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BROUÉ'S CONJECTURE FOR THE PRINCIPAL 5-BLOCK OF THE CHEVALLEY GROUP $G_2(4)$

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§1 Preliminaries

1.1. Let $(\mathcal{K}, \mathcal{O}, k)$ be a splitting p -modular system for all subgroups of the considered groups, that is, \mathcal{O} is a complete discrete valuation ring with unique maximal ideal \mathcal{P} , \mathcal{K} is its quotient field of characteristic 0 and k is its residue field \mathcal{O}/\mathcal{P} of prime characteristic p and we assume that \mathcal{K} and k are both big enough to be splitting fields for all subgroups of the considered groups. The principal p -block $B_0(G)$ of a group G is the indecomposable two-sided ideal of the group ring $\mathcal{O}G$ to which the trivial module belongs. In this paper "modules" always mean finitely generated modules. They are left modules, unless stated otherwise. Given a finite-dimensional k -algebra Λ , $\text{mod-}\Lambda$ denotes the category of finitely generated Λ -modules. All complexes will be cochain complexes. We write \otimes to mean \otimes_k . For a subgroup H of a group G , let U and V be $\mathcal{O}G$ - and $\mathcal{O}H$ -modules, respectively. We write $\text{Res}_H^G U$ or $U_{\downarrow H}$ for the restriction of U to H , namely

$$\text{Res}_H^G U = U_{\downarrow H} =_{\mathcal{O}H} \mathcal{O}G \otimes_{\mathcal{O}G} U$$

and $V^{\uparrow G}$ for the induction of V to G namely

$$V^{\uparrow G} =_{\mathcal{O}G} \mathcal{O}G \otimes_{\mathcal{O}H} V.$$

We use similar notation for kG -modules and kH -modules and for ordinary characters. Let \mathcal{O}_G be the trivial $\mathcal{O}G$ -module and k_G be the trivial kG -module. For $\mathcal{O}G$ -module V we write $\bar{V} = k \otimes_{\mathcal{O}} V$. For an \mathcal{O} -algebra B we write

$$\bar{B} = k \otimes_{\mathcal{O}} B,$$

1.2. Let A and B be two symmetric \mathcal{O} -algebras. According to Rouquier [Ro] we define two types of equivalence. The usual Morita equivalences are a special case of Rickard equivalences. For left A -module U , we denote by U^* the right A -module $\text{Hom}_{\mathcal{O}}(U, \mathcal{O})$.

Definition 1.3. We say that M is an *exact* (A, B) -bimodule if it is projective as an A -module and as a right B -module.

Definition 1.4. Let C^* be a bounded complex of exact (A, B) -bimodules. Assume that we have isomorphisms

$$C^* \otimes_B C^{**} \simeq A \oplus Z_1^* \text{ as complexes of } (A, A)\text{-bimodules}$$

$$C^{**} \otimes_A C^* \simeq B \oplus Z_2^* \text{ as complexes of } (B, B)\text{-bimodules}$$

where A and B are viewed as complexes concentrated in degree 0 and Z_1^* and Z_2^* are homotopy equivalent to 0. Then we say that C^* induces a *Rickard equivalence* between A and B or that C^* is a *Rickard complex*.

Definition 1.5. Let C^* be a complex of (A, B) -bimodules. Assume that we have isomorphisms

$$C^* \otimes_B C^{**} \simeq A \oplus Z_1'^* \text{ as complexes of } (A, A)\text{-bimodules}$$

$$C^{**} \otimes_A C^* \simeq B \oplus Z_2'^* \text{ as complexes of } (B, B)\text{-bimodules}$$

where $Z_1'^*$ and $Z_2'^*$ are homotopy equivalent to complexes of projective bimodules. Then we say that C^* induces a *stable equivalence* between A and B .

§2 Group ring

2.1. Now we concentrate our attention on group rings. Let G be a finite group with an abelian Sylow p -subgroup P . We denote by e the block idempotent of the principal block $B_0(G)$ of $\mathcal{O}G$. Let H be a subgroup of G such that $H \supset N_G(P)$. We denote by f the block idempotent of the principal block $B_0(H)$ of $\mathcal{O}H$.

Definition 2.2. A bounded complex C^\bullet of $(\mathcal{O}Hf, \mathcal{O}Ge)$ -bimodules is called splendid if its components are p -permutation modules whose indecomposable summands have vertices contained in $\Delta P = \{ (x, x) \in H \times G \mid x \in P \}$. Note that any component of a splendid complex is an exact bimodule.

Definition 2.3. Let G be a finite group with a Sylow p -subgroup P , and let $H \leq G$ be a subgroup containing P . A splendid Rickard complex for $B_0(G)$ and $B_0(H)$ is a bounded complex X^\bullet of finitely generated $(B_0(H), B_0(G))$ -bimodules such that

- (i) $X^\bullet \otimes_{B_0(G)} X^{\bullet\bullet}$ is chain homotopy equivalent to $B_0(H)$, considered as a complex of $B_0(H)$ -bimodules,
- (ii) $X^{\bullet\bullet} \otimes_{B_0(H)} X^\bullet$ is chain homotopy equivalent to $B_0(G)$, considered as a complex of $B_0(G)$ -bimodule, and
- (iii) X^\bullet is splendid.

In this case we say that X^\bullet induces a splendid Rickard equivalence between $B_0(G)$ and $B_0(H)$. (If X^\bullet is a splendid Rickard complex, then the functor

$$X^\bullet \otimes_{B_0(G)} ? : D^b(\text{mod-}B_0(G)) \rightarrow D^b(\text{mod-}B_0(H))$$

is an equivalence of triangulated categories, and $X^\bullet \otimes_{B_0(G)} ?$ gives an equivalence between chain homotopy categories, and not just derived categories. $D^b(\text{mod-}B_0(G))$ is a full subcategory of $D(\text{mod-}B_0(G))$ consisting of bounded objects, where $D(\text{mod-}B_0(G))$ is the derived category of the finitely generated module category of $B_0(G)$. We write them $D^b(B_0(G))$ and $D(B_0(G))$ for short.)

Conjecture 2.4. Broué's conjecture ([Br2]). Let G be a finite group with an abelian Sylow p -subgroup P . Then the principal p -block $B_0(G)$ of G and the principal p -block $B_0(N_G(P))$ of $N_G(P)$ are derived equivalent. (Moreover, they are splendidly Rickard equivalent in the refined version by Rickard.)

§3 Results

3.1. Broué's conjecture is known to be true for cyclic Sylow p -subgroups and for elementary abelian Sylow 3-subgroup of order 9 (see [KK]). Holloway proved that Broué's conjecture is true for some specific groups with elementary abelian Sylow 5-subgroups of order 25 in [H]. In particular, he proved it (actually, the splendid

Rickard equivalence) for the principal 5-blocks of J_2 (as well as $2.J_2$). Note that $G_2(4)$ contains a subgroup isomorphic to J_2 and these two groups have a common elementary abelian Sylow 5-subgroup P of order 25, and the common normalizer of P . We prove the splendid Rickard equivalence of the principal 5-blocks of $G_2(4)$ and J_2 . See Theorem 3.2. On the other hand, the first author already proved the splendid Morita equivalence between the principal 5-blocks of some family of the Chevalley groups $G_2(2^n)$ including $G_2(4)$. See Theorem 3.3. With Holloway's work we obtain following Corollary 3.4. (In fact the normalizer of P in $G_2(2^n)$ depends on n , but the factor group by its maximal normal p' -subgroup does not depend on n .)

Theorem 3.2. (Usami, Yoshida 2003). *The principal 5-blocks of $G_2(4)$ and J_2 are splendidly Rickard equivalent.*

Theorem 3.3. (Usami [U] 2001). *Assume that*

$$5 \text{ divides } 2^n + 1 \text{ but } 5^2 \text{ does not divide it.} \quad (3.1)$$

Then the principal 5-blocks of $G_2(2^n)$ and the principal 5-block of $G_2(4)$ are Morita equivalent. Here a $\Delta(P)$ -projective trivial source $G_2(4) \times G_2(2^n)$ -module and its \mathcal{O} -dual induce this Morita equivalence as bimodules, where P is a common abelian Sylow 5-subgroup of $G_2(2^n)$ and $G_2(4)$ and $\Delta(P) = \{(x, x) \in G_2(4) \times G_2(2^n) | x \in P\}$.

Corollary 3.4. *Broué's conjecture is true for the principal 5-blocks of $G_2(2^n)$ with n satisfying (3.1).*

§4 General Methods

4.1. With G , P and H in 2.1 we proceed according to the following steps :

Step 1. Construct a local splendid Rickard complex between $B_0(C_G(Q))$ and $B_0(C_H(Q))$ for each nontrivial p -subgroup Q of P .

Step 2. Construct a splendid complex which induces a stable equivalence between $B_0(G)$ and $B_0(H)$.

Step 3. Construct a global splendid Rickard complex between $B_0(G)$ and $B_0(H)$.

Here we introduce a general functor (from global to local) and we also introduce a useful theorem for Step 2.

Definition 4.2. [Br1]. For an $\mathcal{O}G$ -module V and any p -subgroup P of G , we set

$$\mathrm{Br}_P(V) = V^P / (\sum_Q \mathrm{Tr}_Q^P(V^Q) + \mathcal{P}V^P) \quad (4.1)$$

where V^P denotes the set of fixed points of V under P and Q runs over all proper subgroups of P and

$$\mathrm{Tr}_Q^P(v) = \sum_{x \in P/Q} x(v) \quad (4.2)$$

for a p -subgroup Q of P and $v \in V^Q$.

Remark 4.3. Let Q be a nontrivial p -subgroup of G . We can see Br_Q is a functor between the following categories :

$$\mathrm{Br}_Q : \{\mathcal{O}G\text{-modules}\} \rightarrow \{kN_G(Q)\text{-modules}\}$$

and then

$$\mathrm{Br}_Q : \{\text{complexes of } \mathcal{O}G\text{-modules}\} \rightarrow \{\text{complexes of } kN_G(Q)\text{-modules}\}.$$

Remark 4.4. With the notation in 2.1 and any nontrivial subgroup Q of P note that

$$\mathrm{Br}_{\Delta Q}(\mathcal{O}G) = kC_G(Q)$$

and

$$\mathrm{Br}_{\Delta Q}(f\mathcal{O}Ge) = \bar{f}_Q kC_G(Q) \bar{e}_Q$$

as bimodules, where \bar{f}_Q and \bar{e}_Q are the principal block idempotents of $kC_H(Q)$ and $kC_G(Q)$ respectively.

Theorem 4.5. (Rouquier and Bou, see Theorem 5.6 in [Ro]). *With the notation in 2.1 let C^\bullet be a splendid complex of $(\mathcal{O}Hf, \mathcal{O}Ge)$ -bimodules. The following assertions are equivalent.*

- (i) C^\bullet induces a stable equivalence between $\mathcal{O}Ge$ and $\mathcal{O}Hf$.
- (ii) For every nontrivial subgroup Q of P the complex $\mathrm{Br}_{\Delta Q}(C^\bullet)$ induces a Rickard equivalence between $kC_G(Q)\bar{e}_Q$ and $kC_H(Q)\bar{f}_Q$.

4.6. We can consider a direct summand of a permutation module over k as well as over \mathcal{O} . Then we can define a splendid complex over k similarly to Definition 2.2, and the definitions of a splendid Rickard complex and splendid Rickard equivalence still make sense if we replace the coefficient ring \mathcal{O} by the field k . A splendid Rickard equivalence over \mathcal{O} induces a splendid Rickard equivalence over k just by applying the functor $k \otimes_{\mathcal{O}} ?$ to a splendid Rickard complex. Note that any direct summand of a permutation module and any map between such modules can be lifted from k to \mathcal{O} . Then by Theorem 2.8 in [Ri1] a splendid Rickard complex over k can be lifted to a splendid Rickard complex over \mathcal{O} that is unique up to isomorphism. Then it is sufficient to work over k in order to prove the refined version of Broué's conjecture.

Theorem 4.7. (Rickard) (see [Ri2, Theorem 6.1] and [H, Theorem 4.4]) *Suppose that C^\bullet is a complex of $(kH\bar{f}, kG\bar{e})$ -bimodules that induces a splendid stable equivalence between $kG\bar{e}$ and $kH\bar{f}$ and let $\{S_1, \dots, S_r\}$ be a set of representatives for the isomorphism classes of simple $kG\bar{e}$ -modules. If there are objects $X_1^\bullet, \dots, X_r^\bullet$ of $D^b(kH\bar{f})$ such that, for each $1 \leq i \leq r$, X_i^\bullet is stably isomorphic to $C^\bullet \otimes_{kG\bar{e}} S_i$ and such that*

- (a) $\text{Hom}_{D^b(kH\bar{f})}(X_i^\bullet, X_j^\bullet[m]) = 0$ for $m < 0$,
- (b) $\text{Hom}_{D^b(kH\bar{f})}(X_i^\bullet, X_j^\bullet) = \begin{cases} 0 & \text{if } i \neq j, \\ k & \text{if } i = j, \end{cases}$ and
- (c) $X_1^\bullet, \dots, X_r^\bullet$ generate $D^b(kH\bar{f})$ as a triangulated category,

then there is a splendid Rickard complex X^\bullet that lifts C^\bullet and induces a splendid Rickard equivalence between $kG\bar{e}$ and $kH\bar{f}$ such that, for each $1 \leq i \leq r$, $C^\bullet \otimes_{kG\bar{e}} S_i \cong X_i^\bullet$ in $D^b(kH\bar{f})$.

§5 Steps 1 and 2 for Theorem 3.2

5.1. In this section we set

$$G = G_2(4), G \supset J \supset N_G(P) \text{ where } J \cong J_2, \quad (5.1)$$

where P is a common elementary abelian Sylow 5-subgroup of G and J of order 25. We have

$$N_G(P) = N_J(P) \cong P : D_{12}$$

, that is, a semi-direct product of P by the dihedral group D_{12} of order 12. Fusion of the subgroups of P is controlled by $N_G(P)$ and

there are, up to conjugacy in $N_G(P)$, two nontrivial cyclic 5-subgroups of P , where only one, Q has distinct centralizers in G and J . (5.2)

Q is generated by a 5-element in conjugate class $5C$ in the character tables of J_2 and also of $G_2(4)$ in Atlas [CCNPW]. We fix Q from now on. We set

$$k \otimes_{\mathcal{O}} B_0(G) = k \otimes_{\mathcal{O}} \mathcal{O}Ge = kG\bar{e} \text{ and } k \otimes_{\mathcal{O}} B_0(J) = k \otimes_{\mathcal{O}} \mathcal{O}Jf = kJ\bar{f}.$$

5.2. Before we go further we review the principal 5-block of A_5 . A_5 contains a subgroup isomorphic to D_{10} which is a normalizer of a fixed cyclic Sylow subgroup of order 5. As (kD_{10}, kA_5) -bimodule $\bar{B}(A_5)$ is indecomposable and its projective cover is

$$\bar{R}_0 \otimes \bar{P}_0 \oplus \bar{R}_1 \otimes \bar{P}_1 \rightarrow \bar{B}(A_5) \rightarrow 0 \quad (5.3)$$

where \bar{P}_0 and \bar{R}_0 are the projective covers of the trivial kA_5 -module and the trivial kD_{10} -module, respectively, and \bar{P}_1 is the projective indecomposable module of the principal block of kA_5 , that is not isomorphic to \bar{P}_0 , and \bar{R}_1 is the unique projective indecomposable kD_{10} -module which is not isomorphic to \bar{R}_0 . The splendid Rickard equivalence between the principal blocks of kA_5 and kD_{10} is induced by the splendid Rickard complex

$$\cdots 0 \rightarrow 0 \rightarrow \bar{R}_1 \otimes \bar{P}_1 \rightarrow \bar{B}(A_5) \rightarrow 0 \rightarrow 0 \cdots \quad (5.4)$$

which we can obtain by deleting the first term of (5.3). Keeping (5.4) in mind we construct a splendid Rickard complex between $kC_G(Q)\bar{e}_Q$ and $kC_J(Q)\bar{f}_Q$. See (5.6) below. Then we seek a splendid complex C^* which induces a stable equivalence between $B_0(G)$ and $B_0(J)$. (By Theorem 4.5 it is just to find C^* such that $\text{Br}_{\Delta(Q)}(C^*)$ is equal to (5.6).)

Lemma 5.3. *Let Q be a nontrivial subgroup of P such that Q has distinct centralizers in G and J . Then we have the following.*

(i) $C_G(Q) = Q \times A_5$ and $C_J(Q) = Q \times D_{10}$.

(ii) Tensoring (kQ, kQ) -bimodule kQ to (5.3) we obtain minimal $\Delta(Q)$ -projective cover of indecomposable $\bar{f}_Q kC_G(Q)\bar{e}_Q \cong kQ \otimes \bar{f}_Q kA_5\bar{e}_Q$:

$$kQ \otimes \bar{R}_0 \otimes \bar{P}_0 \oplus kQ \otimes \bar{R}_1 \otimes \bar{P}_1 \rightarrow kQ \otimes \bar{f}_Q kA_5\bar{e}_Q \rightarrow 0 \quad (5.5)$$

(iii) Deleting the first term of (5.5) we obtain the following splendid complex which induces the splendid Rickard equivalence between the principal blocks $kC_G(Q)\bar{e}_Q$ and $kC_J(Q)\bar{f}_Q$.

$$\dots 0 \rightarrow 0 \rightarrow kQ \otimes \bar{R}_1 \otimes \bar{P}_1 \rightarrow \bar{f}_Q kC_G(Q)\bar{e}_Q \rightarrow 0 \rightarrow 0 \dots \quad (5.6)$$

(iv) The following is the minimal ΔQ -projective cover of $k_{\Delta Q,2} \uparrow^{(Q \times Q),2} \otimes \bar{f}_Q kA_5 \bar{e}_Q$:

$$\begin{aligned} k_{\Delta Q,2} \uparrow^{(Q \times Q),2} \otimes \bar{R}_0 \otimes \bar{P}_0 \oplus k_{\Delta Q,2} \uparrow^{(Q \times Q),2} \otimes \bar{R}_1 \otimes \bar{P}_1 \\ \rightarrow k_{\Delta Q,2} \uparrow^{(Q \times Q),2} \otimes \bar{f}_Q kA_5 \bar{e}_Q \rightarrow 0. \end{aligned} \quad (5.7)$$

Furthermore we have

$$k_{\Delta Q,2} \uparrow^{(Q \times Q),2} \downarrow_{Q \times Q} \cong kQ$$

and then the restriction of (5.7) to $C_J(Q) \times C_G(Q)$ is (5.5).

Lemma 5.4. (i) There exists an exact sequence (with M^0 as the Scott module of $J \times G$ with vertex ΔP , $\text{Scott}(J \times G, \Delta P)$)

$$\text{Scott}(J \times G, \Delta Q) \oplus M^{-1} \oplus (\text{some projective bimodule}) \rightarrow M^0 \rightarrow 0 \quad (\text{exact}) \quad (5.8)$$

such that $k \otimes (5.8)$:

$$\overline{\text{Scott}}(J \times G, \Delta Q) \oplus \bar{M}^{-1} \oplus (\text{some projective bimodule}) \rightarrow \bar{M}^0 \rightarrow 0 \quad (\text{exact})$$

is the minimal $\Delta(Q)$ -projective cover of \bar{M}^0 , where \bar{M}^{-1} is the indecomposable trivial source module with vertex $\Delta(Q)$ which corresponds to the second term with vertex ΔQ in (5.7).

(ii) Deleting the Scott module and the projective summand from (5.8) we obtain a splendid complex

$$\dots 0 \rightarrow 0 \rightarrow M^{-1} \rightarrow M^0 \rightarrow 0 \rightarrow 0 \dots$$

which induces a splendid stable equivalence between $B_0(G)$ and $B_0(J)$.

§6 Step 3 for Theorem 3.2

6.1. We obtain a candidate of a splendid Rickard complex between $B_0(G)$ and $B_0(J)$: (We use a perfect isometry between the sets of their ordinary characters to search some candidates.)

$$X^\bullet : \cdots 0 \longrightarrow 0 \longrightarrow (\text{a projective bimodule}) \longrightarrow (\text{a projective bimodule}) \\ \longrightarrow M^{-1} \longrightarrow M^0 \longrightarrow 0 \longrightarrow \cdots$$

Set

$$X^\bullet \otimes_{\mathcal{O}G_e} S_i = \overline{X}^\bullet \otimes_{kG_e} S_i = X_i$$

for simple $\mathcal{O}G_e$ -modules $\{S_i \mid 1 \leq i \leq 6\}$. We have only to check conditions (a), (b) and (c) in Rickard's Theorem (Theorem 4.7).

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