \times \{\alpha\}$. For $\beta \in \omega_2$, a neighborhood base at $\langle \beta, \omega_1 \rangle$ is the family of all sets of the form $H_\beta \setminus F$, where F is a finite subset of $\{\beta\} \times \omega_1$. All other points of X are isolated in X. The construction of X is based on a example in [11]. Let $Y = X \setminus \big((\omega_2 \times \{\omega_1\}) \cup (\{\omega_2\} \times \omega_1)\big)$. Then Y is 1-paracompact but not 1-subparacompact in X.

References

[1] R.A. Alò and H.L. Shapino, *Normal Topological Spaces*, Cambridge Univ. Press, Cambridge, 1974.

[2] A.V. Arhangel'skiĭ, *Relative topological properties and relative topological spaces*, Topology Appl., 70 (1996), 87–99.

[3] A.V. Arhangel'skiĭ, *From classic topological invariants to relative topological properties*, Sci. Math. Japon., **55** (2002), 153–201.

[4] A.V. Arhangel'skiĭ and H.M.M. Genedi, *Beginnings of the theory of relative topological properties*, in: General Topology. Spaces and Mappings, MGU, Moscow, 1989, 3–48.

[5] A.V. Arhangel'skiĭ and I.Ju Gordienko, *Relative symmetrizability and metrizability*, Comm. Math. Univ. Carolinae, **37** (1996), 757–774.

[6] C.E. Aull, *Paracompact subsets*, Proc. the Second Prague Topological Symposium, 45–51, Prague, 1966.

[7] C.E. Aull, *Collectionwise normal subsets*, J. London Math. Soc., **1** (1969), 155–162.

[8] A. Bella and I.V. Yaschenko, *Lindelöf property and absolute embeddings*, Proc. Amer. Math. Soc., **127** (1999), 907–913.

[9] R.L. Blair, *A cardinal generalization of z-embedding*, in: Rings of continuous functions, Lecture Notes in Pure and Appl. Math., Vol. **95**, Marcel Dekker, New York, 1985, 7–78.

[10] R.H. Bing, *Metrization of topological spaces*, Canad. J. Math., **3** (1951), 175–186.

[11] D.K. Burke, *Covering Properties*, in: K. Kunen and J.E. Vaughan eds., Handbook of the Set-Theoretic Topology, North-Holand, Amsterdam, 1984, 347–422.

[12] R. Engelking, *General Topology*, Heldermann Verlag, Berlin, 1989.

[13] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, Princeton, 1960.

[14] I.Ju Gordienko, *On relative properties of paracompactness and normality type*, Moscow Univ. Math. Bull., **46** (1991), 31–32.

[15] I.Ju Gordienko, *A characterization of relative Lindelöf property by relative paracompactness*, in: General Topology. Spaces, mappings and functors, MUG, Moscow, 1992, 40–44.

[16] E.M. Grabner, G.C. Grabner and K. Miyazaki, *On properties of relative metacompactness and paracompactness type*, Topology Proc., **25** (2000), 145–177.

[17] T. Hoshina, *Extensions of mappings* II, in: K. Morita and J. Nagata eds., Topics in General Topology, North-Holland, Amsterdam, 1989, 41–80.

[18] T. Hoshina and K. Yamazaki, *Weak C-embeddeing and P-embedding, and product spaces*, Topology Appl., **125** (2002), 233–247.

[19] T. Hoshina and K. Yamazaki, *A characterization of weak P-embedding*, unpublished.

[20] S. Kawaguchi and R. Sokei, *Some relative properties on normality and paracompactness, and their absolute embeddings*, submitted.

[21] L.D. Kočinac, *Some relative topological properties*, Math. Vestnik, **44** (1992), 33–44.

[22] F.G. Lupiañez, *On covering properties*, Math. Nachr. **141** (1989), 37–43.

[23] F.G. Lupiañez and E. Outerelo, *Paracompactness and closed subsets*, Tsukuba J. Math., **13** (1989), 483–493.

[24] M.V. Matveev, *A survey on star covering properties*, Topology Atlas, preprint No.**330** (1998).

[25] M.V. Matveev, O.I. Pavlov and J. Tartir, *On relatively normal spaces, relatively regular spaces, and on relative property* (a), Topology Appl., **93** (1999), 121–129.

[26] Z. Qu and Y. Yasui, *Relatively subparacompct spaces*, Sci. Math. Japon., **54** (2001), 281–287.

[27] B.M. Scott, *Pseudocompact, metacompact spaces are compact*, Topology Proc., **4** (1979), 577–587.

[28] S. Watson, *Pseudocompact metacompact spaces are compact*, Proc. Amer. Math. Soc., **81** (1981), 151–152.

[29] K. Yamazaki, *Absolute weak C-embedding in Hausdorff spaces*, Topology Appl., **131** (2003), 273–279.

[30] K. Yamazaki, *Aull-paracompactness and strong star-normality of subspaces in topological spaces*, Comm. Math. Univ. Carolinae, **45** (2004), 743–747.

[31] K. Yamazaki, an unpublished note.

Doctoral Program in Mathematics,
Graduate School of Pure and Applied Sciences,
University of Tsukuba, Tsukuba, Ibaraki 305-8571, Japan
E-mail: shinji@math.tsukuba.ac.jp

Tokyo Gakugei University Senior High School,
Setagaya, Tokyo, 154-0002, Japan
E-mail: ryoken@gakugei-hs.setagaya.tokyo.jp