Title	On the density of the set of primes which are related to decimal expansion of rational numbers (Analytic Number Theory and Surrounding Areas)
Author(s)	Kitaoka, Yoshiyuki; Nozaki, Michihiro
Citation	数理解析研究所講究録 (2004), 1384: 175-177
Issue Date	2004-07
URL	http://hdl.handle.net/2433/25748
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録 1384 巻 2004 年 175-177

On the density of the set of primes which are related to decimal expansion of rational numbers

名城大学 北岡 良之 (Yoshiyuki Kitaoka)
Meijo University
知多東高校 野崎 通弘 (Michihiro Nozaki)
Chitahigashi High School

We give several conjectures on the set of prime numbers which are closely related to 10-adic decimal expansion of rational numbers. The starting point is the following theorem.

Theorem 1 *Let* $p\,(\neq 2,5)$ *be a prime number.* $1/p$ *has a purely periodic decimal expansion*

$$1/p = 0.\dot{c}_1\cdots\dot{c}_e = 0.c_1\cdots c_e c_1\cdots c_e\cdots, \quad (0\leq c_i\leq 9)$$

where we assume that e *is the minimal length of periods, i.e.* $e =$ *the order of* $10 \mod p$. *Suppose* $e = nk$ *for natural numbers* $n\,(>1), k$. *We divide the period to* n *parts of equal length and add them. Then we have*

$$\begin{aligned} & c_1\cdots c_k + c_{k+1}\cdots c_{2k} + \cdots + c_{(n-1)k+1}\cdots c_{nk} \\ = \quad & 9\cdots 9 \times \begin{cases} n/2 \text{ if } n \text{ is even,} \\ s(p) \text{ if } n \text{ is odd,} \end{cases} \end{aligned}$$

where $9\cdots 9 = 10^k - 1$ *and* $s(p)$ *is an integer such that* $1 \leq s(p) \leq n-2$.

We are concerned with the density of the set of primes for given n and $s = s(p)$. Hereafter we assume that $n\,(\geq 3)$ is an odd natural number and $1 \leq s \leq n-2$. Put

$$P(n,s,x) = \frac{\#\{p \mid p\leq x, n|e, s(p)=s\}}{\#\{p \mid p \leq x, n|e\}},$$

where $p\ (\neq 2,5)$ stands for a prime number and $e =$ the order of 10 mod p.

The following table of $P(n,s,10^9)$ is made by computer.

s	n = 5	n = 9	n = 11
1	0.1666	0	0.0000
2	0.6667	0	0.0014
3	0.1667	0.2499	0.0403
4		0.5001	0.2432
5		0.2500	0.4301
6		0	0.2433
7		0	0.0403
8			0.0014
9			0.0000

As a matter of fact, the graph of $P(n, s, x)$ in x is almost straight line. The ratios are symmetric at $(n-1)/2$. In the table, 0.0000 means that primes which take the values $s = 1, 9$ are very rare in the case of $n = 11$, and 0 for $n = 9$ means that the set is empty, which can be proven. The first conjecture is

Conjecture 1 $\lim_{x\to\infty} P(n, s, x)$ *exists, and by denoting it by* $P(n, s)$

$$P(n, s) = P(n, n-1-s) \text{ for } 1 \le s \le n-2.$$

Moreover $P(n, s) > 0$ *holds if* n *is an odd prime number.*

Moreover the table above looks like normal distribution. Let us recall notations of statistics. For the table of frequency distribution

value	x_1	x_2	$\cdots$	x_m	sum
relative frequency	r_1	r_2	$\cdots$	r_m	1

define the average μ and the standard deviation σ by

$$\mu = \sum_{i=1}^{m} x_i r_i, \quad \sigma = \sqrt{\sum_{i=1}^{m} x_i^2 r_i - \mu^2}.$$

Then we get

n	μ	σ
5	2.0001	0.5774
9	4.0002	0.7070
11	5.0002	0.9132
37	18.0010	1.7325

This table suggests

Conjecture 2

$$\lim_{x\to\infty} \mu = (n-1)/2.$$

To formulate being normal distribution, we denote the density function of normal distribution of average μ and standard deviation σ by

$$f_{\mu,\sigma}(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$$

and compare the ratio with it. The table is

n	$\max_{1\le s\le n-2} \lvert P(n,s,x) - f_{\mu,\sigma}(s)\rvert$
5	0.0243
9	0.0641
11	0.0067
37	0.0006

This table and more general table for odd $n \le 101$ suggest

Conjecture 3

$$\lim_{n\to\infty}\lim_{x\to\infty}\max_{1\le s\le n-2} \lvert P(n,s,x) - f_{\mu,\sigma}(s)\rvert = 0.$$

We considered 10-adic expansion. But in the proof of Theorem 1, the number 10 is not important. It is generalized as follows:

Theorem 2 *Let a ($\neq 0, \pm 1$) be an integer and p a prime number. Put e = the order of a mod p and suppose $e = nk$, where $n \ge 3$ and $(a^k - 1, p) = 1$. Define an integer r_i by*

$$r_i \equiv a^{ki} \mod p, \quad 0 \le r_i < p.$$

Then $s(p) = (\sum_{i=0}^{n-1} r_i)/p$ is an integer such that $1 \le s(p) \le n-2$.

The former part is the case of $a = 10$. Similarly as above, we put

$$P_a(n,s,x) = \frac{\#\{p \mid p \le x, n|e, s(p) = s\}}{\#\{p \mid p \le x, n|e\}}.$$

The numerical data suggest the final

Conjecture 4

$$\lim_{x\to\infty} P_a(n,s,x) = \lim_{x\to\infty} P_{10}(n,s,x)(= P(n,s)).$$

The proof of theorems are easy and other probably new observations will be included in 本格的に代数を学ぶ前に.