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Fuzzy rough sets, gradual decision rules and approximate reasoning

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Abstract: We have proposed a fuzzy rough set approach without using any fuzzy logical connectives to extract gradual decision rules from decision tables. In this paper, we discuss the use of these gradual decision rules within modus ponens and modus tollens inference patterns. We show that these patterns are very similar and, moreover, we generalize them to formalize approximate reasoning based on the extracted gradual decision rules. We demonstrate that approximate reasoning can be performed by manipulation of modifier functions associated with the gradual decision rules.
Keywords: Rough sets, Fuzzy sets, Gradual decision rules, Modifier function, Approximate reasoning

1. Introduction

Rough set theory deals mainly with the ambiguity of information caused by granular description of objects, while fuzzy set theory treats mainly the uncertainty of concepts and linguistic categories. Because of the difference in the treatment of uncertainty, fuzzy set theory and rough set theory are complementary and their various combinations have been studied by many researchers (see for example Cattaneo 1998, Dubois, Prade 1992b, Greco, Matarazzo, Slowinski 1999, 2000a,b, Inuiguchi, Tanino 2003, Nakamura, Gao 1991, Polkowski 2002, Slowinski 1995, Slowinski, Stefanowski 1996, Yao 1997). Most of them involved some fuzzy logical connectives (t-norm, t-conorm, fuzzy implication) to define fuzzy set operations. It is known, however, that selection of the "right" fuzzy logical connectives is not an easy task and that the results of fuzzy rough set analysis are sensitive to this selection. The authors (Greco, Inuiguchi, Slowinski 2003a) have proposed fuzzy rough sets without using any fuzzy logical connectives to extract gradual decision rules from decision tables. Within this approach, lower and upper approximations, are defined using modifier functions following from a given decision table.

This paper presents results of a fundamental study concerning utilization of knowledge obtained by the fuzzy rough set approach proposed in (Greco, Inuiguchi, Slowinski 2003a). Since the obtained knowledge is represented by gradual decision rules, we discuss inference patterns (modus ponens and modus tollens) for gradual decision rules. We show that the modus ponens and modus tollens are very similar in our approach. Moreover, we discuss inference patterns of the generalized modus ponens as a basis for approximate reasoning. The results demonstrate that approximate reasoning can be performed by manipulation of modifier functions associated with the extracted gradual decision rules.

In the next section, we review gradual decision rules extracted from a decision table and underlying fuzzy rough sets. We describe fuzzy-rough modus ponens and modus tollens with respect to the extracted gradual decision rules in Section 3. We show the high similarity between fuzzy-rough modus ponens and modus tollens. In Section 4, we generalize the modus ponens and modus tollens in order to make inference using different fuzzy sets in the gradual decision rules. We demonstrate that all inference can be done by manipulation of modifier functions. Finally, we give concluding remarks in Section 5.

2. Gradual decision rules extracted from a decision table

In a given decision table, we may found some gradual decision rules of the following types (Greco, Inuiguchi, Slowinski 2003a):

- *lower-approximation rules with positive relationship* (LP-rule): "if condition X has credibility $C(X)\geq\alpha$, then decision Y has credibility $C(Y)\geq f_{Y|X}^{+}(\alpha)$";
- *lower-approximation rules with negative relationship* (LN-rule): "if condition X has credibility $C(X)\leq\alpha$, then decision Y has credibility $C(Y)\geq f_{Y|X}^{-}(\alpha)$";
- *upper-approximation rule with positive relationship* (UP-rule): "if condition X has credibility $C(X)\leq\alpha$, then decision Y could have credibility $C(Y)\leq g_{Y|X}^{+}(\alpha)$";
- *upper-approximation rule with negative relationship* (UN-rule): "if condition X has credibility $C(X)\geq\alpha$, then decision Y could have credibility $C(Y)\leq g_{Y|X}^{-}(\alpha)$",

where X is a given condition (premise), Y is a given decision (conclusion) and $f_{Y|X}^{+}:[0,1]\rightarrow[0,1]$, $f_{Y|X}^{-}:[0,1]\rightarrow[0,1]$, $g_{Y|X}^{+}:[0,1]\rightarrow[0,1]$ and $g_{Y|X}^{-}:[0,1]\rightarrow[0,1]$ are functions relating the credibility of X with the credibility of Y in lower- and upper-approximation rules, respectively. Those functions can be seen as modifier functions (see, for example, Inuiguchi, Greco, Slowinski, Tanino 2003). An LP-rule can be regarded as a gradual decision rule (Dubois, Prade 1992a); it can be interpreted as: "the more object x is X, the more it is Y". In this case, the relationship between credibility of premise and conclusion is positive and certain. LN-rule can be interpreted in turn as: "the less object x is X, the more it is Y", so the relationship is negative and certain. On the other hand, the UP-rule can be interpreted as: "the more object x is X, the more it could be Y", so the relationship is positive and possible. Finally, UN-rule can be interpreted as: "the less object x is X, the more it could be Y", so the relationship is negative and possible.

Table 1. A decision maker's evaluation of sample cars

Car:	A	B	C	D	E	F	G	H	I	J
mileage (km/l)	12	12	13	14	15	9	11	8	14	13
$\mu_{gas\_saving\_car}$	0.5	0.5	0.67	0.83	1	0	0.33	0	0.83	0.67
acceptability	0.6	0.5	0.6	0.8	0.9	0.3	0.5	0.3	0.8	0.6

Example 1. Let us consider a decision table about hypothetical car selection problem in which the mileage is used for evaluation of cars. We may define a fuzzy set X of *gas_saving_cars* by the following membership function:

$$\mu_{gas\_saving\_car}(x)=\begin{cases}0 & \text{if } mileage(x)<9\\ (mileage(x)-9)/6 & \text{if } 9\leq mileage(x)<15\\ 1 & \text{if } mileage(x)\geq 15\end{cases}.$$

From Table 1, we may find the following gradual decision rules:

- LP-rule: "if x is *gas_saving_car* with credibility $\mu_{gas\_saving\_car}(mileage(x))\geq\alpha$, then x is *acceptable_car* with credibility $\mu_{acceptable\_car}(x)\geq f_{Y|X}^{+}(\alpha)$";
- UP-rule: "if x is *gas_saving_car* with credibility $\mu_{gas\_saving\_car}(mileage(x))\leq\alpha$, then x is *acceptable_car* with credibility $\mu_{acceptable\_car}(x)\leq g_{Y|X}^{+}(\alpha)$",

where $f_{Y|X}^{+}$ and $g_{Y|X}^{+}$ are defined by

$$f_{Y|X}^{+}(\alpha)=\begin{cases}0 & \text{if } \alpha=0\\ 0.3 & \text{if } 0<\alpha<0.33\\ 0.5 & \text{if } 0.33\leq\alpha<0.67\\ 0.6 & \text{if } 0.67\leq\alpha<0.83\\ 0.8 & \text{if } 0.83\leq\alpha<1\\ 0.9 & \text{if } \alpha=1\end{cases} \quad\text{and}\quad g_{Y|X}^{+}(\alpha)=\begin{cases}0.3 & \text{if } \alpha=0\\ 0.5 & \text{if } 0<\alpha\leq 0.33\\ 0.6 & \text{if } 0.33<\alpha\leq 0.67\\ 0.8 & \text{if } 0.67<\alpha\leq 0.83\\ 0.9 & \text{if } 0.83<\alpha<1\\ 1 & \text{if } \alpha=1\end{cases}.$$

In Example 1, we consider a fuzzy set of gas saving cars as condition of rules but if we would consider a fuzzy set of gas guzzler cars as condition of rules, we would obtain LN- and UN-rules. As illustrated in this example, the condition X and decision Y can be represented by fuzzy sets.

The functions $f_{Y|X}^{+}(\cdot)$, $f_{Y|X}^{-}(\cdot)$, $g_{Y|X}^{+}(\cdot)$ and $g_{Y|X}^{-}(\cdot)$ are related to specific definitions of lower and upper approximations considered within rough set theory (Pawlak 1991). Suppose that we want to approximate knowledge contained in Y using knowledge about X. Let us also adopt the hypothesis that X is positively related to Y. Then, we can define the lower approximation $\underline{App}^{+}(X,Y)$, and upper approximation $\overline{App}^{+}(X,Y)$ of Y by the following membership functions:

$$\mu[\underline{App}^{+}(X,Y),x] = \begin{cases} \inf\limits_{z\in U:\mu_X(z)\geq\mu_X(x)}\{\mu_Y(z)\}, & \text{if } \mu_X(x)>0, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mu[\overline{App}^{+}(X,Y),x] = \begin{cases} \sup\limits_{z\in U:\mu_X(z)\leq\mu_X(x)}\{\mu_Y(z)\}, & \text{if } \mu_X(x)<1, \\ 1, & \text{otherwise.} \end{cases}$$

Similarly, if we adopt the hypothesis that X is negatively related to Y, then we can define the lower approximation $\underline{App}^{-}(X,Y)$, and upper approximation $\overline{App}^{-}(X,Y)$ of Y by the following membership functions:

$$\mu[\underline{App}^{-}(X,Y),x] = \begin{cases} \inf\limits_{z\in U:\mu_X(z)\leq\mu_X(x)}\{\mu_Y(z)\} & \text{if } \mu_X(x)<1 \\ 0 & \text{otherwise} \end{cases},$$

$$\mu[\overline{App}^{-}(X,Y),x] = \begin{cases} \sup\limits_{z\in U:\mu_X(z)\geq\mu_X(x)}\{\mu_Y(z)\} & \text{if } \mu_X(x)>0 \\ 1 & \text{otherwise} \end{cases}.$$

The lower and upper approximations defined above can serve to induce certain and approximate decision rules in the following way. Let us remark that inferring lower and upper credibility rules is equivalent to finding modifiers $f_{Y|X}^{+}(\cdot)$, $f_{Y|X}^{-}(\cdot)$, $g_{Y|X}^{+}(\cdot)$ and $g_{Y|X}^{-}(\cdot)$.

These functions can be defined as follows: for each $\alpha\in[0,1]$

$$f_{Y|X}^{+}(\alpha) = \sup_{\mu_X(x)\leq\alpha}\left\{\mu\left[\underline{App}^{+}(X,Y),x\right]\right\} = \begin{cases} \sup\limits_{x\in U:\mu_X(x)\leq\alpha}\left(\inf\limits_{z\in U:\mu_X(z)\geq\mu_X(x)}\mu_Y(z)\right) & \text{if } \alpha>0 \\ 0 & \text{if } \alpha=0 \end{cases},$$

$$f_{Y|X}^{-}(\alpha) = \sup_{\mu_X(x)\geq\alpha}\left\{\mu\left[\underline{App}^{-}(X,Y),x\right]\right\} = \begin{cases} \sup\limits_{x\in U:\mu_X(x)\geq\alpha}\left(\inf\limits_{z\in U:\mu_X(z)\leq\mu_X(x)}\mu_Y(z)\right) & \text{if } \alpha>0 \\ 0 & \text{if } \alpha=0 \end{cases},$$

$$g_{Y|X}^{+}(\alpha) = \inf_{\mu_X(x)\geq\alpha}\left\{\mu\left[\overline{App}^{+}(X,Y),x\right]\right\} = \begin{cases} \inf\limits_{z\in U:\mu_X(x)\geq\alpha}\left(\sup\limits_{z\in U:\mu_X(z)\leq\mu_X(x)}\mu_Y(z)\right) & \text{if } \alpha<1 \\ 0 & \text{if } \alpha=1 \end{cases},$$

$$g_{Y|X}^{-}(\alpha) = \inf_{\mu_X(x)\leq\alpha}\left\{\mu\left[\overline{App}^{-}(X,Y),x\right]\right\} = \begin{cases} \inf\limits_{z\in U:\mu_X(x)\leq\alpha}\left(\sup\limits_{z\in U:\mu_X(z)\geq\mu_X(x)}\mu_Y(z)\right) & \text{if } \alpha<1 \\ 0 & \text{if } \alpha=1 \end{cases}.$$

We may define a fuzzy rough set by a pair of lower and upper approximations. Some properties of fuzzy rough sets have been investigated in (Greco, Inuiguchi, Slowinski 2003a).

3. Fuzzy-rough modus ponens and modus tollens

Given a decision table, we may induce gradual decision rules from X to Y expressed by functions $f_{Y|X}^{+}(\cdot)$ and $g_{Y|X}^{+}(\cdot)$ or by functions $f_{Y|X}^{-}(\cdot)$ and $g_{Y|X}^{-}(\cdot)$. Remark that we may also induce gradual decision rules from Y to X in the same way. For example, when we have a rule "if the speed of a truck is high, then its damage in a crash is big", we may obtain a rule "if the damage of a truck is big, then its speed had been high before the crash" at the same time.

Such invertibility often occurs when X and Y strongly coincide each other; in other words, Y can be explained by X almost completely. In order to clarify the differences between gradual decision rules from X to Y and from Y to X, we are using the following notation. By $f_{Y|X}^{+}(\cdot)$, $f_{Y|X}^{-}(\cdot)$, $g_{Y|X}^{+}(\cdot)$ and $g_{Y|X}^{-}(\cdot)$, we denote modifier functions corresponding to gradual decision rules from X to Y. Analogously, by $f_{X|Y}^{+}(\cdot)$, $f_{X|Y}^{-}(\cdot)$, $g_{X|Y}^{+}(\cdot)$ and $g_{X|Y}^{-}(\cdot)$, we denote modifier functions corresponding to gradual decision rules from Y to X. The first four modifiers are defined on the basis of rough approximations $\underline{App}^{+}(X,Y)$, $\overline{App}^{+}(X,Y)$, $\underline{App}^{-}(X,Y)$ and $\overline{App}^{-}(X,Y)$, respectively, while the last four modifiers are defined analogously on the basis of rough approximations $\underline{App}^{+}(Y,X)$, $\overline{App}^{+}(Y,X)$, $\underline{App}^{-}(Y,X)$ and $\overline{App}^{-}(Y,X)$.

While the previous sections concentrated on the issues of representation, rough approximation and gradual decision rule extraction, this section is devoted to inference with a generalized *modus ponens* (*MP*) and a generalized *modus tollens* (*MT*).

Classically, *MP* has the following form,

if	$X \rightarrow Y$	is true
and	X	is true
then	Y	is true

MP has the following interpretation: assuming an implication $X \rightarrow Y$ (true decision rule) and a fact X (premise), we obtain another fact Y (conclusion). If we replace the classical decision rule above by our four kinds of gradual decision rules, then we obtain the following four fuzzy-rough *MP*:

(*LP-MP*)

if	$\mu_X(x)\geq\alpha \rightarrow \mu_Y(x)\geq f_{Y|X}^{+}(\alpha)$
and	$\mu_X(x)\geq\alpha'$
then	$\mu_Y(x)\geq f_{Y|X}^{+}(\alpha')$

(*LN-MP*)

if	$\mu_X(x)\leq\alpha \rightarrow \mu_Y(x)\geq f_{Y|X}^{-}(\alpha)$
and	$\mu_X(x)\leq\alpha'$
then	$\mu_Y(x)\geq f_{Y|X}^{-}(\alpha')$

(*UP-MP*)

if	$\mu_X(x)\leq\alpha \rightarrow \mu_Y(x)\leq g_{Y|X}^{+}(\alpha)$
and	$\mu_X(x)\leq\alpha'$
then	$\mu_Y(x)\leq g_{Y|X}^{+}(\alpha')$

(*UN-MP*)

if	$\mu_X(x)\geq\alpha \rightarrow \mu_Y(x)\leq g_{Y|X}^{-}(\alpha)$
and	$\mu_X(x)\geq\alpha'$
then	$\mu_Y(x)\leq g_{Y|X}^{-}(\alpha')$

In the classical *MP*, the inference pattern is applicable only when the given fact X is same as the premise X of the rule $X \rightarrow Y$, in fuzzy-rough *MP*, however, the inference pattern is applicable when the given fact has the same form of the inequality relation as the premise of the rule. Moreover, in the real world, we may apply these inference patterns to get the information about $\mu_Y(x)$ of a new object x due to rules we obtained from a given decision table and due to an observed value of $\mu_X(x)$. This means that the above reasoning is a kind of extrapolation. Therefore, we assume $x\in\hat{U}$ and $\hat{U} \supseteq U$.

On the other hand, the classical *MT* has the following form,

if	$X \rightarrow Y$	is true
and	Y	is false
then	X	is false

In the same way as we did in fuzzy-rough *MP*, we would like to obtain fuzzy-rough *MT* such as

$$\begin{array}{lll} (LP\text{-}MT) & \text{if} & \mu_X(x)\geq\alpha \rightarrow \mu_Y(x)\geq f_{Y|X}{}^{+}(\alpha) \\ & \text{and} & \mu_Y(x)<\beta \\ \hline & \text{then} & \mu_X(x)<\varphi(\beta) \end{array} \qquad (1)$$

We should find a proper function φ:[0,1]→[0,1] which validates (1). The following theorem gives answers to this problem.

Theorem. The following assertions are true:

1) Knowing rule $\mu_X(x)\geq\alpha \rightarrow \mu_Y(x)\geq f_{Y|X}{}^{+}(\alpha)$ and $\mu_Y(x)<\beta$, we get $\mu_X(x)<g_{X|Y}{}^{+}(\beta)$.
2) Knowing rule $\mu_X(x)\leq\alpha \rightarrow \mu_Y(x)\geq f_{Y|X}{}^{-}(\alpha)$ and $\mu_Y(x)<\beta$, we get $\mu_X(x)>f_{X|Y}{}^{-}(\beta)$.
3) Knowing rule $\mu_X(x)\leq\alpha \rightarrow \mu_Y(x)\leq g_{Y|X}{}^{+}(\alpha)$ and $\mu_Y(x)>\beta$, we get $\mu_X(x)>f_{X|Y}{}^{+}(\beta)$.
4) Knowing rule $\mu_X(x)\geq\alpha \rightarrow \mu_Y(x)\leq g_{Y|X}{}^{-}(\alpha)$ and $\mu_Y(x)>\beta$, we get $\mu_X(x)<g_{X|Y}{}^{-}(\beta)$.
5) Knowing rule $\mu_X(x)\geq\alpha \rightarrow \mu_Y(x)\geq f_{Y|X}{}^{+}(\alpha)$ and $\mu_Y(x)\leq\beta$, we get $\mu_X(x)\leq\inf\{g_{X|Y}{}^{+}(\gamma)|\,\gamma>\beta\}$.
6) Knowing rule $\mu_X(x)\leq\alpha\rightarrow\mu_Y(x)\geq f_{Y|X}{}^{-}(\alpha)$ and $\mu_Y(x)\leq\beta$, we get $\mu_X(x)\geq\sup\{f_{X|Y}{}^{-}(\gamma)|\,\gamma>\beta\}$.
7) Knowing rule $\mu_X(x)\leq\alpha \rightarrow \mu_Y(x)\leq g_{Y|X}{}^{+}(\alpha)$ and $\mu_Y(x)\geq\beta$, we get $\mu_X(x)\geq\sup\{f_{X|Y}{}^{+}(\gamma)|\,\gamma<\beta\}$.
8) Knowing rule $\mu_X(x)\geq\alpha \rightarrow \mu_Y(x)\leq g_{Y|X}{}^{-}(\alpha)$ and $\mu_Y(x)\geq\beta$, we get $\mu_X(x)\leq\inf\{g_{X|Y}{}^{-}(\gamma)|\,\gamma<\beta\}$.

Assertions 1) to 4) of the Theorem imply the following four fuzzy-rough *MT*:

$$\begin{array}{lll} (LP\text{-}MT) & \text{if} & \mu_X(x)\geq\alpha \rightarrow \mu_Y(x)\geq f_{Y|X}{}^{+}(\alpha) \\ & \text{and} & \mu_Y(x)<\beta \\ \hline & \text{then} & \mu_X(x)<g_{X|Y}{}^{+}(\beta) \end{array} \qquad \begin{array}{lll} (LN\text{-}MT) & \text{if} & \mu_X(x)\leq\alpha \rightarrow \mu_Y(x)\geq f_{Y|X}{}^{-}(\alpha) \\ & \text{and} & \mu_Y(x)<\beta \\ \hline & \text{then} & \mu_X(x)>f_{X|Y}{}^{-}(\beta) \end{array}$$

$$\begin{array}{lll} (UP\text{-}MT) & \text{if} & \mu_X(x)\leq\alpha \rightarrow \mu_Y(x)\leq g_{Y|X}{}^{+}(\alpha) \\ & \text{and} & \mu_Y(x)>\beta \\ \hline & \text{then} & \mu_X(x)>f_{X|Y}{}^{+}(\beta) \end{array} \qquad \begin{array}{lll} (UN\text{-}MT) & \text{if} & \mu_X(x)\geq\alpha \rightarrow \mu_Y(x)\leq g_{Y|X}{}^{-}(\alpha) \\ & \text{and} & \mu_Y(x)>\beta \\ \hline & \text{then} & \mu_X(x)<g_{X|Y}{}^{-}(\beta) \end{array}$$

Thus, for *(LP-MT)* in (1), we have $\varphi(\beta) = g_{X|Y}{}^{+}(\beta)$. As we obtain a conclusion $\mu_X(x)<g_{X|Y}{}^{+}(\beta)$ from a fact $\mu_Y(x)<\beta$, we may remark that *(LP-MT)* is very similar to *(UP-MP)*, with the exchange between *X* and *Y*, i.e.,

$$\begin{array}{lll} (UP\text{-}MP) & \text{if} & \mu_Y(x)\leq\alpha \rightarrow \mu_X(x)\leq g_{X|Y}{}^{+}(\alpha) \\ & \text{and} & \mu_Y(x)\leq\beta \\ \hline & \text{then} & \mu_X(x)\leq g_{X|Y}{}^{+}(\beta) \end{array}$$

In this sense, rule $\mu_X(x)\geq\alpha \rightarrow \mu_Y(x)\geq f_{Y|X}{}^{+}(\alpha)$ is very similar to rule $\mu_Y(x)\leq\alpha \rightarrow \mu_X(x)\leq g_{X|Y}{}^{+}(\alpha)$, however, from the fact $\mu_Y(x)\leq\beta$, we do not obtain the same conclusion. The difference is shown in Assertion 5). When $g_{X|Y}{}^{+}(\cdot)$ is lower semi-continuous, it is the same as $\inf\{g_{X|Y}{}^{+}(\gamma)|\,\gamma>\beta\}$. However, $g_{X|Y}{}^{+}(\cdot)$ is not upper semi-continuous, as can be seen in Example 1, where it is only lower semi-continuous. The difference can occur only in extreme points of segments of $g_{X|Y}{}^{+}(\cdot)$. By the same reasoning, *(LN-MT)*, *(UP-MT)* and *(UN-MT)* are very similar to *(LN-MP)*, *(LP-MP)* and *(UN-MP)* with the exchange between *X* and *Y*, respectively. Then, rules $\mu_X(x)\leq\alpha \rightarrow \mu_Y(x)\geq f_{Y|X}{}^{-}(\alpha)$, $\mu_X(x)\leq\alpha \rightarrow \mu_Y(x)\leq g_{Y|X}{}^{+}(\alpha)$ and $\mu_X(x)\geq\alpha \rightarrow \mu_Y(x)\leq g_{Y|X}{}^{-}(\alpha)$ are very similar to rules $\mu_Y(x)\leq\alpha \rightarrow \mu_X(x)\geq f_{X|Y}{}^{-}(\alpha)$, $\mu_Y(x)\leq\alpha \rightarrow \mu_X(x)\leq g_{X|Y}{}^{+}(\alpha)$ and $\mu_Y(x)\geq\alpha \rightarrow \mu_X(x)\leq g_{X|Y}{}^{-}(\alpha)$, respectively. Therefore, the generalized fuzzy-rough *MT* is very similar to the generalized fuzzy-rough *MP* with an exchange between the premise and the conclusion.

4. Generalized fuzzy-rough modus ponens for approximate reasoning

Generalized modus ponens is formalized as

if	$X \rightarrow Y$	is true
and	X'	is true
then	Y'	is true

Namely, the fact X' is not always the same as the premise X of rule $X \rightarrow Y$. Such an inference we might often apply in the real life. For example, we may infer "the tomato is very ripe" from a fact "the tomato is very red", using our knowledge represented by the rule "if a tomato is red then it is ripe". Such an inference has been treated in fuzzy reasoning (Zadeh, 1973). In this section, we propose to formalize this generalization using our fuzzy-rough *MP* and *MT*. (*LP-MP*) and (*LP-MT*) can be generalized as follows:

(*LP-LP-MP*) if $\mu_X(x) \geq \alpha \rightarrow \mu_Y(x) \geq f_{Y|X}^{+}(\alpha)$
and $\mu_{X'}(x) \geq \alpha'$
then $\mu_{Y'}(x) \geq f_{Y|X}^{+}(\alpha')$

(*LP-LP-MT*) if $\mu_X(x) \geq \alpha \rightarrow \mu_Y(x) \geq f_{Y|X}^{+}(\alpha)$
and $\mu_{Y''}(x) < \alpha'$
then $\mu_{X''}(x) < g_{X|Y}^{+}(\alpha')$

(*LP-LP-MP*) generalizes (*LP-MP*) by replacing X and Y with X' and Y', respectively, while (*LP-LP-MT*) generalizes (*LP-MT*) by replacing X and Y with X'' and Y'', respectively. Remark that Y' and X'' are not given here, but X' and Y'' are. Therefore, our problem is to get to know Y' and X''. Since it is often difficult to get an explicit answer, we consider the following alternative inference patterns:

(*LP-LP-MPw*) if $\mu_X(x) \geq \alpha \rightarrow \mu_Y(x) \geq f_{Y|X}^{+}(\alpha)$
and $\mu_{X'}(x) \geq \alpha'$
then $\mu_Y(x) \geq \psi(\alpha')$

(*LP-LP-MTw*) if $\mu_X(x) \geq \alpha \rightarrow \mu_Y(x) \geq f_{Y|X}^{+}(\alpha)$
and $\mu_{Y''}(x) < \alpha'$
then $\mu_X(x) < \theta(\alpha')$

We assume that there is a relation between X and X' in (*LP-LP-MP*) and a relation between Y and Y'' in (*LP-LP-MT*). Moreover, we suppose that these relations are known at least to some extent. For example, we may have another decision table with object set $U' \subseteq \hat{U}$ which gives a relation between X and X'. Analogously, we may have another decision table with object set $U'' \subseteq \hat{U}$ which gives a relation between Y and Y''. We may then represent the relation between X and X' by gradual decision rules using functions $f_{X|X'}^{+}(\cdot)$, $f_{X|X'}^{-}(\cdot)$, $g_{X|X'}^{+}(\cdot)$ and $g_{X|X'}^{-}(\cdot)$ derived from the decision table with object set $U' \subseteq \hat{U}$. For example, consider the "red tomato" example. Assume that we collected a set U' of tomatoes with different shades of red. Then, to each tomato we may assign a degree of membership to fuzzy set of "red tomatoes" and a degree of membership to fuzzy set of "very red tomatoes". Arranging that information into a table, we obtain a decision table with a decision attribute specifying "the degree of very red" and a condition attribute specifying "the degree of red". Applying our rough-fuzzy approach to this table, we obtain the modifier functions $f_{X|X'}^{+}(\cdot)$, $f_{X|X'}^{-}(\cdot)$, $g_{X|X'}^{+}(\cdot)$ and $g_{X|X'}^{-}(\cdot)$. In the same manner, the relation between Y and Y'' is represented by gradual decision rules using functions $f_{Y|Y''}^{+}(\cdot)$, $f_{Y|Y''}^{-}(\cdot)$, $g_{Y|Y''}^{+}(\cdot)$ and $g_{Y|Y''}^{-}(\cdot)$ derived from the decision table with object set $U'' \subseteq \hat{U}$.

To infer Y, we should obtain information of the type $\mu_X(x) \geq \alpha''$ from $\mu_{X'}(x) \geq \alpha'$. This can be done through the following (*LP-MP*) with respect to X' and X:

if $\mu_{X'}(x)\geq\alpha \rightarrow \mu_X(x)\geq f_{X|X'}{}^{+}(\alpha)$
and $\mu_{X'}(x)\geq\alpha'$
---
then $\mu_X(x)\geq f_{X|X'}{}^{+}(\alpha')$

Applying (*LP-MP*) with respect to X and Y to the inference result $\mu_X(x)\geq f_{X|X'}{}^{+}(\alpha')$, we obtain $\mu_Y(x)\geq f_{Y|X}{}^{+}(f_{X|X'}{}^{+}(\alpha'))$. Thus, we get $\psi(\alpha')=f_{Y|X}{}^{+}(f_{X|X'}{}^{+}(\alpha'))$ in (*LP-LP-MPw*), i.e.,

(*LP-LP-MPw*) if $\mu_X(x)\geq\alpha \rightarrow \mu_Y(x)\geq f_{Y|X}{}^{+}(\alpha)$
and $\mu_{X'}(x)\geq\alpha'$
---
then $\mu_Y(x)\geq f_{Y|X}{}^{+}(f_{X|X'}{}^{+}(\alpha'))$.

The conclusion of this inference pattern is discussed below. When X and Y are defined through attribute values $a(x)$ and $b(x)$, namely, $\mu_X(x)=\mu_{(X)}(a(x))$ and $\mu_Y(x)=\mu_{(Y)}(b(x))$, this inference pattern is useful to know the possible range of attribute value $b(x)$ from the information about attribute value $a(x)$, as $\mu_{(X)}(a(x))\geq\alpha'$. Actually, the possible range can be obtained as $\{b(x) \mid \mu_{(Y)}(b(x))\geq f_{Y|X}{}^{+}(f_{X|X'}{}^{+}(\alpha'))\}$.

To have inference pattern (*LP-LP-MP*), we should utilize the following equivalence:

$$\alpha\geq f_{Y|X}{}^{+}(\beta) \text{ if and only if } \hat{g}^{+}_{X|Y}(\alpha)=\sup\{\mu_X(x) \mid \mu_Y(x)\leq\alpha\}\geq\beta \text{ and there exists } y\in U \text{ such that } \mu_X(y)=\beta. \quad (2)$$

This implication is valid not only for relation between Y and X but also for relation between X and X'. The conclusion is the same for two given facts $\mu_{X'}(x)\geq\alpha'$ and $\mu_{X'}(x)\geq h_{X'}(\alpha')=\sup\{\mu_{X'}(z) \mid \mu_{X'}(z)\leq\alpha', z\in U\}$, since we have $f_{X|X'}{}^{+}(\alpha')=f_{X|X'}{}^{+}(h_{X'}(\alpha'))$. Moreover, $\beta\geq h_{X'}(\alpha')$ implies $\bar{k}_{X'}(\beta)=\inf\{\mu_{X'}(z)|\mu_{X'}(z)>\beta, z\in U\}>\alpha'$. Therefore, we can draw the following chain of inferences: $\mu_Y(x)\geq f_{Y|X}{}^{+}(f_{X|X'}{}^{+}(\alpha'))$ if and only if $\hat{g}^{+}_{X|Y}(\mu_Y(x))\geq f_{X|X'}{}^{+}(\alpha')$. $\hat{g}^{+}_{X|Y}(\mu_Y(x))\geq f_{X|X'}{}^{+}(\alpha')$ is equivalent to $\hat{g}^{+}_{X|Y}(\mu_Y(x))\geq f_{X|X'}{}^{+}(h_{X'}(\alpha'))$. $\hat{g}^{+}_{X|Y}(\mu_Y(x))\geq f_{X|X'}{}^{+}(h_{X'}(\alpha'))$ if and only if $\hat{g}_{X'|X}(\hat{g}^{+}_{X|Y}(\mu_Y(x)))\geq h_{X'}(\alpha')$. Finally, $\hat{g}_{X'|X}(\hat{g}^{+}_{X|Y}(\mu_Y(x)))\geq h_{X'}(\alpha')$ implies $\bar{k}_{X'}(\hat{g}_{X'|X}(\hat{g}^{+}_{X|Y}(\mu_Y(x))))>\alpha'$. Since $f_{Y|X}{}^{+}(\cdot)$ is non-decreasing, we have $f^{+}_{Y|X}(\bar{k}_{X'}(\hat{g}_{X'|X}(\hat{g}^{+}_{X|Y}(\mu_Y(x)))))\geq f^{+}_{Y|X}(\alpha')$. Hence, we obtain $\mu_{Y'}(x)=f^{+}_{Y|X}(\hat{g}_{X'|X}(\hat{g}^{+}_{X|Y}(\mu_Y(x))))$, i.e.,

(*LP-LP-MP*) if $\mu_X(x)\geq\alpha \rightarrow \mu_Y(x)\geq f_{Y|X}{}^{+}(\alpha)$
and $\mu_{X'}(x)\geq\alpha'$
---
then $f^{+}_{Y|X}(\bar{k}_{X'}(\hat{g}_{X'|X}(\hat{g}^{+}_{X|Y}(\mu_Y(x)))))\geq f^{+}_{Y|X}(\alpha')$.

The conclusion of this inference pattern is more ambiguous than that of (*LP-LP-MPw*) because the relation between $\beta\geq h_{X'}(\alpha')$ and $\bar{k}_{X'}(\beta)>\alpha'$ is a one-way implication and we applied $f_{Y|X}{}^{+}(\cdot)$ which is not strictly increasing. However, the inference pattern may be useful to know approximately how a conclusion fuzzy set Y is modified when a premise fuzzy set X is modified to X'.

When deriving (*LP-LP-MP*), we obtained another inference pattern as follows:

(*LP-LP-MPm*) if $\mu_X(x)\geq\alpha \rightarrow \mu_Y(x)\geq f_{Y|X}{}^{+}(\alpha)$
and $\mu_{X'}(x)\geq\alpha'$
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then $\bar{k}_{X'}(\hat{g}_{X'|X}(\hat{g}^{+}_{X|Y}(\mu_Y(x))))>\alpha'$ (which implies $\bar{k}_{X'}(\hat{g}_{X'|X}(\hat{g}^{+}_{X|Y}(\mu_Y(x))))\geq\alpha'$).

The conclusion of this inference pattern is more ambiguous than that of (*LP-LP-MPw*) but it is more specific than that of (*LP-LP-MP*). This inference pattern is useful when we would like to know the image of a fuzzy set X' through the rule $\mu_X(x)\geq\alpha \rightarrow \mu_Y(x)\geq f_{Y|X}{}^{+}(\alpha)$, given fuzzy sets X and Y.

Now, let us move to a discussion on (*LP-LP-MT*). Analogously, we obtain $\theta(\alpha')=g_{X|Y}{}^{+}(g_{Y|Y''}{}^{+}(\alpha'))$, i.e.,

(*LP-LP-MTw*)	if	$\mu_X(x) \geq \alpha \rightarrow \mu_Y(x) \geq f_{Y\mid X}^{+}(\alpha)$
	and	$\mu_{Y''}(x) < \alpha'$
	then	$\mu_{X''}(x) < g_{X\mid Y}^{+}(g_{Y\mid Y''}^{+}(\alpha'))$.

Similarly to (2), we obtain

$$\alpha < g_{X\mid Y}^{+}(\beta) \text{ if and only if } \bar{f}_{Y\mid X}^{+}(\alpha) = \inf\{\mu_Y(x) \mid \mu_X(x) > \alpha\} \leq \beta \text{ and there exists } y \in U \text{ such that } \mu_Y(y) = \beta. \quad (3)$$

At the first glance, we may think that similar results to (*LP-LP-MP*) will be obtained. However, we should notice that it is not $\bar{f}_{Y\mid X}^{+}(\alpha) < \beta$ in (3) but $\bar{f}_{Y\mid X}^{+}(\alpha) \leq \beta$. By this difference, we cannot obtain (*LP-LP-MT*) but (*LP-LP-MTm*) corresponding to (*LP-LP-MPm*). We obtain only the following inference patterns:

(*LP-LP-MT'*)	if	$\mu_X(x) \geq \alpha \rightarrow \mu_Y(x) \geq f_{Y\mid X}^{+}(\alpha)$
	and	$\mu_{Y''}(x) < \alpha'$
	then	$g_{X\mid Y}^{+}(\bar{h}_{Y''}(\hat{f}_{Y''\mid Y}(\bar{f}_{Y\mid X}^{+}(\mu_X(x))))) \leq g_{X\mid Y}^{+}(\alpha')$,
(*LP-LP-MTm'*)	if	$\mu_X(x) \geq \alpha \rightarrow \mu_Y(x) \geq f_{Y\mid X}^{+}(\alpha)$
	and	$\mu_{Y''}(x) < \alpha'$
	then	$\bar{h}_{Y''}(\hat{f}_{Y''\mid Y}(\bar{f}_{Y\mid X}^{+}(\mu_X(x)))) < \alpha'$,

where $\hat{f}_{Y\mid X}^{+}(\alpha) = \inf\{\mu_Y(x) \mid \mu_X(x) \geq \alpha\}$ and $\bar{h}_{Y''}(\beta) = \sup\{\mu_{Y''}(z) \mid \mu_{Y''}(z) < \beta\}$. Since we have $\bar{f}_{Y\mid X}^{+}(\alpha) \leq \hat{f}_{Y\mid X}^{+}(\alpha)$ for any $\alpha \in [0,1]$. The conclusions of those inference patterns are less ambiguous than the extended inference patterns with respect to (*UP-MP*), whose conclusions are obtained as $g_{X\mid Y}^{+}(\bar{h}_{Y''}(\hat{f}_{Y''\mid Y}(\hat{f}_{Y\mid X}^{+}(\mu_X(x))))) \leq g_{X\mid Y}^{+}(\alpha')$ and $\bar{h}_{Y''}(\hat{f}_{Y''\mid Y}(\hat{f}_{Y\mid X}^{+}(\mu_X(x)))) \leq \alpha'$.

5. Conclusions and further research directions

In this paper we discussed fuzzy-rough inference patterns with gradual decision rules extracted from a decision table. We showed that fuzzy-rough modus tollens is very similar to fuzzy-rough modus ponens and that all inference can be done by proper manipulations of modifier functions. If in the premise of the gradual decision rule fuzzy set X is defined with multiple attributes, the inference by manipulations of modifier functions are much easier than the direct inference method which requires manipulations of multidimensional fuzzy sets. Therefore, we plan to apply fuzzy-rough inference also to gradual decision rules defined with multiple attributes (Greco, Inuiguchi, Slowinski 2003b). Moreover, we can apply the proposed fuzzy-rough inference to case based reasoning problems. These would be the topics of our future studies.

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References

Cattaneo, G. (1998), Fuzzy extension of rough sets theory, [in] L. Polkowski, A. Skowron (eds.), *Rough Sets and Current Trends in Computing*, LNAI 1424, Springer, Berlin, 275-282

Dubois, D., Prade, H. (1992a), Gradual inference rules in approximate reasoning, *Information Sciences*, 61, 103-122

Dubois, D., Prade, H. (1992b), Putting rough sets and fuzzy sets together, [in] R. Slowinski (ed.), *Intelligent Decision Support: Handbook of Applications and Advances of the Sets Theory*, Kluwer, Dordrecht, pp. 203-232

Greco, S., Inuiguchi, M., Slowinski, R. (2003a), Rough sets and gradual decision rules, [in]: G. Wang, Q. Liu, Y. Yao, A. Skowron (eds.), *Rough Sets, Fuzzy Sets, Data Minig, and Granular Computing*, Lecture Notes in Artificial Intelligence, vol. 2639, Springer-Verlag, Berlin, pp. 156-164

Greco, S., Inuiguchi, M., Slowinski, R. (2003b), Fuzzy rough sets and multiple premise gradual decision rules, [in]: V. Di Gesù, F. Masulli, A. Petrosino (eds.), Proceedings of WILF 2003 International Workshop on *Fuzzy Logic and Applications*, Napoli, Italy October 9-11, 2003

Greco, S., Matarazzo, B., Slowinski R. (1999), The use of rough sets and fuzzy sets in MCDM, [in] T.Gal, T.Stewart, T.Hanne (eds.), *Advances in Multiple Criteria Decision Making*, Kluwer Academic Publishers, Boston, 14.1-14.59

Greco, S., Matarazzo, B., Slowinski, R. (2000a), Rough set processing of vague information using fuzzy similarity relations, [in]: C.S. Calude and G. Paun (eds.), *Finite Versus Infinite – Contributions to an Eternal Dilemma*, Springer-Verlag, London, 149-173

Greco, S., Matarazzo, B., Slowinski, R. (2000b), Fuzzy extension of the rough set approach to multicriteria and multiattribute sorting, [in] J. Fodor, B. De Baets and P. Perny (eds.), *Preferences and Decisions under Incomplete Knowledge*, Physica-Verlag, Heidelberg, 131-151

Inuiguchi, M., Greco, S., Slowinski, R., Tanino, T (2003), Possibility and necessity measure specification using modifiers for decision making under fuzziness, *Fuzzy Sets and Systems*, **137**, 151-175

Inuiguchi, M., Tanino, T. (2003), New fuzzy rough sets based on certainty qualification, [in]: S. K. Pal, L. Polkowski and A. Skowron (eds.): *Rough-Neural Computing: Techniques for Computing with Words*, Springer-Verlag, Berlin, 2003, 278-296

Nakamura, A., Gao, J.M. (1991), A logic for fuzzy data analysis, *Fuzzy Sets and Systems*, 39, 127-132

Pawlak, Z. (1991), *Rough Sets*, Kluwer, Dordrecht

Polkowski, L. (2002), *Rough Sets: Mathematical Foundations*, Physica-Verlag, Heidelberg

Slowinski R. (1995), Rough set processing of fuzzy information, [in]: T.Y.Lin, A.Wildberger (eds.), *Soft Computing: Rough Sets, Fuzzy Logic, Neural Networks, Uncertainty Management, Knowledge Discovery*. Simulation Councils, Inc., San Diego, CA, 142-145

Slowinski, R., Stefanowski, J. (1996), Rough set reasoning about uncertain data, *Fundamenta Informaticae* 27, 229-243

Yao, Y.Y. (1997), Combination of rough and fuzzy sets based on α-level sets, [in]: T.Y. Lin and N. Cercone (eds.), *Rough Sets and Data Mining: Analysis for Imprecise Data*, Kluwer, Boston, 301-321

Zadeh, L.A. (1973), Outline of a new approach to the analysis of complex systems and decision processes, *IEEE Trans. Systems Man Cybernet.*, 3, 1973, 28-44