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Characterization of the Global Convergence for the XOR Boolean Networks

Juei-Ling Ho

Tainan Woman's College of Arts and Technology, Tainan, Taiwan

We propose to study the dynamics of XOR Boolean network. We focus on the characterization of global convergence in the entire space of a unique attractor. We study the global dynamic behavior by exploring certain properties of synaptic matrix and then we can get the relationship between the XOR Boolean network and the general Boolean networks.

Key Words: XOR Boolean network; global convergence; unique attractor; Global dynamic behavior.

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E-mail : t20054@ms.twcat.edu.tw

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1. INTRODUCTION

The automata networks are introduced by W. McCulloch, S. Ulam and J. von Neumann to model phenomena studied in physics and biology. In 1943 the neurophysiologist W. McCulloch and a mathematician W. Pitts published a paper entitled "A Logical Calculus of the Ideas Immanent in Nervous Activity"[1]. In that paper, McCulloch and Pitts claimed that the brain could be modelled as a network of logical operations such as AND, OR, XOR, NOT, and so forth. We call this particular case of automata networks are the McCulloch-Pitts automata. Actually, the McCulloch-Pitts automata is the neural networks. S. Ulam and J. von Neumann introduced another particular cases of automata networks, the Cellular automata[2,3,4].

Because from a mathematical point of view, automata networks are discrete dynamical systems, in time and apace. We are interested in the dynamical behaviour of automata networks. In 1986, F. Robert proposed the theory of discrete iterations which is one of the tools available to characterize the dynamical behaviour of automata networks[5,6]. The F. Robert' model gave a characterization of the local convergence for the Boolean networks. In 1999, M.-H. Shih and J.-L. Ho gave a complete solution to a difficult problem: The Markus-Yamabe problem in Boolean networks [7]. In that paper, we characterize the global convergence for the Boolean networks. In this article, we shall give a characterization of the global convergence for the XOR Boolean networks, and we shall use the notions of derivative, spectrum, iteration graph, Hamming metric, and von Neumann neighborhoods to study the global behavior of the iterations of XOR Boolean networks.

2. THE XOR BOOLEAN NETWORKS

Let us begin with some notions and notations. Let $A = (G, Q, (f_i : i \in J_n))$ be a finite automaton where G is the graph on J_n with $J_n = \{1, 2, \cdots, n\}$, and $Q = \{0, 1\}$ is the set of states. The automaton's global transition function $F_A : Q^n \to Q^n$, defined on the set of configurations Q^n, is constructed with the local transition functions $(f_i : i \in J_n)$ and with the synchronous updating rule, that is the Boolean mappings. There are three operations on A, one is $1 \oplus 1 = 0 \oplus 0 = 0$ (XOR), one is $1 \cdot 1 = 1, 1 \cdot 0 = 0 \cdot 1 = 0 \cdot 0 = 0$ (AND), another is unary operation $^{-}$ with $\bar{0} = 1$, and $\bar{1} = 0$(NOT), we usually suppress "•" and substitute ab for $a \cdot b$. We denote B to be the other finite automaton with the same sets G, Q and J_n. There are three operations on B with OR, AND and NOT, where $1 + 1 = 1 + 0 = 0 + 1 = 1, 0 + 0 = 0$(OR). Let $F_B : Q^n \to Q^n$ to be this automaton's global transition function with $F_B(x) = F_A(x)$ for all x in Q^n.

The automaton's global transition function F_B is called the Boolean network. Hence we call the automaton's global transition function F_A to be the XOR Boolean network. For the parallel iteration, since $F_B(x) = F_T(x)$ for all x in Q^n, the trajectories of these Boolean networks are the same. Starting at x^0 in Q^n, It is the sequence $\{x^t\}_{t\geq 0}$ in Q^n such that :

$$\forall t \geq 0,\ x^{t+1} = F_A(x^t) = F_B(x^t)$$

For $x \in Q^n$, the von Neumann neighborhood of x is the set $V_x = \{x, \tilde{x}^1, \cdots, \tilde{x}^n\}$ where $\tilde{x}^j$ is the j-th neighbor of x, that is, $\tilde{x}^j = (x_1, \cdots, \bar{x}_j, \cdots, x_n)$ and for $x \in Q^n$, the derivative of $F(F = F_A$ or $F_B)$ evaluated at x is given by $F'(x) = (f_{ij}(x))$, where $f_{ij}(x) = 1$ if $f_i(x) \neq f_i(\tilde{x}^j), f_{ij}(x) = 0$ otherwise. For a 01-matrix of order n, denote by M_n, the set of $n \times n$ 01-matrix. Denote $\sigma_A(C)$ and $\sigma_B(C)$ the spectrum of C in M_n are associated to automata A and B, respectively. For any $C \in M_n$, denote $C(\alpha)$ the principal submatrix of C that lies in rows and columns indexed by a nonempty subset $\alpha \subseteq J_n$. Here we also let C_j be the j-th column of the matrix C and let e_j be the j-th unit vector, for j in J_n.

Let $\xi = F(\xi)$ be a fixed point of $F(F = F_A$ or $F_B)$ in Q^n (that is a stable configuration for F). Then ξ is called to be the attractor in its von Neumann neighborhoods V_ξ if (1) $F(V_\xi) \subset V_\xi$ and (2) For all x^0 in V_ξ, the trajectory $x^{t+1} = F(x^t)$ reaches ξ in at most n steps.

3. GLOBAL CONVERGENCE FOR THE XOR BOOLEAN NETWORKS

In 1996, F. Robert developed the characterization of local convergence for the Boolean networks and he got the following conclusions ([6], p.103) :

Theorem 3.1 Let $F_B : Q^n \to Q^n$ be the Boolean network with $\xi = F(\xi)$. Suppose the following two conditions are valid :

(1) $F(V_\xi) \subset V_\xi$
(2) $\sigma_B(F'(\xi)) = \{0\}$

Then ξ is an attractor in its von Neumann neighborhoods V_ξ.

Actually, we were interested in global convergence for the Boolean networks. In 1999 M.-H. Shih and J.-L. Ho ([7], p.73 & p.66)) have developed the characterization of global convergence for the Boolean networks :

Theorem 3.2 Let $F_B : Q^n \rightarrow Q^n$ be the Boolean network. Suppose the following two conditions are valid :

(1) $F_B(V_x) \subset V_{F_B(x)}$ for all x in Q^n
(2) $\sigma_B(F'_B(x)) = \{0\}$ for all x in Q^n.

Then there exists a unique attractor ξ in Q^n, such that $\forall x^0 \in Q^n$, the trajectory $x^{t+1} = F_B(x^t)$ reaches ξ in finite steps.

Now, we give the characterization of global convergence for the XOR Boolean networks :

Theorem 3.3 Let $F_A : Q^n \rightarrow Q^n$ be the XOR Boolean network. Suppose the following two conditions are valid :

(1) $F_A(V_x) \subset V_{F_A(x)}$ for all x in Q^n
(2) $1 \notin \sigma_A(F'_A(x))$ for all x in Q^n.

Then there exists a unique attractor ξ in Q^n, such that $\forall x^0 \in Q^n$, the trajectory $x^{t+1} = F_A(x^t)$ reaches ξ in finite steps.

We shall prove the main Theorem 3.3 in the next section.

4. PROOF OF THE MAIN THEOREM

In order to prove Theorem 3.3 we shall employ the following lemmas.

Lemma 4.1 Let $C \in M_n$. Then $0 \in \sigma_A(C)$
if and only if
there exists a nonempty subset $\alpha \subseteq J_n$ such that $\sum_{j\in\alpha} C_j = \mathbf{0}$ (the zero vector).

Proof. ($\Rightarrow$) Let u be an eigenvector of C associated with the eigenvalue 0
$\because u \neq \mathbf{0}$
$\therefore \exists$ nonempty subset $\alpha \subseteq J_n$ such that $u = \sum_{j\in\alpha} e_j$

Hence

$$\sum_{j\in\alpha} C_j = \sum_{j\in\alpha} Ce_j = C\left(\sum_{j\in\alpha} e_j\right) = Cu = 0 \cdot u = \mathbf{0}$$

($\Leftarrow$) Let α be the nonempty subset of J_n such that $\sum_{j\in\alpha} C_j = \mathbf{0}$

Choose $u = \sum_{j\in\alpha} e_j$, then clearly

$$Cu = C\left(\sum_{j\in\alpha} e_j\right) = \sum_{j\in\alpha} Ce_j = \sum_{j\in\alpha} C_j = \mathbf{0}. \ \square$$

Lemma 4.2 Let $C \in M_n$. Then $1 \in \sigma_A(C)$
if and only if

there exists a nonempty subset $\alpha \subseteq J_n$ such that $\sum_{j\in\alpha} (C \oplus I)_j = \mathbf{0}$ (I is the identity matrix).

Proof. Let u be an eigenvector of A associated with the eigenvalue 1

Then

$$\begin{aligned} & Cu = 1 \cdot u = u \\ \Rightarrow\ & Cu \oplus u = \mathbf{0} \\ \Rightarrow\ & (C \oplus I)u = \mathbf{0} \\ \Rightarrow\ & 0 \in \sigma_A(C) \end{aligned}$$

By Lemma 4.1, we have

$$\exists\, \alpha \subseteq J_n, \alpha \neq \phi$$

such that

$$\sum_{j\in\alpha} (C \oplus I)_j = \mathbf{0}$$

Conversely, if there is a nonempty subset $\alpha \subseteq J_n$ such that

$$\sum_{j\in\alpha} (C \oplus I)_j = \mathbf{0}$$

By Lemma 4.1, we have

$$\begin{aligned} & 0 \in \sigma_A(C \oplus I) \\ \Rightarrow\ & \exists u \neq \mathbf{0} \text{ such that } (C \oplus I)u = 0 \cdot u = \mathbf{0} \\ \Rightarrow\ & Cu \oplus u = \mathbf{0} \\ \Rightarrow\ & Cu = u \\ \Rightarrow\ & 1 \in \sigma_A(C). \ \square \end{aligned}$$

Lemma 4.3 Let $C \in M_n$. Then $\sigma_A(C) = \phi$
if and only if
for any nonempty subset $\alpha \subseteq J_n$, $\sum_{j\in\alpha} A_j \neq \mathbf{0}$ and $\sum_{j\in\alpha} (A \oplus I)_j \neq \mathbf{0}$

Proof. It's not difficult to get this conclusion from Lemma 4.1 and Lemma 4.2. □

We have known any Boolean matrix $C \in M_n$ has a nonempty spectrum $\sigma_B(C)$ (see [6], p.48), but now we want to propose the spectrum $\sigma_A(C)$ maybe empty.

Lemma 4.4 Let $F_A : Q^n \rightarrow Q^n$ be the XOR Boolean network and let $y \in Q^n$.
Suppose the following two conditions are valid :

(1) $F_A(V_y) \subset V_{F_A(y)}$
(2) $1 \notin \sigma_A(F'_A(y))$.

Then each entry in the diagonal of $F'_A(y)$ is 0.

Proof. By condition (1), we get

$F'_A(y)$ has at most one 1 in each column

$\Rightarrow \forall i \in J_n, \exists j \in J_n$ such that $[F'_A(y)]_i = 0$ or e_j

Suppose $i = j$, then

the i-th column of $F'_A(y) \oplus I$ equals zero

By Lemma 4.2, we have

$$1 \in \sigma_A(F'_A(y))$$

This contradicts condition (2). Therefore $i \neq j$, this implies

$$f_{ii}(y) = 0 \quad \text{for all } i \in J_n. \ \square$$

Lemma 4.5 Let $F_A : Q^n \rightarrow Q^n$ be the XOR Boolean network, let $F_B : Q^n \rightarrow Q^n$ be the Boolean network and such that $F_B(x)= F_A(x)$ for all x in Q^n and let $y \in Q^n$.

Suppose the following two conditions are valid :

(1) $F_A(V_y) \subset V_{F_A(y)}$
(2) $1 \notin \sigma_A(F'_A(y))$.

Then for any principal submatrix $F'_B(y)(\alpha)$ of $F'_B(y)$, there exists an index $i \in \alpha$ such that the i-th row of $F'_B(y)(\alpha)$ equals zero.

Proof. Let $F'_A(y)= (f_{ij}(y))_{n\times n}$. Suppose there is a nonempty subset $\alpha \subseteq J_n$ such that $F'_B(y)(\alpha)$ has no zero rows. Since the discrete derivative of F_A at y equals to the discrete derivative of F_B at y, this implies

$F'_A(y)(\alpha)$ has no zero rows

Hence by Lemma 4.4 with the condition (1), we get, for any $i \in \alpha$, there exists unique $j \in \alpha, j \neq i$ such that

$$f_{ij}(y)= 1$$

and for any $j \in \alpha$, there exists unique $i \in \alpha,\ i \neq j$ such that

$$f_{ij}(y)= 1$$

This implies $F'_A(y)(\alpha)$ has no zero columns and

$$\sum_{i\in\alpha} [F'_A(y)(\alpha) \oplus I(\alpha)]_i = \mathbf{0}$$

Since condition (1) and $F'_A(y)(\alpha)$ has no zero columns that ensures for any $i \in \alpha$, $f_{ij}(y)= 0$ if $j \notin \alpha$, we have

$$\sum_{i\in\alpha} [F'_A(y) \oplus I]_i = \mathbf{0}$$

Choose $u = \sum_{i\in\alpha} e_i$, then

$$[F'_A(y) \oplus I]u = [F'_A(y) \oplus I]\left(\sum_{i\in\alpha} e_i\right) = \sum_{i\in\alpha} [F'_A(y) \oplus I]_i = \mathbf{0}$$

By Lemma 4.2, we have

$$1 \in \sigma_A(F'_A(y))$$

This contradicts condition (2). Therefore, for any nonempty subset $\alpha \subseteq J_n$

$F'_A(y)(\alpha)$ has at least one zero row.

This implies for any nonempty subset $\alpha \subseteq J_n$

$F'_B(y)(\alpha)$ has at least one zero row. □

Next, we want to propose $\sigma_A(F'_A(y)) \neq \phi$.

Lemma 4.6 Let $F_A : Q^n \rightarrow Q^n$ be the XOR Boolean network and let $y \in Q^n$. Suppose the following two conditions are valid :

(1) $F_A(V_y) \subset V_{F_A(y)}$
(2) $1 \notin \sigma_A(F'_A(y))$.

Then $0 \in \sigma_A(F'_A(y))$.

Proof. Let $F'_A(y)= (f_{ij}(y))_{n\times n}$.

Suppose $0 \notin \sigma_A(F'_A(y))$.

By Lemma 4.1 with the condition (1), we get for any $i \in J_n, \exists! j \in J_n$ such that

$$[F'_A(y)]_i = e_j$$

and $[F'_A(y)]_{i_1} \neq [F'_A(y)]_{i_2}$ if $i_1 \neq i_2$.

By Lemma 4.4, we have

$f_{ii}(y)= 0 \quad$ for all $i \in J_n$

Hence

$$\sum_{i \in J_n} [F'_A(y) \oplus I]_i = \mathbf{0}$$

By Lemma 4.2, we have

$$1 \in \sigma_A(F'_A(y))$$

This contradicts condition (2), hence $0 \in \sigma_A(F'_A(y))$. □

Lemma 4.7 Let $F_B : Q^n \to Q^n$ be the Boolean network, let $F_A : Q^n \to Q^n$ be the XOR Boolean network and such that $F_A(x)= F_B(x)$ for all x in Q^n and let $y \in Q^n$. Suppose

$$1 \notin \sigma_B(F'_B(y))$$

Then for any nonempty subset $\alpha \subseteq J_n$

$$\sum_{i \in \alpha} [F'_A(y) \oplus I]_i \neq 0.$$

Proof. Let $F'_A(y)= (f_{ij}(y))_{n \times n}$ and $F'_B(y)= (h_{ij}(y))_{n \times n}$. The conditions imply (see [7], Theorem 2.2, p.64)

$h_{ii}(y)= 0 \quad$ for all $i \in J_n$

Hence we get,

$f_{ii}(y)= 0 \quad$ for all $i \in J_n$

Suppose there is a nonempty subset $\alpha \subseteq J_n$ such that

$$\sum_{i \in \alpha} [F'_A(y) \oplus I]_i = 0,$$

then for any $i \in \alpha$, $\exists j \in \alpha, j \neq i$ such that

$$1 \oplus f_{ij}(y)= 0$$

it means,

$$f_{ij}(y)= 1$$

and then $F'_A(y)(\alpha)$ is a principal submatrix of $F'_A(y)$ which has no zero rows. Since the discrete derivative of F_B at y equals to the discrete derivative of F_A at y, we get $F'_B(y)(\alpha)$ is a principal submatrix of $F'_B(y)$ which has no zero rows. Hence we have (see [6], Theorem 3, p.47)

$$1 \in \sigma_B(F'_B(y))$$

This contradicts the conditions, hence the conclusion follows. □

Lemma 4.8 Let $F_A : Q^n \to Q^n$ be the XOR Boolean network, let $F_B : Q^n \to Q^n$ be the Boolean network and such that $F_B(x)= F_A(x)$ for all x in Q^n and let $y \in Q^n$. Suppose the following two conditions are valid :

(1) $F_A(V_y) \subset V_{F_A(y)}$
(2) $1 \notin \sigma_A(F'_A(y))$

if and only if the following *two conditions are valid :*

(a) $F_B(V_y) \subset V_{F_B(y)}$
(b) $\sigma_B(F'_B(y)) = \{0\}$.

Proof. First, by the definition, we have $F_B(x)= F_A(x)$ for all x in Q^n, this means conditions (1) and (a) are equivalent. Now if conditions (1) and (2) hold, then the Lemma 4.5 concludes the condition (b). Conversely, if the condition (b) hold, then the Lemma 4.7 implies the condition (1) is true. Therefore we complete this proof. □

We now turn to the proof of Theorem 3.3. Let $F_A : Q^n \to Q^n$ be the XOR Boolean network, let $F_B : Q^n \to Q^n$ be the Boolean network and such that $F_B(x)= F_A(x)$ for all x in Q^n. Then the Lemma 4.8 shows the following conditions hold for all x in Q^n :

(a) $F_B(V_x) \subset V_{F_B(x)}$
(b) $\sigma_B(F'_B(x)) = \{0\}$.

Now, by Theorem 3.2, we can conclude there exists a unique attractor ξ in Q^n, such that $\forall x^0 \in Q^n$, the trajectory $x^{t+1} = F_B(x^t)$ reaches ξ in finite steps.. Since $F_B(x) = F_A(x)$ for all x in Q^n, the trajectory $x^{t+1} = F_A(x^t)$ have the same orbits with the trajectory $x^{t+1} = F_B(x^t)$. Therefore, there exists a unique attractor ξ in Q^n, such that $\forall x^0 \in Q^n$, the trajectory $x^{t+1} = F_A(x^t)$ reaches ξ in finite steps.. This complete the proof of Theorem 3.3.

5. THE EXISTENCES OF FIXED POINT

Let $F_B : Q^n \to Q^n$ be the Boolean network. From Theorem 3.2 we know the fixed point exists if both the following conditions hold :

(a) $F_B(V_x) \subset V_{F_B(x)}$ for all x in Q^n
(b) $\sigma_B(F'_B(x)) = \{0\}$ for all x in Q^n

Indeed, this condition (b) is enough to conclude the existences of fixed point for this Boolean network[8]. Now we are curious to the existences of fixed point for the XOR Boolean network when the condition(2) hold in Lemma 4.8. Let $F_A : Q^n \to Q^n$ be the XOR Boolean network. Suppose $1 \notin \sigma_A(F'_A(y))$. Can we say there exists a fixed point for this XOR Boolean network? Unfortunately, we cannot. Now we want to present two counterexamples. First, we consider $\sigma_A(F'_A(x)) = \phi$ for all x in Q^n :

Let $F_A : Q^3 \to Q^3$ be defined by

$$F_A(x) = F_A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \stackrel{def}{=} \begin{pmatrix} x_2 \oplus x_1\bar{x}_3 \oplus x_2 x_1 \bar{x}_3 \\ x_1\bar{x}_2 \oplus x_3(\overline{x_1 \oplus x_2}) \oplus x_1\bar{x}_2 x_3(\overline{x_1 \oplus x_2}) \\ \overline{x_2 x_3}(x_1 \oplus x_2 \oplus x_3) \end{pmatrix}$$

Then F_A is given by Table 5.1. The discrete derivatives of F_A are the following,

$$F'_A(0,0,0) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad F'_A(1,0,0) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

$$F'_A(0,1,0) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \qquad F'_A(0,0,1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix},$$

$$F'_A(1,1,0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad F'_A(1,0,1) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

$$F'_A(0,1,1)=\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad F'_A(1,1,1)=\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

Then for any x in Q^3 and for any nonempty subset $\alpha \subseteq J_3$, $\sum_{j\in\alpha} [F'_A(x)]_j \neq \mathbf{0}$ and $\sum_{j\in\alpha} [F'_A(x) \oplus I]_j \neq \mathbf{0}$. even the Lemma 4.3 shows $\sigma_A(F'_A(x)) = \phi$ for all x in Q^n, but F_A still has no any fixed point.

TABLE 5.1

Bit string x	000	001	010	011	100	101	110	111
Bit string $F(x)$	001	010	100	100	110	011	101	110

Moreover, even we consider $\sigma_A(F'_A(x)) = \{0\}$ for all x in Q^n, it's no use for the next counterexample :

Let $F_A : Q^3 \to Q^3$ be defined by

$$F_A(x)= F_A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \stackrel{def}{=} \begin{pmatrix} \bar{x}_1\bar{x}_2 \oplus x_3(x_1 \oplus x_2) \oplus \bar{x}_1\bar{x}_2x_3(x_1 \oplus x_2) \\ \bar{x}_1(x_2 \oplus x_3 \oplus x_2x_3) \\ x_1x_2 \oplus \bar{x}_3(x_1 \oplus x_2) \oplus x_1x_2\bar{x}_3(x_1 \oplus x_2) \end{pmatrix}$$

Then F_A is given by Table 5.2. The discrete derivatives of F_A are the following,

$$F'_A(0,0,0)=\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \qquad F'_A(1,0,0)=\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

$$F'_A(0,1,0)=\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \qquad F'(0,0,1)=\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$F'(1,1,0)=\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad F'_A(1,0,1)=\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

$$F'_A(0,1,1)=\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \qquad F'_A(1,1,1)=\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix},$$

Then for any x in Q^3 and for any nonempty subset $\alpha \subseteq J_3$, $\sum_{j\in\alpha} [F'_A(x) \oplus I]_j \neq \mathbf{0}$. It means, $1 \notin \sigma_A(F'_A(x))$ for all x in Q^3 (see Lemma 4.2). But $\sum_{j\in\alpha} [F'_A(x)]_j = \mathbf{0}$ where $\alpha = J_3 = \{1,2,3\}$ as $x = (0,0,0),(0,0,1),(1,0,0),(0,1,0)$; $\alpha = \{2,3\}$ as $x = (1,0,1),(1,1,0)$; $\alpha = \{2\}$ as $x = (0,1,1)$; and $\alpha = \{3\}$ as $x = (1,1,1)$. Hence $0 \in \sigma_A(F'_A(x))$ for all x in Q^3 (see Lemma 4.1), and then $\sigma_A(F'_A(x)) = \{0\}$ for all x in Q^3, but F_A still has no fixed point.

TABLE 5.1

Bit string x	000	001	010	011	100	101	110	111
Bit string $F(x)$	100	110	011	110	001	100	001	001

Therefore, by Theorem 3.3, we get the conclusion : The unique fixed point exists in the XOR Boolean network $F_A : Q^n \to Q^n$ if both the following two conditions are valid :

(1) $F_A(V_x) \subset V_{F_A(x)}$ for all x in Q^n
(2) $1 \notin \sigma_A(F'_A(x))$ for all x in Q^n.

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