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非局所非線形境界値問題の厳密解と大域的解構造
Exact solutions and the global solution structure of nonlocal nonlinear boundary problems

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We are interested in the global structure of all solutions of several nonlocal nonlinear boundary problems arising in various fileds. We show four examples.

The first problem is related with the Oseen's spiral flow [11].

Find a function $U(x)$ such that

$$(\mathrm{O})\quad \begin{cases} \{U_{xx}+AU-U^2\}_x=0, & x\in(-\pi,\pi),\\ U(-\pi)=U(\pi), \quad U_x(-\pi)=U_x(\pi), & \\ \displaystyle\int_{-\pi}^{\pi} U(x)\,\mathrm{d}x=0, & \end{cases}$$

for arbitrarily fixed A.

It is easily seen that $U\equiv 0$ is the trivial solution of the above problem for any fixed A. Okamoto [10] started to investigate the global bifurcation structure of this problem. Moreover, Ikeda-Mimura-Okamoto [4] obtained the asymptotic shape of solutions as $A\to-\infty$.

Let us recall the standard notation of complete elliptic integrals:

$$\begin{aligned} K(k) &:= \int_0^{\pi/2}\frac{\mathrm{d}\varphi}{\sqrt{1-k^2\sin^2\varphi}}, && k\in[0,1),\\ E(k) &:= \int_0^{\pi/2}\sqrt{1-k^2\sin^2\varphi}\,\mathrm{d}\varphi, && k\in[0,1). \end{aligned}$$

Jacobi's elliptic functions $\mathrm{sn}(x,k)$ and $\mathrm{cn}(x,k)$ with the modulus k are defined as follows:

$$\mathrm{sn}^{-1}(z,k) := \int_0^z \frac{\mathrm{d}\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}}, \quad z\in[0,1],\ k\in[0,1),$$

and

$$\mathrm{cn}^2(z,k)=1-\mathrm{sn}^2(z,k).$$

We note that

$$E(0)=K(0)=\frac{\pi}{2}, \quad E(1)=1, \quad K(k)\sim\frac{1}{2}\log\left(\frac{16}{1-k^2}\right) \ \text{as } k\to 1.$$

Ikeda-Kondo-Okamoto-Yotsutani [3] have parameterized all solutions (A, U) of (O) in terms of the elliptic functions, and clarified the global bifurcation structure by the following Theorems 1 and 2.

Theorem 1 *All the solution (A, U) of (O) are parameterized by*

$$\left\{\left(n^2 A(k),\ n^2 U(nx - x_0; A(k))\right) : 0 < k < 1,\ -\pi < x_0 \leq \pi,\ n = 1, 2, 3 \cdots\right\},$$

where

$$\begin{aligned} A(k) &:= \frac{4K(k)}{\pi^2}\left(3E(k) + (k^2 - 2)K(k)\right), \\ U(x; A(k)) &:= -\frac{6k^2 K(k)^2}{\pi^2}\,\mathrm{cn}^2\left(\frac{K(k)}{\pi}x, k\right) \\ &\quad + \frac{6K(k)}{\pi^2}\{E(k) - (1 - k^2)K(k)\}. \end{aligned}$$

Theorem 2 *The function $A(k)$ is strictly monotone decreasing in $k \in (0, 1)$. It also satisfies $\lim_{k \to 0} A(k) = 1$ and $\lim_{k \to 1} A(k) = -\infty$.*

The second problem is related with structure of stationary solutions in S^1 of the Ginzburg-Landau equation.

Find a function $u(x)$ such that

$$(\mathrm{P}) \quad \begin{cases} u_{xx} - \dfrac{C^2}{u^3} + \lambda(1 - u^2)u = 0 \quad \text{in } [-\pi, \pi], \\ C := 2m\pi \left\{\displaystyle\int_{-\pi}^{\pi} \frac{1}{u^2} dx\right\}^{-1}, \\ u(-\pi) = u(\pi), \quad u_x(-\pi) = u_x(\pi), \\ u > 0 \ \text{ in } [-\pi, \pi], \end{cases}$$

where m is a given integer and λ is a bifurcation parameter.

The structure of solutions is similar to that of Oseen's spiral flow, though the analysis is more difficult. Kosugi-Morita-Yotsutani [5] have clarified the global bifurcation structure of this problem.

We briefly explain about the original equation. Consider the following Ginzburg-Landau equation:

$$\begin{cases} \psi_{xx} + \lambda(1 - |\psi|^2)\psi = 0, \quad x \in (-\pi, \pi), \\ \psi(-\pi) = \psi(\pi), \quad \psi_x(-\pi) = \psi_x(\pi). \end{cases}$$

We here assume that $|\psi| > 0$ and ψ is written as the form

$$\psi = u(x)\exp(i\theta(x)),$$

where u and θ are both real-valued smooth functions. Clearly the equation is equivalent the following system:

$$\begin{cases} u_{xx} - (\theta_x)^2 u + \lambda(1-u^2)u = 0, \quad x \in (-\pi,\pi), \\ (u^2\theta_x)_x = 0, \quad x \in (-\pi,\pi), \\ u(-\pi) = u(\pi), \quad u_x(-\pi) = u_x(\pi), \\ \theta(\pi) - \theta(-\pi) = 2m\pi, \quad \theta_x(-\pi) = \theta_x(\pi), \end{cases}$$

where m is an integer. Thus, $\theta_x = C/u^2$ for a constant C and hence we obtain (P).

The third problem is related to find the minimum energy curve for given the length L and area M, which K.Watanabe [13] started to investigate.

For given $L > 0$ and $M > 0$ with $L^2 - 4\pi M > 0$, find a function $\kappa(s)$ such that

$$(\mathrm{E}) \begin{cases} \left\{\kappa_{ss} + \frac{1}{2}\kappa^3 + \mu\kappa\right\}_s = 0 \quad \text{in } [0,L], \\ \mu := \frac{1}{L^2 - 4\pi M}\left\{M\int_0^L \kappa(s)^3 ds - \frac{L}{2}\int_0^L \kappa(s)^2 ds\right\}, \\ \kappa(0) = \kappa(L), \quad \kappa_s(0) = \kappa_s(L), \\ \int_0^L \kappa(s)ds = 2\pi. \end{cases}$$

Murai-Matsumoto-Yotsutani [9] have completely clarified the global bifurcation strucure of this problem, though we need terribly complicated calculations and arguments. This result is written by Minoru Murai in this lecture note.

The final problem is a limiting equation for the Shigesada-Kawasaki-Teramoto model with cross-diffusion [12]. This problem is the hardest.

Find $(v(x), \tau)$ such that $\tau > 0$, and

$$(\mathrm{S}) \begin{cases} \int_0^1 \frac{\tau}{v}\left(a_1 - b_1\frac{\tau}{v} - c_1 v\right) dx = 0, \\ d_2\, v_{xx} + v\left(a_2 - b_2\frac{\tau}{v} - c_2 v\right) = 0 \quad \text{in } (0,1), \\ v_x(0) = 0, \quad v_x(1) = 0, \\ v > 0 \quad on \quad [0,1] \end{cases}$$

where $a_1, a_2, b_1, b_2, c_1, c_2, d_2$ are given positive constants.

We briefly explain the original equation. In 1979, Kawasaki–Shigesada–Teramoto proposed a cross-diffusion system

$$\begin{cases} u_t = \{(d_1 + \rho_{12} v)u\}_{xx} + u\ (a_1 - b_1 u - c_1 v) \quad (0 < x < 1, t > 0), \\ v_t = \{(d_2 + \rho_{21} u)v\}_{xx} + v\ (a_2 - b_2 u - c_2 v) \quad (0 < x < 1, t > 0), \\ v_x(0,t) = 0\ (t > 0), \quad v_x(1,t) = 0\ (t > 0), \\ u(x,0) = u_0(x) \geq 0\ (0 < x < 1),\ v(x,0) = v_0(x) \geq 0\ (0 < x < 1), \end{cases}$$

where $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$ are positive constants, ρ_{12} and ρ_{21} are nonnegative constants, and $u_0(x)$ and $v_0(x)$ are nonnegative initial data. This is a mathematical model to explain the segregation phenomena. Mathematical study of cross-diffusion equations was begun by M. Mimura in 1980 (see, e.g., [8]). There are various results concerning the existence of solutions to time-dependent problem (see, e.g., [1], [2] and references therein), and stationary problems. Sharp existence and non-existence results of stationary solutions are not known.

The limiting equation (S) was discovered by Lou-Ni [6] as a limiting equation when cross-diffusion effect $\rho_{12} \to \infty$. Actually, we see from the numeriacl computations that solutions of (S) approximate stable stationary solutions of the original time-dependent problem. Thus, it is important to know the structure of solutions of (S).

Lou-Ni-Yotsutani [7] have almost clarified the existence and the shape of solutions as follows. Let us put

$$A := \frac{a_1}{a_2}, \quad B := \frac{b_1}{b_2}, \quad C := \frac{c_1}{c_2}.$$

We concentrate on the case $B < C$ (strong competition case).

Theorem 3 (Existence) *Suppose that* $B < C$. *If*

$$\max\{0, \frac{B + C - 2A}{C - B}\} \frac{a_2}{\pi^2} < d_2 < \frac{a_2}{\pi^2},$$

then there exists a solution $(v(x), \tau)$ *of* (S).

Theorem 4 (Nonexistence) *Suppose that* $B < C$.

(i) If $d_2 \geq \frac{a_2}{\pi^2}$, *then there exists no solution of* (S).

(ii) If $A < \frac{B+C}{2}$, *then there exists a* $d_2^* = d_2^*(A, B, C, a_2) > 0$ *such that there exiists no solution of* (S) *for* $d_2 \in (0, d_2^*]$.

(iii) If $A < B$, *there exists no solution of* (S).

The following theorems give the shape of solutions.

Theorem 5 (Shape of solutions as $d_2 \to a_2/\pi^2$) *Let* $(v(x,d_2),\tau(d_2))$ *be solutions of* (S). *If* $A \geq B$, *then*

$$v(x;d_2) \to 0,$$
$$\frac{v(x;d_2)-v(0;d_2)}{v(1;d_2)-v(0;d_2)} \to \sin^2(\frac{\pi}{2}x),$$
$$\frac{\tau(d_2)}{v(x;d_2)} \to \frac{a_2}{b_2}\cdot\frac{A/B+\sqrt{(A/B)^2-A/B}}{1+2\{A/B-1+\sqrt{(A/B)^2-A/B}\}\sin^2(\frac{\pi}{2}x)},$$

uniformly on $[0,1]$ *as* $d_2 \to a_2/\pi^2$.

Theorem 6 (Shape of solutions as $d_2 \to 0$ for $A < \frac{B+3C}{4}$) *Let* $(v(x,d_2),\tau(d_2))$ *be solutions of* (S). *If* $A < \frac{B+3C}{4}$ *and* $B \neq C$, *then*

$$v(0;d_2) \to 2\cdot\frac{a_2}{c_2}\cdot\frac{\frac{B+3C}{4}-A}{C-B}, \qquad v(x;d_2) \to \frac{a_2}{c_2}\cdot\frac{A-B}{C-B} \quad \text{for } x>0,$$
$$\frac{\tau(d_2)}{v(0;d_2)} \to \frac{a_2}{2c_2}\cdot\frac{C-A}{C-B}\cdot\frac{A-B}{\frac{B+3C}{4}-A}, \qquad \frac{\tau(d_2)}{v(x;d_2)} \to \frac{a_2}{b_2}\cdot\frac{C-A}{C-B} \quad \text{for } x>0,$$

as $d_2 \to 0$.

Theorem 7 (Shape of solutions as $d_2 \to 0$ for $A \geq \frac{B+3C}{4}$) *Let* $(v(x,d_2),\tau(d_2))$ *be solutions of* (S). *If* $B < C$ *and* $A \geq \frac{B+3C}{4}$, *then*

$$v(0;d_2) \to 0, \qquad v(x;d_2) \to \frac{3a_2}{4c_2} \quad \text{for } x>0,$$
$$\frac{\tau(d_2)}{v(0;d_2)} \to \infty, \qquad \frac{\tau(d_2)}{v(x;d_2)} \to \frac{a_2}{4c_2} \quad \text{for } x>0,$$

as $d_2 \to 0$.

Now, We will discuss about the uniqueness and non-uniquess. The following result is a part of joint projects with W.-M. Ni.

Theorem 8. *Suppose that* $B < C$. *If* d_2 *is sufficiently small, the solution* $(v(x),\tau)$ *is* <u>*unique*</u> *for any given* A.

Idea of a proof of Theorem 8. All solutions $(v(x), \tau)$ of

$$\begin{cases} d_2\ v_{xx} + v(a_2 - b_2\frac{\tau}{v}\ - c_2 v) = 0 \text{ in } (0,1), \\ v_x(0) = 0, \quad v_x(1) = 0, \\ v > 0 \ \ on \ \ [0,1] \end{cases}$$

are represented by one paramer k. We denote it by $(v(x; k, d_2), \tau(k, d_2))$. Rewrite

$$\int_0^1 \frac{1}{v}\ (a_1 - b_1\frac{\tau}{v}\ - c_1 v)\ dx = 0$$

as

$$\frac{b_1\tau \int_0^1 \frac{1}{v^2} dx + c_1}{\int_0^1 \frac{1}{v} dx} = a_1.$$

We put the left hand side by $\breve{a}_1(k, d_2)$. For given a_1 and sufficiently small d_2, we show that there exists the unique solution k of $\breve{a}_1(k, d_2) = a_1$ by using Theorems 6 and 7.

Theorem 9. *Suppose that $C > 7B/3$. There exists an open set D such that* (S) *has* <u>*at least two solutions*</u> $(v(x), \tau)$ *for* $d_2 \in D$.

Idea of a proof of Theorem 9. We investigate the fuction $\breve{a}_1(k, d_2)$. We see that the Taylor expansion with respect to k at $k = 0$ of $\breve{a}_1(k, d_2)$ becomes

$$\breve{a}_1(k, d_2) = constant + \left\{ \left(13\frac{C}{B} + 35\right) \pi^4 d_2^2 - 14\left(\frac{C}{B} - 1\right)\pi^2 d_2 + \frac{C}{B} - 1 \right\} k^4 . + \cdots$$

The coefficient of k^4 becomes negative for some d_2, if $C/B > 7/3$. This implies the non-uniqueness of the solutions.

It seems that the following conjecture holds in view of the above theorems and the numerical computation.

Conjecture 1: Suppose that $B < C$. For any d_2 with

$$\max\{0, \frac{B + C - 2A}{C - B}\} \frac{a_2}{\pi^2} < d_2 < \frac{a_2}{\pi^2},$$

there exists the <u>unique</u> solution $(v(x), \tau)$ of (S).

Conjecture 2: Suppose that $B < C \leq 7B/3$. (S) has a solution $(v(x), \tau)$ if and only if d_2 satisfies

$$\max\{0, \frac{B + C - 2A}{C - B}\} \frac{a_2}{\pi^2} < d_2 < \frac{a_2}{\pi^2}.$$

Moreover, the solution is unique.

Conjecture 3: Suppose that $C > 7B/3$. There exists the only one connected non-empty open set D such that (S) has exactly two solutions $(v(x), \tau)$ if and only if $d_2 \in D$.

References

[1] Y.S. Choi, R. Lui and Y. Yamada, *Existence of global solutions for Shigesada-Kawasaki-Teramoto model with weak cross-diffusion*, Discrete Contin. Dyn. Syst. 9 (2003), 1193-1200.

[2] Y.S. Choi, R. Lui and Y. Yamada, *Existence of global solutions for Shigesada-Kawasaki-Teramoto model with strongly coupled cross-diffusion*, preprint.

[3] H. Ikeda, K. Kondo, H. Okamoto and S. Yotsutani, *On the global branches of the solutions to a nonlocal boundary-value problem arising in Oseen's spriral flows*, Commun. Pure Appl. Anal., 3 (2003), 381-390.

[4] H. Ikeda, M. Mimura and H. Okamoto, *A Singular perturbation problem arising in Oseen's spiral flows*, Japan J. Indust. Appl. Math. 18 (2001), 393-403.

[5] S. Kosugi, Y. Morita and S. Yotsutani, *A complete bifurcation diagram of the Ginzburg-Landau equation with periodic boundary condition*, preprint.

[6] Y. Lou and W.-M. Ni, *Diffusion vs. cross-diffusion: An elliptic approach*, J. Differential Equations 154 (1999), 157-190.

[7] Y. Lou, W.-M. Ni and S. Yotsutani, *On a limiting system in the Lotka–Voltera competition with cross-diffusion*, Discrete Contin. Dyn. Syst. 10 (2004), 435-458.

[8] M. Mimura, Y. Nishiura, A. Tesei and T. Tsujikawa, *Coexistence problem for two competing species models with density-dependent diffusion*, Hiroshima Math. J. (1984), 425-449.

[9] M. Murai, W. Matsumoto and S. Yotsutani, in preparation.

[10] H. Okamoto, *Localization of singularities in inviscid limit – numerical examples*, Proceedings of Navier-Stokes Equations: Theory and Numerical Methods (ed. R. Salvi), Longman, Pitman Reserch Notes in Methamatics Series 388 (1998), 220-236.

[11] C. W. Oseen, *Exakte Lösungen der hydrodynamischen Differentialgleichungen. I.*, Arkiv Mat. Astr. Fysik, 20 (1927–1928), No. 14, pp. 1-14: ibid. II., ibid, No. 22, 1–9.

[12] N. Shigesada, K. Kawasaki and E. Teramoto, *Spatial segregation of interacting species*, J. Theoretical Biology 79 (1979), 83-99.

[13] K. Watanabe, *Plane domains which are spectrally determined*, Ann. Global. Anal. Geom. 18 (2000), 447-475.