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RADIUS OF STARLIKENESS AND CONVEXITY 2

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ABSTRACT. In this note, we will investigate radius of starlikeness and covexity for certain analytic functions on the unit disk.

1. INTRODUCTION AND NOTATIONS

Let $U=\{|z|<1\}$ and let B denote the family of functions $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ which are analytic in U, and which satisfy the condition $|a_n|\leq n$ $(n=2,3,4,\cdots)$. For functions $f\in B$, we consider the radius of univalence , starlikeness and convexity i.e.,

$$\begin{aligned} R_U &= max\{r: f\in B\Rightarrow\ f\ is\ univalent\ in\ |z|<r<1\} \\ R_S &= max\{r: f\in B\Rightarrow\ f\ is\ starlike\ in\ |z|<r<1\} \\ R_C &= max\{r: f\in B\Rightarrow\ f\ is\ convex\ in\ |z|<r<1\} \end{aligned}$$

For R_U, Yong Chan Kim and Mamoru Nunokawa obtained in [2] the following beautiful result:

Theorem A. $R_U=0.164878\cdots$, *where* R_U *is the root of the equation* $2r^3-6r^2+7r-1=0$.

And for R_S, Shigeyoshi Owa and Nunokawa [1] showed

Theorem B. $R_S>0.08998\cdots$.

The present author has showed in [3]

$$R_S>0.104\cdots,\quad \text{and}\quad R_C>0.056\cdots.$$

In this note, we shall improve the above two results.

2. MAIN RESULTS

Theorem 1. $R_S>0.137788\cdots$.

Theorem 2. $R_C>0.075450\cdots$.

Proof of theorem 1. Noting $\frac{f(z)}{z}$ is analytic in U, we will need two results. The first is the following lemma due to Owa and Nunokawa:

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Lemma 1.

$$(1)\qquad \begin{cases} \frac{1-4r+2r^2}{(1-r)^2} \le \left|\frac{f(z)}{z}\right| \le \frac{1}{(1-r)^2} & (0 \le |z| = r < R_1) \\ 0 \le \left|\frac{f(z)}{z}\right| \le \frac{1}{(1-r)^2} & (R_1 \le r < 1) \end{cases}$$

where $R_1 = 1 - \frac{1}{\sqrt{2}} = 0.29289\cdots$

The second is Poisson-Jensen's Formula.

$$(2)\qquad \log \frac{f(z)}{z} = \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{f(\zeta)}{\zeta}\right| \cdot \frac{\zeta + z}{\zeta - z} d\phi$$

where $\zeta = \rho e^{i\phi}$, $z = re^{i\theta}$ *and* $0 \le r < \rho < 1$. By logarithmic differentiation, we find from (2) that

$$(3)\qquad R_e\frac{zf'(z)}{f(z)} = 1 + \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{f(\zeta)}{\zeta}\right| R_e\frac{2\zeta z}{(\zeta - z)^2} d\phi$$

where

$$R_e\frac{2\zeta z}{(\zeta - z)^2} = 2\rho r\frac{(\rho^2 + r^2)\cos(\phi - \theta) - 2\rho r}{\{\rho^2 + r^2 - 2\rho r\cos(\phi - \theta)\}^2}.$$

For brevity, putting

$$(4)\qquad \lambda = \frac{2\rho r}{\rho^2 + r^2}, \quad \alpha = \phi - \theta$$

and

$$(5)\qquad \beta = \cos^{-1}\lambda, \qquad g(\alpha) = \frac{\lambda(\cos\alpha - \lambda)}{(1 - \lambda\cos\alpha)^2}$$

we have from (3)

(6)

$$R_e\frac{zf'(z)}{f(z)} = 1 + \frac{1}{2\pi}\int_{-\beta}^{\beta} g(\alpha)\log\left|\frac{f(\zeta)}{\zeta}\right| d\phi + \frac{1}{2\pi}\int_{\beta}^{2\pi-\beta} g(\alpha)\log\left|\frac{f(\zeta)}{\zeta}\right| d\phi.$$

Since $R_1 > R_U$, we may assume $r < R_1$. And since

$$\begin{cases} g(\alpha) \ge 0 & (-\beta \le \alpha \le \beta) \\ g(\alpha) \le 0 & (\beta \le \alpha \le 2\pi\beta), \end{cases}$$

we deduce from (1) and (5)

$$R_e\frac{zf'(z)}{f(z)} \geq 1+\frac{1}{2\pi}\int_{-\beta}^{\beta} g(\alpha)\,\log\frac{1-4\rho+2\rho^2}{(1-\rho)^2}d\alpha$$
$$+\frac{1}{2\pi}\int_{\beta}^{2\pi-\beta} g(\alpha)\,\log\frac{1}{(1-\rho)^2}d\alpha$$

Evidently

$$\int g(\alpha)d\alpha = \int \frac{\lambda(\cos\alpha-\lambda)}{(1-\lambda\cos\alpha)^2}d\alpha = \lambda\frac{\sin\alpha}{1-\lambda\cos\alpha},$$

So we have

$$(7)\quad R_e\frac{zf'(z)}{f(z)} \geq 1-\frac{\lambda}{\pi}\left\{\log\frac{(1-\rho)^2}{1-4\rho+2\rho^2}+\log\frac{1}{(1-\rho)^2}\right\}\cdot\frac{\sin\beta}{1-\lambda\cos\beta}$$

It deduce

$$R_e\frac{zf'(z)}{f(z)} \geq 1-\frac{\lambda}{\pi\sqrt{1-\lambda^2}}\cdot\log\frac{1}{1-4\rho+2\rho^2},$$
$$R_e\frac{zf'(z)}{f(z)} \geq 1-\frac{2\rho r}{\pi(\rho^2-r^2)}\cdot\log\frac{1}{1-4\rho+2\rho^2}.$$

We set

$$\phi(r,\rho) = 1-\frac{2\rho r}{\pi(\rho^2-r^2)}\cdot\log\frac{1}{1-4\rho+2\rho^2}.$$

According to

$$\phi(r,\rho_0)\nearrow \quad (\text{ for fixed } \rho_0 \text{ })$$

We deduce

$$\phi(r_0,\rho_0)\geq 0 \Longrightarrow \phi(r,\rho_0)\geq 0 \ (\text{ for } r\leq r_0 \text{ }).$$

On the other hand

$$\phi(r_3,\rho_3) = 9.47656\times 10^{-6} > 0$$

So we have

$$R_S \geq r_3 = 0.137788$$

Proof of theorem 2. The proof is almost similar. We begin with the following lemma:

Lemma 2.

$$(8)\qquad \begin{cases} \frac{1-7r+6r^2-2r^3}{(1-r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3} & (0\leq r<R_3) \\ 0 \leq |f'(z)| \leq \frac{1+r}{(1-r)^3} & (R_3\leq r<1) \end{cases}$$

where, $R_3 = R_U = 0.164878\cdots$. *It's very easy to prove the lemma.*

First,

$$|f'(z)| \le \sum_{n=1}^{\infty} n^2 r^{n-1} = \frac{1+r}{(1-r)^3}.$$

Next,

$$\begin{aligned}|f'(z)| &\ge 1 - \sum_{n=2}^{\infty} n^2 r^{n-1} \\ &\ge 2 - \sum_{n=1}^{\infty} n^2 r^{n-1} = \frac{1 - 7r + 6r^2 - 2r^3}{(1-r)^3}.\end{aligned}$$

From Poisson-Jensen's formula, we obtain

$$\log f'(z) = \frac{1}{2\pi}\int_0^{2\pi} \log|f'(\zeta)| \cdot \frac{\zeta + z}{\zeta - z} d\phi, \tag{9}$$

which implies that

$$\begin{aligned}R_e\frac{zf''(z)}{f'(z)} + 1 = 1 &+ \frac{1}{2\pi}\int_{-\beta}^{\beta} g(\alpha)\log|f'(\zeta)|\,d\alpha \\ &+ \frac{1}{2\pi}\int_{\beta}^{2\pi-\beta} g(\alpha)\log|f'(\zeta)|\,d\alpha.\end{aligned} \tag{10}$$

By virtue of (8) and (10), we obtain that

$$\begin{aligned}R_e\frac{zf''(z)}{f'(z)} + 1 \ge 1 &- \frac{\lambda}{\pi}\left\{\log\frac{(1-\rho)^3}{1 - 7\rho + 6\rho^2 - 2\rho^3}\right. \\ &+ \left.\log\frac{1+\rho}{(1-\rho)^3}\right\}\cdot\frac{\sin\beta}{1 - \lambda\cos\beta}.\end{aligned}$$

$$R_e\frac{zf''(z)}{f'(z)} + 1 \ge 1 - \frac{2\rho r}{\pi(\rho^2 - r^2)}\cdot\log\frac{1+\rho}{1 - 7\rho + 6\rho^2 - 2\rho^3} \tag{11}$$

We write the right hand side equation of (11) by $\psi(r,\rho)$. Finally, if we put $r_4 = 0.075450$ *and* $\rho_4 = 0.127$, then $\psi(r_4, \rho_4) = 2.23722 \times 10^{-5}$ It deduce that

$$R_e\frac{zf''(z)}{f'(z)} + 1 > 0 \quad (0 \le r_4).$$

So we have $R_C > 0.075450$.

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