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## On weak dividing in $n$ -simple theories

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Weak dividing was originally defined by Shelah in [1]. After a long time, Dolich characterized that notion in simple context in [2]. Then Kim and Shi continued the investigation, in particular they proved that a theory  $T$  is stable if and only if weak dividing is symmetric in [3]. Recently, the class of simple theory was split into  $\omega + 1$  subclasses by Kolesnikov in [4]. He used the notion of  $n$ -simplicity for  $n \leq \omega$ . I studied his paper and had some consideration about the relation with weak dividing.

At first, we recall some definitions in [4].

For  $n \geq 2$ , let the symbol  $\text{Ind}(x; y_0, \dots, y_{n-1})$  denote the type expressing that  $y_0, \dots, y_{n-1}$  are indiscernible over  $x$ .

**Definition 1** Fix  $1 \leq n \leq k < \omega$ . Take a formula  $\varphi(x, y_0, \dots, y_{n-1})$  and a partial type  $p(x)$ . Define  $D_n[p, \varphi, k] \geq \alpha$  by induction on  $\alpha$ .

- (1)  $D_n[p, \varphi, k] \geq 0$  if  $p$  is consistent.
- (2) for  $\alpha$  limit,  $D_n[p, \varphi, k] \geq \alpha$  if  $D_n[p, \varphi, k] \geq \beta$  for all  $\beta < \alpha$ ;
- (3)  $D_n[p, \varphi, k] \geq \alpha + 1$  if for every finite  $r \subseteq p(x)$  there is a sequence  $\{a_i | i < \omega\}$  such that for all  $\bar{i} \in [\omega]^n$

$$D_n[r \cup \{\varphi(x, \bar{a}_{\bar{i}})\} \cup \text{Ind}(x; \bar{a}_{\bar{i}}), \varphi, k] \geq \alpha$$

and the set  $\{\varphi(x, \bar{a}_{\bar{i}}) | \bar{i} \in [\omega]^n\}$  is  $[k]^n$ -contradictory.

The expressions  $D_n[p, \varphi, k] = \alpha$ ,  $D_n[p, \varphi, k] = -1$ , and  $D_n[p, \varphi, k] = \infty$  are defined as usual.

**Definition 2** Let  $\alpha \leq \omega$ . We say that a complete theory  $T$  is  $\alpha$ -simple if for all  $n < \alpha$ , for all  $\varphi(x, y_0, \dots, y_n)$  and  $k > n + 1$  the rank  $D_{n+1}[x = x, \varphi, k]$  is bounded.(i.e. is less than  $\infty$ .)

**Definition 3** (1) A formula  $\varphi(x, y_0, \dots, y_{n-1})$ , a set of sequences  $\{I_\eta | \eta \in ([\omega]^n)^{<\omega}\}$ , and  $k < \omega$  witness the  $n$ -tree property if for every  $\eta \in ([\omega]^n)^\omega$ , the type  $\{\varphi(x, \bar{a}_{\eta[l]}^l) | l < \omega\}$  is realized by  $\bar{b}_\eta$  such that sequences  $\bar{a}_{\eta[l]}^l$  are

indiscernible over  $b_\eta$  for each  $l < \omega$  and for every  $\eta \in ([\omega]^n)^{<\omega}$  the set  $\{\varphi(x, \bar{a}_\eta^l) \mid \bar{z} \in [\omega]^n\}$  is  $[k]^n$ -contradictory where  $\bar{a}_{\eta[l]}^{\eta[l]} := \{a_{i_0}^{\eta[l]}, \dots, a_{i_{n-1}}^{\eta[l]}\}$  for  $\eta[l] = i_0 \cdots i_{n-1}$ .

(2) A theory  $T$  has the  $n$ -tree property if there exist a formula, a set of parameters, and a number  $k$  witnessing the  $n$ -tree property.

The next proposition is proved by the definitions.

**Proposition 4** ([4]) *A theory  $T$  is  $\alpha$ -simple if and only if it does not have an  $(n+1)$ -tree property for any  $n < \alpha$ .*

Kolesnikov defined some notion of dividing for  $n$ -simple context.

**Definition 5** For  $n < \omega$ , we say that a formula  $\varphi(x, a_0, \dots, a_{n-1})$   $n$ -divides over  $A$  if there is an indiscernible sequence  $\{a_i \mid i < \omega\}$  over  $A$  and  $b \models \varphi(x, a_0, \dots, a_{n-1})$  such that  $\{a_0, \dots, a_{n-1}\}$  are indiscernible over  $b$  and the set  $\{\varphi(x, \bar{a}_\eta^l) \mid \bar{z} \in [\omega]^n\}$  is  $[k]^n$ -contradictory for some  $k$ .

**Remark 6** It is clear that for  $n = 1$  the definition is the same as that of dividing.

We recall the definition of weak dividing to make sure.

**Definition 7** We say that  $p(x) = \text{tp}(a/B)$  weakly divides over  $A (\subseteq B)$  if there is a formula  $\psi(x_1, \dots, x_n)$  over  $A$  such that  $[p]^\psi := p(x_1) \cup \dots \cup p(x_n) \cup \{\psi(x_1, \dots, x_n)\}$  is inconsistent while  $[q]^\psi$  is consistent where  $q(x) = \text{tp}(a/A)$ .

The next facts are easily checked.

**Fact 8** Let  $A \subset B$  and  $\varphi(x_0, \dots, x_{n-1}, b)$  be a formula over  $B$ . Suppose that there is an indiscernible sequence  $\{a_i \mid i < \omega\}$  over  $A$  satisfying:  
 $\models \varphi(a_0, \dots, a_{n-1}, b)$  and  $\{a_i \mid i < n\}$  are indiscernible over  $b$ . If the type  $\{\varphi(x_0, \dots, x_{n-1}, b)\} \cup \text{Ind}(B; \{x_i \mid i < \omega\})$  is inconsistent, then there is a formula  $\psi(a_0, \dots, a_{n-1}, z)$  such that  $\psi(a_0, \dots, a_{n-1}, z)$   $n$ -divides over  $A$ .

**Fact 9** Let  $A \subset B$  and  $p(x) = \text{tp}(a/B)$ . Suppose that there is a formula  $\varphi(x_0, \dots, x_{n-1})$  over  $A$  and an infinite indiscernible sequence  $\{a_i \mid i < \omega\}$  over  $A$  with  $\text{tp}(a_0/A) = p \upharpoonright A$  such that

$\models \varphi(a_0, \dots, a_{n-1})$  and

the type " $\{\varphi(x_0, \dots, x_{n-1})\} \cup \text{Ind}(A; \{x_i : i < \omega\}) \cup \bigcup_{i < \omega} p(x_i)$ " is inconsistent.

Then  $p$  weakly divides over  $A$ .

Moreover if  $T$  is simple, then  $p$  divides over  $A$ .

The case is problematic when realizations of the formula can not be extended to an infinite indiscernible sequence over the original parameters. I tried to use the facts above for the argument of weak dividing in  $n$ -simple theories, but I have no result to show here.

We can define an analogy of weak dividing for  $n$ -dividing.

**Definition 10** Let  $A \subset B$ . And let  $p(x_0, \dots, x_{n-1})$  be a complete type over  $B$  such that  $p(x_0, \dots, x_{n-1}) \vdash \text{Ind}(A; x_0, \dots, x_{n-1})$ . We say that  $p(x_0, \dots, x_{n-1})$  "weakly  $n$ -divides over  $A$ " if there are  $k < \omega$  and a formula  $\psi(x_0, \dots, x_{k-1})$  over  $A$  such that  $\{\psi(x_0, \dots, x_{k-1})\} \cup \bigcup_{\bar{i} \in [k]^n} p(\bar{x}_{\bar{i}}) \upharpoonright A$  is consistent while  $\{\psi(x_0, \dots, x_{k-1})\} \cup \bigcup_{\bar{i} \in [k]^n} p(\bar{x}_{\bar{i}})$  is inconsistent where  $p(\bar{x}_{\bar{i}}) = p(x_{i_0}, \dots, x_{i_{n-1}})$  for  $i_0 < i_1 < \dots < i_{n-1} < k$  and  $k > n$ .

**Remark 11** When  $n = 1$ , "weak 1-dividing" is the same as "weak dividing".

#### Notation

From now, we denote  $[p]^\psi$  for the type  $\{\psi(x_0, \dots, x_{k-1})\} \cup \bigcup_{\bar{i} \in [k]^n} p(\bar{x}_{\bar{i}})$ .

**Fact 12** Let  $A \subset B \subset C$ .

- (1) If  $\text{tp}(a/C)$  does not weakly  $n$ -divide over  $B$ , then  $\text{tp}(a/B)$  does not weakly  $n$ -divide over  $A$ .
- (2) If  $\text{tp}(a/C)$  does not weakly  $n$ -divide over  $B$  and  $\text{tp}(a/B)$  does not weakly  $n$ -divide over  $A$ , then  $\text{tp}(a/C)$  does not weakly  $n$ -divide over  $A$ .

**Fact 13** Weak  $n$ -dividing has the local character.

**Fact 14** If  $\text{tp}(b/Aa_0 \dots a_{n-1})$   $n$ -divides over  $A$ , then  $\text{tp}(a_0 \dots a_{n-1}/Ab)$  weakly  $n$ -divides over  $A$ .

**Remark 15** Naturally, we define weak  $n$ -dividing for complete types as follows:

a complete type  $p$  weakly  $n$ -divides over  $A$  if it implies a formula which weakly  $n$ -divides over  $A$ .

**Lemma 16** Let  $A \subset B$ . And let  $p(x_0, \dots, x_{n-1})$  be a complete type over  $B$  such that  $p(x_0, \dots, x_{n-1}) \vdash \text{Ind}(A; x_0, \dots, x_{n-1})$ .

Then the following are equivalent;

- (1)  $p$  does not weakly  $n$ -divide over  $A$ .
- (2) For any set  $C := \{a_i \mid i \in I\}$  satisfying that for any  $n$ -sequence  $a_{i_0}, \dots, a_{i_{n-1}} \in C$  with  $i_0 < i_1 < \dots < i_{n-1}$ ,  $\models p \upharpoonright A(a_{i_0}, \dots, a_{i_{n-1}})$ , there is  $B'$  such that  $\text{tp}(B/A) = \text{tp}(B'/A)$  and for any  $a_{i_0}, \dots, a_{i_{n-1}} \in C$  with  $i_0 < i_1 < \dots < i_{n-1}$ ,  $\text{tp}(B'/a_{i_0} \dots a_{i_{n-1}}A) = \text{tp}(B/a_{i_0} \dots a_{i_{n-1}}A)$ .

The further characterization needs to investigate the relation between  $n$ -simple theories and  $n$ -dividing more.

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