

Title	Mass formula for Jacobi weight enumerators of type II binary codes and some relationships of it with Jacobi forms (Analytic Number Theory and Surrounding Areas)
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Citation	数理解析研究所講究録 (2004), 1384: 1-9
Issue Date	2004-07
URL	<a href="http://hdl.handle.net/2433/25724">http://hdl.handle.net/2433/25724</a>
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

# Mass formula for Jacobi weight enumerators of type II binary codes and some relationships of it with Jacobi forms

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29. Sep. 2003

## 1 Definitions from binary linear codes

### 1.1 Binary codes

Let  $\mathbb{F}_2 = GF(2)$  be the field of 2 elements. Let  $V = \mathbb{F}_2^n$  be the vector space of dimension  $n$  over  $\mathbb{F}_2$ . A linear  $[n, k]$  code  $C$  is a vector subspace of  $V$  of dimension  $k$ . An element  $\mathbf{x}$  in  $C$  is called a codeword of  $C$ . The inner product on  $V$ , which is denoted by  $\mathbf{x} \cdot \mathbf{y}$  for  $\mathbf{x}, \mathbf{y}$  in  $V$ , is defined as usual. Two codes  $C_1$  and  $C_2$  are said to be equivalent if and only if after a suitable change of coordinate positions of  $C_1$  all the codewords in both codes coincide.

Let  $C$  be a binary code of length  $n$ . An automorphism  $\sigma$  of the code  $C$  is an element of the permutation group of  $n$  letters  $S_n$  which leaves  $C$  invariant. All automorphisms of the code  $C$  form a group and it is denoted by  $Aut(C)$ .

The dual code  $C^\perp$  of  $C$  is defined by

$$C^\perp = \{\mathbf{u} \in V \mid \mathbf{u} \cdot \mathbf{v} = 0 \quad \forall \mathbf{v} \in C\}.$$

The code  $C$  is called self-orthogonal if it satisfies  $C \subseteq C^\perp$ , and the code  $C$  is called self-dual if it satisfies  $C = C^\perp$ . Self-dual codes exist only if  $n \equiv 0 \pmod{2}$ . For even  $n$  we let  $S_n$  denote the set of all self-dual binary codes of length  $n$ . Let

$$\mathbf{x} = (x_1, x_2, \dots, x_n)$$

be a vector in  $V$ , then the Hamming weight  $wt(\mathbf{x})$  of the vector  $\mathbf{x}$  is defined to be the number of  $i$ 's such that  $x_i \neq 0$ . The Hamming distance  $d$  on  $V$  is also defined by  $d(\mathbf{x}, \mathbf{y}) = wt(\mathbf{x} - \mathbf{y})$ . Let  $C$  be a code, then  $d$  of the code  $C$  is defined by

$$\begin{aligned} d &= \text{Min}_{\mathbf{x}, \mathbf{y} \in C, \mathbf{x} \neq \mathbf{y}} d(\mathbf{x}, \mathbf{y}) \\ &= \text{Min}_{\mathbf{x} \in C, \mathbf{x} \neq \mathbf{0}} wt(\mathbf{x}). \end{aligned}$$

Let  $C$  be a self-dual binary code, then the weight  $wt(\mathbf{x})$  of each codeword  $\mathbf{x}$  in  $C$  is even. Further, if the weight of each codeword  $\mathbf{x}$  in  $C$  is divisible by 4, then the code is called doubly even. It is known that a doubly even self-dual binary codes  $C$  exist only when the length  $n$  of  $C$  is a multiple of 8. In short a doubly even self-dual binary code is type II binary code.

Let  $C$  be a self-dual doubly even code of length  $n$ , which are embedded in  $\mathbb{F}_2^n$ . Let  $\mathbf{u} = (u_1, u_2, \dots, u_n), \mathbf{v} = (v_1, v_2, \dots, v_n)$  be any pair of vectors in  $\mathbb{F}_2^n$ , then the number of common 1's of the corresponding coordinates for  $\mathbf{u}$  and  $\mathbf{v}$  is denoted by  $\mathbf{u} * \mathbf{v}$ . This is called the intersection number of  $\mathbf{u}$  and  $\mathbf{v}$ , and  $\mathbf{u} * \mathbf{u}$  is nothing else  $wt(\mathbf{u})$ .

Let  $C$  be a type II binary  $[n, \frac{n}{2}]$  code. The homogeneous weight enumerator  $W_C(x, y)$  of the code  $C$  is defined by

$$W_C(x, y) = \sum_{\mathbf{v} \in C} x^{n-wt(\mathbf{v})} y^{wt(\mathbf{v})}$$

Following identity is known as the MacWilliams identity:

$$\begin{aligned} W_{\mathbf{C}}(x, y) &= \frac{1}{2^{\frac{n}{2}}} W_{\mathbf{C}}(x+y, x-y) \\ &= W_{\mathbf{C}}\left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}\right), \end{aligned} \quad (1)$$

Since  $\mathbf{C}$  is doubly even, each codeword  $u$  of  $\mathbf{C}$  has weight divisible by 4, and we know that

$$W_{\mathbf{C}}(x, iy) = W_{\mathbf{C}}(x, y). \quad (2)$$

Let  $G_1$  be the group generated by

$$\sigma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \text{ and } \sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$

The above two equations (1) and (2) show that the homogeneous weight enumerator of a type II binary code is invariant under linear action of the elements of the group  $G_1$ . Let  $\mathbb{C}[x, y]$  be the polynomial ring over the field of complex numbers  $\mathbb{C}$ . We let  $\mathbb{C}[x, y]^{G_1}$  to denote the subring of  $\mathbb{C}[x, y]$  consisting of all elements in  $\mathbb{C}[x, y]$  invariant under linear action of  $G_1$ . The following theorem is due to A. Gleason [9]

**Theorem 1.1** *It holds that*

$$\mathbb{C}[x, y]^{G_1} = \mathbb{C}[W_{e_8}(x, y), W_{gol_{24}}(x, y)],$$

where  $W_{e_8}(x, y)$  is the weight enumerator of the extended Hamming code of length 8, and  $W_{gol_{24}}(x, y)$  is the weight enumerator of the binary Golay code of length 24.

Let  $H_1$  be a subgroup of  $G_1$  generated by  $\sigma_1\sigma_2\sigma_1$  and  $\sigma_1$ . This subgroup is of index 2 in  $G_1$ . Let  $\mathbb{C}[x, y]^{H_1}$  be the ring of invariants for  $H_1$ . Then it is known that (see for instance [19])

**Theorem 1.2** *It holds that*

$$\mathbb{C}[x, y]^{H_1} = \mathbb{C}[W_{e_8}(x, y), E_{12}(x, y)],$$

where  $E_{12}(x, y) = x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12}$ .

## 1.2 Jacobi weight enumerator

**Definition: Jacobi polynomials for binary codes**

Jacobi polynomial  $Jac(\mathbf{C}, \mathbf{v} \mid X, Z)$  for  $\mathbf{C}$  with respect to  $\mathbf{v} \in \mathbb{F}_2^n$  is defined by

$$Jac(\mathbf{C}, \mathbf{v} \mid X, Z) = \sum_{u \in \mathbf{C}} X^{u \cdot \mathbf{v}} Z^{u \cdot \mathbf{v}}.$$

The homogeneous form of  $Jac(\mathbf{C}, \mathbf{v} \mid X, Z)$  is given by

$$Jac(\mathbf{C}, \mathbf{v}; x, y, u, v) = \sum_{t \in \mathbf{C}} x^{n - wt(t) - t \cdot \mathbf{v}} y^{wt(t) + t \cdot \mathbf{v}} u^{wt(t) - t \cdot \mathbf{v}} v^{t \cdot \mathbf{v}}.$$

**Theorem 1.3** *Let the notations be as above, then we have*

$$Jac(\mathbf{C}, \mathbf{v}; x', y', u', v') = Jac(\mathbf{C}, \mathbf{v}; x, y, u, v), \quad (3)$$

where

$$\begin{pmatrix} x' \\ y' \\ u' \\ v' \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix}$$

It may be remarked here that it holds

$$Jac(\mathbb{C}, \mathbf{v}; x, iy, u, iv) = Jac(\mathbb{C}, \mathbf{v}; x, y, u, v) \quad (4)$$

Let  $G_1 \oplus G_1$  be the group generated by  $diag(\sigma_1, \sigma_1)$  and  $diag(\sigma_2, \sigma_2)$ , and  $\mathbb{C}[x, y, u, v]$  be the polynomial ring in 4 independent variables over  $\mathbb{C}$ . We let  $\mathbb{C}[x, y, u, v]^{G_1 \oplus G_1}$  to denote the subring of  $\mathbb{C}[x, y, u, v]$  invariant under the linear action of each element of  $G_1 \oplus G_1$ . The above equations (3) and (4) implies that  $Jac(\mathbb{C}, \mathbf{v}; x, y, u, v)$  belongs to  $\mathbb{C}[x, y, u, v]^{G_1 \oplus G_1}$ . We have a Gleason type result for  $\mathbb{C}[x, y, u, v]^{G_1 \oplus G_1}$  ([4]).

Let  $H_1 \oplus H_1$  be the group generated by  $diag(\sigma_1, \sigma_1)$  and  $diag(\sigma_1 \sigma_2 \sigma_1, \sigma_1 \sigma_2 \sigma_1)$ , and  $R = \mathbb{C}[x, y, u, v]^{H_1 \oplus H_1}$  be the ring of invariants for the group  $H_1 \oplus H_1$ . We also have a Gleason type result for  $R$ . Here we briefly describe the result. When a polynomial  $f(x, y, u, v)$  of total degree  $n$  belongs to  $R$  we call the partial degree of  $f$  with respect to the variables  $u$  and  $v$  the index of  $f$ . The Molien series for  $H_1 \oplus H_1$  is given by

$$\begin{aligned} \Phi_{H_1 \oplus H_1}(t) &= \sum_{n \geq 0} \dim_{\mathbb{C}}(FJac_n) t^n \\ &= \frac{1 + 8t^8 + 18t^{12} + 21t^{16} + 19t^{20} + 21t^{24} + 7t^{28} + t^{32}}{(1 - t^8)^2 (1 - t^{12})^2} \\ &= 1 + 10t^8 + 20t^{12} + 40t^{16} + 75t^{20} + 130t^{24} + 179t^{28} + 283t^{32} + \\ &\quad 383t^{36} + 513t^{40} + 678t^{44} + 883t^{48} + 1078t^{52} + 1372t^{56} + \\ &\quad + 1658t^{60} + 1994t^{64} + 2385t^{68} + 2836t^{72} + \dots \end{aligned}$$

We decompose this ring  $R$  into a direct sum :

$$R = \bigoplus_{n \geq 0} R_n,$$

where  $R_n$  is the  $n$ -th homogeneous part of  $R$ . Further we decompose  $R_n$  as

$$R_n = \bigoplus_{0 \leq m \leq n} R_{n,m},$$

where  $R_{n,m}$  is the set of polynomials  $f(x, y, u, v) \in R_n$  with partial degree with respect to  $u$  and  $v$  equal to  $m$ . This set  $R_{n,m}$  forms a vector subspace of  $R$ .

## 2 Jacobi forms

### 2.1 Definition of Jacobi forms

Let  $\mathbb{H}$  be the complex upper half plane and  $\tau$  be a variable on  $\mathbb{H}$ . Let  $\mathbb{C}$  be the complex plane and  $z$  be a variable on  $\mathbb{C}$ . A complex valued holomorphic function  $\phi(\tau, z)$  defined on  $\mathbb{H} \times \mathbb{C}$  is called a Jacobi form of weight  $k$  and index  $h$  with respect to the pair  $(SL_2(\mathbb{Z}), \mathbb{Z})$  if it satisfies the conditions (5), (6) and (7) below:

$$\phi(\tau, z) = (c\tau + d)^{-k} e^{2\pi i h \left( \frac{-cz^2}{c\tau + d} \right)} \phi \left( \frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) \text{ holds for } \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \quad (5)$$

$$\phi(\tau, z) = e^{2\pi i h (\lambda^2 \tau + 2\lambda z)} \phi(\tau, z + \lambda\tau + \mu) \text{ for } \lambda, \mu \in \mathbb{Z} \quad (6)$$

$\phi(\tau, z)$  has a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{n \geq r^2/4h} c(n, r) q^n \zeta^r \quad (7)$$

## 2.2 Eisenstein Jacobi forms

One major construction method of Jacobi forms is Eisenstein Jacobi forms (c.f. [8], pages 17-18).

$$\begin{aligned}
E_{k,m}(\tau, z) &= \\
&= \frac{1}{2} \sum_{\substack{c, d \in \mathbb{Z} \\ (c, d) = 1}} \sum_{\lambda \in \mathbb{Z}} (c\tau + d)^{-k} e^{m\lambda} \left( \lambda^2 \frac{a\tau + b}{c\tau + d} + 2\lambda \frac{z}{c\tau + d} - \frac{cz^2}{c\tau + d} \right) \\
&= \sum_{\substack{n, r \in \mathbb{Z} \\ 4nm \geq r^2}} e_{k,m}(n, r) q^n \zeta^r
\end{aligned}$$

where  $a, b$  are chosen so that  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ .

## 3 Massformula for Jacobi weight enumerators

### 3.1 Mass formula for ordinary weight enumerators

For  $1 \leq h < \frac{n}{2}$  let  $C_0$  be a binary self-orthogonal code of length  $n$  and dimension  $h$  containing all one vector  $\mathbf{1}$  in  $\mathbb{F}_2^n$ . We denote by

$$\nu(n, h) = \#\{C \in \mathcal{D}_n \mid C \supset C_0\}.$$

This is independent of the choice of  $C_0$ .

We recall that  $\mathcal{S}_n$  is the set of all binary self-dual codes of length  $n$  for each even integer  $n$ . We denote by

$$\mu(n, h) = \#\{C \in \mathcal{S}_n \mid C \supset C_0\}.$$

We quote a well-known result

**Proposition 3.1** ([20]) *It holds that*

$$\nu(n, h) = \prod_{j=0}^{\frac{n}{2}-h-1} (2^j + 1).$$

**Proposition 3.2** ([20]) *For  $h$  with  $1 \leq h < \frac{n}{2}$  it holds that*

$$\mu(n, h) = \prod_{j=1}^{\frac{n}{2}-h} (2^j + 1).$$

J.G. Thompson [20] proved that

$$\sum_{C \in \mathcal{D}_n} W_1(x, y; C) = \nu(n, 1)(x^n + y^n) + \nu(n, 2) \sum_{\substack{0 < j < n \\ 4|j}} \binom{n}{j} x^{n-j} y^j.$$

If we define

$$\begin{aligned}
W_1^{(n)}(x, y) &= \sum_{4|j} \binom{n}{j} x^{n-j} y^j \\
&= \frac{1}{4}((x+y)^n + (x-y)^n + (x+iy)^n + (x-iy)^n),
\end{aligned}$$

then

$$\sum_{C \in \mathcal{D}_n} W_1(x, y; C) = \nu(n, 2)(2^{n/2-2}(x^n + y^n) + W_1^{(n)}(x, y)).$$

Recall that the root system  $D_4$  consists of the 24 roots listed below:

$$\begin{aligned} & \pm\sqrt{2}e_j \quad (j = 1, 2, 3, 4), \\ & \frac{1}{\sqrt{2}}(\pm 1, \pm 1, \pm 1, \pm 1). \end{aligned}$$

We imbed these vectors into  $\mathbb{C}^2$  as follows.

$$\begin{aligned} & i^k\sqrt{2}e_j \quad (j = 1, 2, k = 0, 1, 2, 3), \\ & \zeta^j e_1 + \zeta^k e_2 \quad (j, k = 1, 3, 5, 7), \end{aligned}$$

where  $\zeta = e^{\pi i/4}$ . Now, let  $D_4$  denote the set of 24 vectors above. If  $n \equiv 0 \pmod{4}$ , then

$$\begin{aligned} \sum_{\alpha \in D_4} (\alpha_1 x + \alpha_2 y)^n &= 2^{n/2+2}(x^n + y^n) + \sum_{j,k=1,3,5,7} (\zeta^j x + \zeta^k y)^n \\ &= 2^{n/2+2}(x^n + y^n) + (-1)^{n/4} \sum_{j,k=0,2,4,6} (\zeta^j x + \zeta^k y)^n \\ &= 16(2^{n/2-2}(x^n + y^n) + (-1)^{n/4} W_n^{(1)}(x, y)). \end{aligned} \quad (8)$$

### 3.2 A Theorem

Using the notation introduced in the previous section, the mass formula for the Jacobi weight enumerator polynomial can easily be established. The Jacobi weight enumerator polynomial for a code  $C$  with respect to a reference vector  $u$  is defined by

$$\text{Jac}(C, u; x_{00}, x_{01}, x_{10}, x_{11}) = \sum_{v \in C} X(u, v).$$

Denote by  $\overline{\text{Jac}}_{n,k}$  the sum of the Jacobi weight enumerator polynomial with respect to a fixed reference vector of weight  $k$  for all  $C \in \mathcal{D}_n$ . Note that  $\overline{\text{Jac}}_{n,k}$  is independent of the choice of  $u$ . We prove

**Theorem 3.3** (Munemasa-Ozeki [13])

$$\overline{\text{Jac}}_{n,k} = \frac{1}{16} \nu(n, 2) \sum_{\alpha=(\alpha_1, \alpha_2) \in D_4} (\alpha_1 x_{00} + \alpha_2 x_{01})^{n-k} (\alpha_1 x_{10} + \alpha_2 x_{11})^k. \quad (9)$$

## 4 An application of the mass formula to the construction of Jacobi forms

### 4.1 Some instances

If we apply the so called Bannai-Ozeki map (c.f. [2]) to the right hand side of (9), we obtain many important Jacobi forms of weight  $n/2$  and index  $k$ . As the mass formula the both hands are meaningful only when  $n$  is divisible by 8. However the polynomials in the right hand side are useful even if  $n \equiv 4 \pmod{8}$  in constructing Jacobi forms. Here we give few instances of the construction.

To do this we recall Jacobi's theta functions:

$$\begin{aligned} \theta_0(\tau, z) &= \sum_{n \in \mathbb{Z}} (-1)^n e^{\pi n^2 \tau + 2n\pi iz}, \\ \theta_2(\tau, z) &= \sum_{n \in \mathbb{Z}} e^{\pi(n+1/2)^2 \tau + (2n+1)\pi iz}, \\ \theta_3(\tau, z) &= \sum_{n \in \mathbb{Z}} e^{\pi n^2 \tau + 2n\pi iz}. \end{aligned}$$

We put  $\varphi_i(\tau, z) = \theta(2\tau, 2z)$ , and  $\varphi_i(\tau) = \varphi_i(\tau, 0)$  ( $i = 2, 3$ ).

When  $n = 8$  and  $k = 1$  the right hand side of (9) becomes 30 times of

$$7y^4x^3u + 7x^4y^3v + y^7v + x^7u.$$

Substituting  $x = \varphi_2(\tau)$ ,  $y = \varphi_3(\tau)$ ,  $u = \varphi_2(\tau, z)$ ,  $v = \varphi_3(\tau, z)$  in this polynomial we get a Jacobi form of weight 4 and index 1:

$$\begin{aligned} \psi_{4,1} = & 1 + (\zeta^2 + 56\zeta + 56\zeta^{-1} + \zeta^{-2} + 126)q \\ & + (126\zeta^2 + 576\zeta + 576\zeta^{-1} + 126\zeta^{-2} + 756)q^2 \\ & + (56\zeta^3 + 756\zeta^2 + 1512\zeta + 1512\zeta^{-1} + 756\zeta^{-2} + 56\zeta^{-3} + 2072)q^3 \\ & + (\zeta^4 + 576\zeta^3 + 2072\zeta^2 + 4032\zeta + 4032\zeta^{-1} + 2072\zeta^{-2} + 576\zeta^{-3} + \zeta^{-4} + 4158)q^4 \\ & (126\zeta^4 + 1512\zeta^3 + 4158\zeta^2 + 5544\zeta + 5544\zeta^{-1} + 4158\zeta^{-2} + 1512\zeta^{-3} + 126\zeta^{-4} + 7560)q^5 + \dots \end{aligned}$$

When  $n = 8$  and  $k = 2$  the right hand side of (9) becomes 30 times of

$$3x^4y^2v^2 + 8x^3y^3uv + 3x^2y^4u^2 + x^6u^2 + y^6v^2.$$

The last polynomial leads to a Jacobi form of weight 4 and index 2:

$$\begin{aligned} \psi_{4,2} = & 1 + [14(\zeta^2 + \zeta^{-2}) + 64(\zeta + \zeta^{-1}) + 84]q + \\ & + [\zeta^4 + \zeta^{-4} + 64(\zeta^3 + \zeta^{-3}) + 280(\zeta^2 + \zeta^{-2}) + 448(\zeta + \zeta^{-1}) + 574]q^2 \\ & + (84\zeta^4 + 448\zeta^3 + 840\zeta^2 + 1344\zeta + 1344\zeta^{-1} + 840\zeta^{-2} + 448\zeta^{-3} + 84\zeta^{-4} + 1288)q^3 \\ & + [64(\zeta^5 + \zeta^{-5}) + 574(\zeta^4 + \zeta^{-4}) + 1344(\zeta^3 + \zeta^{-3}) + 2368(\zeta^2 + \zeta^{-2}) + 2688(\zeta + \zeta^{-1}) + 3444]q^4 \\ & + [14(\zeta^6 + \zeta^{-6}) + 448(\zeta^5 + \zeta^{-5}) + 1288(\zeta^4 + \zeta^{-4}) + 2688(\zeta^3 + \zeta^{-3}) \\ & + 3542(\zeta^2 + \zeta^{-2}) + 4928(\zeta + \zeta^{-1}) + 4424]q^5 + \dots \end{aligned}$$

When  $n = 12$  and  $k = 1$  the right hand side of (9) is a polynomial that is 4050 times of

$$-22x^4y^7v - 11x^8y^3v - 11y^8x^3u - 22y^4x^7u + x^{11}u + y^{11}v.$$

This leads to a Jacobi form of weight 6 and index 1

$$\begin{aligned} \psi_{6,1} = & 1 + (\zeta^2 - 88\zeta - 88\zeta^{-1} + \zeta^{-2} - 330)q \\ & + (-330\zeta^2 - 4224\zeta - 4224\zeta^{-1} - 330\zeta^{-2} - 7524)q^2 \\ & + (-88\zeta^3 - 7524\zeta^2 - 30600\zeta - 30600\zeta^{-1} - 7524\zeta^{-2} - 88\zeta^{-3} - 46552)q^3 \\ & + (\zeta^4 - 4224\zeta^3 - 46552\zeta^2 - 130944\zeta \\ & - 130944\zeta^{-1} - 46552\zeta^{-2} - 4224\zeta^{-3} + \zeta^{-4} - 169290)q^4 \\ & (-330\zeta^4 - 30600\zeta^3 - 169290\zeta^2 - 355080\zeta \\ & - 355080\zeta^{-1} - 169290\zeta^{-2} - 30600\zeta^{-3} - 330\zeta^{-4} - 464904)q^5 + \dots \end{aligned}$$

When  $n = 12$  and  $k = 2$  the right hand side of (9) is 4050 times of the polynomial

$$-14y^4x^6u^2 - 14y^6x^4v^2 - 3y^2x^8v^2 - 3y^8x^2u^2 + x^{10}u^2 + y^{10}v^2 - 16y^3x^7uv - 16y^7x^3uv.$$

This leads to a Jacobi form of weight 6 and index 2:

$$\begin{aligned}
\psi_{6,2} = & 1 + (-10\zeta^2 - 128\zeta - 128\zeta^{-1} - 10\zeta^{-2} - 228)q \\
& + (\zeta^4 - 128\zeta^3 - 1496\zeta^2 - 3968\zeta \\
& - 3968\zeta^{-1} - 1496\zeta^{-2} - 128\zeta^{-3} + \zeta^{-4} - 5450)q^2 \\
& + (-228\zeta^4 - 3968\zeta^3 - 14088\zeta^2 - 27264\zeta \\
& - 27264\zeta^{-1} - 14088\zeta^{-2} - 3968\zeta^{-3} - 228\zeta^{-4} - 31880)q^3 \\
& + (-128\zeta^5 - 5450\zeta^4 - 27264\zeta^3 - 67712\zeta^2 - 103680\zeta \\
& - 103680\zeta^{-1} - 67712\zeta^{-2} - 27264\zeta^{-3} - 5450\zeta^{-4} - 128\zeta^{-5} - 124260)q^4 \\
& (-10\zeta^6 - 3968\zeta^5 - 31880\zeta^4 - 103680\zeta^3 - 197650\zeta^2 - 292480\zeta \\
& - 292480\zeta^{-1} - 197650\zeta^{-2} - 103680\zeta^{-3} - 31880\zeta^{-4} - 3968\zeta^{-5} - 10\zeta^{-6} - 316168)q^5 + \dots
\end{aligned}$$

In this way we obtain an infinite family of Jacobi forms of various weights and various indices.

## 4.2 A comparison of two constructions

In [8] only the values  $e_{k,m}(n,r)$  ( $k \leq 8, m = 1$ ) of the Fourier coefficients of  $E_{k,m}(\tau, z)$  are given explicitly.

Here we explain a method to compute  $e_{k,m}(n,r)$  for any even  $k$  and  $m \geq 1$ . For this we start from the formula given in [8] page 22:

$$e_{k,m}(n,r) = \frac{\sigma_{k-1}(m)^{-1}}{\zeta(3-2k)} \sum_{d|(n,r,m)} d^{k-1} H(k-1, \frac{4nm-r^2}{d^2}),$$

and

$$e_{k,1}(n,r) = \frac{H(k-1, 4n-r^2)}{\zeta(3-2k)},$$

where  $\zeta(3-2k)$  is the special value of Riemann's zeta function. The quantity  $H(k-1, N)$  is described at page 30 in [8]:

$$H(k-1, N) = \begin{cases} L_{-N}(2-k) & \text{if } N > 0 \text{ and } N \equiv 0 \text{ or } 3 \pmod{4}, \\ \zeta(3-2k) & \text{if } N = 0, \\ 0 & \text{if } N > 0 \text{ and } N \equiv 1 \text{ or } 2 \pmod{4}. \end{cases}$$

When  $-N \equiv 0$  or  $1 \pmod{4}$  we put  $-N = (-N_0)u^2$   $u \in \mathbb{N}$  so that  $-N_0$  is the discriminant of the quadratic number field  $\mathbb{Q}(\sqrt{-N})$ . The number  $L_{-N}(2-k)$  comes from the L-function  $L_{-N_0}(s)$  by way of

$$L_{-N}(s) = L_{-N_0}(s) \sum_{d|u} \mu(d) \left( \frac{-N_0}{d} \right) d^{-s} \sigma_{1-2s} \left( \frac{u}{d} \right),$$

and

$$L_{-N_0}(s) = L(s, \left( \frac{-N_0}{*} \right)) = \sum_{n=1}^{\infty} \frac{\left( \frac{-N_0}{n} \right)}{n^s}.$$

To make the value  $e_{k,1}(n,r)$  explicit it is necessary to know the values  $\zeta(3-2k)$  and  $L_{-N_0}(1-m)$ . As to the values  $\zeta(3-2k)$  there are many literature available and they tell us that

$$\begin{aligned}
\zeta(2k) &= \frac{(-1)^{k-1} (2\pi)^{2k}}{2} \frac{B_{2k}}{(2k)!} \quad (k \geq 1) \\
\zeta(1-n) &= (-1)^{n-1} \frac{B_n}{n} \quad (n = 1, 2, \dots),
\end{aligned}$$

where  $B_n$  is the  $n$ -th Bernoulli number. The beginning few numbers are

$$B_1 = \frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, \dots, B_{\text{odd} \geq 3} = 0.$$



By these one gets

$$\zeta(2) = \frac{\pi^2}{6}, \zeta(4) = \frac{\pi^4}{90}, \zeta(6) = \frac{\pi^6}{945}, \zeta(8) = \frac{\pi^8}{9450}, \zeta(10) = \frac{\pi^{10}}{93555}, \dots$$

$$\zeta(-1) = -\frac{1}{12}, \zeta(-2) = 0, \zeta(-3) = \frac{1}{120}, \zeta(-5) = -\frac{1}{252}, \zeta(-7) = \frac{1}{240}, \zeta(-9) = -\frac{1}{132}, \dots$$

It is much complicated to get the values  $L_{-N_0}(1-m)$ . On reading the book [1] we find a suitable formula to do this task. Note that a similar formula has been given in [11] Chapter XIII in a not straight way.

**Theorem 4.1 (Arakawa-Ibukiyama-Kaneko)** Let  $\chi$  be a primitive character mod  $f$  and  $m$  be a positive integer, then

$$L(1-m, \chi) = -\frac{B_{m,\chi}}{m},$$

where  $B_{m,\chi}$  is the generalized Bernoulli number associated with  $\chi$ :

$$B_{m,\chi} = f^{m-1} \sum_{a=1}^f \chi(a) B_m\left(\frac{a}{f}\right),$$

and  $B_m(x)$  is the Bernoulli polynomial of degree  $m$ .

The Bernoulli polynomials are given by

$$B_m(x) = \sum_{j=0}^m (-1)^j \binom{m}{j} B_j x^{m-j}.$$

With the above Theorem we compute  $e_{k,m}(n,r)$ , and we give small tables of them, that are not contained in [8].

$4n-r^2$	0	3	4	7	8	11	12	15
$e_{10,1}(n,r)$	1	$-\frac{860776}{43867}$	$-\frac{9947070}{43867}$	$-\frac{1159757568}{43867}$	$-\frac{3601586268}{43867}$	$-\frac{53854227000}{43867}$	$-\frac{113044851304}{43867}$	$-\frac{754799931648}{43867}$
$e_{12,1}(n,r)$	1	$\frac{339848}{77683}$	$\frac{6871898}{77683}$	$\frac{2485779648}{77683}$	$\frac{10096500348}{77683}$	$\frac{285849348696}{77683}$	$\frac{713061257096}{77683}$	$\frac{7428376170816}{77683}$

$4n-r^2$	16	19	20
$e_{10,1}(n,r)$	$-\frac{1303792306110}{43867}$	$-\frac{5607166776120}{43867}$	$-\frac{8689286943288}{43867}$
$e_{12,1}(n,r)$	$\frac{14621136806394}{77683}$	$\frac{88801830903192}{77683}$	$\frac{152244273101400}{77683}$

$8n-r^2$	0	4	7	8	12	15	16	20	23	24	28	31	32
$e_{4,2}(n,r)$	1	14	64	84	280	448	574	840	1344	1288	2368	2688	3444
$e_{6,2}(n,r)$	1	-10	-128	-228	-1496	-3968	-5450	-14088	-27264	-31880	-67712	-103680	-124260
$e_{8,2}(n,r)$	1	$\frac{122}{43}$	$\frac{4872}{43}$	$\frac{11052}{43}$	$\frac{3640}{43}$	$\frac{862464}{43}$	$\frac{1015162}{43}$	$\frac{4266360}{43}$	$\frac{10665792}{43}$	$\frac{13948984}{43}$	$\frac{38576704}{43}$	$\frac{74169984}{43}$	$\frac{91963692}{43}$

We remark that the functions  $\psi_{4,1}, \psi_{6,1}, \psi_{8,1}$  respectively coincide with Eisenstein-Jacobi forms  $E_{4,1}, E_{6,1}, E_{8,1}$  respectively of index 1 described in [8] pages 17-23. Explicit Fourier expansions of Eisenstein-Jacobi forms of index  $\geq 2$  are not given in [8]. We have verified that  $\psi_{4,2}, \psi_{6,2}$  also coincide with Jacobi-Eisenstein series of index 2. This is done by using the relation (7) in [8], page 22. Besides these exceptional cases Eisenstein-Jacobi form  $E_{k,m}$  differ from  $\psi_{k,m}$ . One may be interested with a problem to explore the further relations between these two constructions of Jacobi forms.

## 5 Eisenstein type polynomials in more variables

$$E_{k_1, k_2}(x_{00}, x_{01}, x_{10}, x_{11}, y_{10}, y_{11}) = \frac{1}{16} \nu(n, 2) \sum_{\alpha=(\alpha_1, \alpha_2) \in D_4} (\alpha_1 x_{00} + \alpha_2 x_{01})^{n-k_1-k_2} (\alpha_1 x_{10} + \alpha_2 x_{11})^{k_1} (\alpha_1 y_{10} + \alpha_2 y_{11})^{k_2}, \quad (10)$$

$$E_{k_1, k_2, k_3}(x_{00}, x_{01}, x_{10}, x_{11}, y_{10}, y_{11}, z_{10}, z_{11}) = \frac{1}{16} \nu(n, 2) \sum_{\alpha=(\alpha_1, \alpha_2) \in D_4} (\alpha_1 x_{00} + \alpha_2 x_{01})^{n-k_1-k_2-k_3} (\alpha_1 x_{10} + \alpha_2 x_{11})^{k_1} (\alpha_1 y_{10} + \alpha_2 y_{11})^{k_2} (\alpha_1 z_{10} + \alpha_2 z_{11})^{k_3} \quad (11)$$

⋮

where all exponents are non negative integers.

The right-hand side of (10) belongs to  $\mathbb{C}[x_{00}, x_{01}, x_{10}, x_{11}, y_{10}, y_{11}]^{H_1 \oplus H_1 \oplus H_1}$ , and the right-hand side of (11) belongs to  $\mathbb{C}[x_{00}, x_{01}, x_{10}, x_{11}, y_{10}, y_{11}, z_{10}, z_{11}]^{H_1 \oplus H_1 \oplus H_1 \oplus H_1}$ . As discussed in [2] these polynomials contribute to the construction of Jacobi forms.

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