数理解析研究所講究録 1350 巻 2004 年 1-9



# Reconstruction of White Noise Analysis

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**Abstract**

The aim of our study has been the investigation of random complex systems. For this purpose we are suggested to return to J. Bernoulli's idea expressed in his book "Ars Conjectandi". This idea has been followed by P. Lévy and has been realized by his famous formula called *stochastic infinitesimal equation* for a stochastic process, where the significant role is played by the **innovation**. We shall therefore start with an interpretation of the innovation.

We know that the standard type of innovation is given by the time derivative of a Lévy process, that is a general white noise, the Lévy decomposition of which has been well established. A general white noise consists of *idealized elemental random variables*.

In order to discuss the analysis of functionals of a general innovation a suitable space of random variables should be introduced. Then, we come to the study of white noise functionals, in particular, stochastic processes and random fields parameterized by a contour or a closed surface. A natural generalization of the stochastic infinitesimal equation will be given. It is a *stochastic variational equation*, where the innovation is given by the same idea as in the case of a stochastic process.

Some thought on future directions will be touched upon briefly.

**Reduction $\longrightarrow$ Synthesis $\longrightarrow$ Analysis,**

where the *causality* with respect to the time or space-time variable is always involved.

**AMS subject classification :60H40**

## §0. Introduction.

**§0.1.** The *Leitmotive* of our approach are as follows.

1) Remind the idea of Bernoulli to discuss *stochastic*.

The idea appears, either explicitly or implicitly, in the works by P. Lévy, N. Wiener, A.N. Kolmogorov and others.

2) Linear and nonlinear operations on paths.

Some path-wise analysis for stochastic processes are significant. More generally, generalized harmonic analysis in the sense of N. Wiener will be useful.

Typical examples are subordination and continuity problems on paths, Some non-linear predictions require operations on sample functions of a stochastic process, etc.

3) Introduce a new space of random functions, call it **(P)**, where topologies are either almost sure convergence or convergence in probability. Often the quasi convergence is used.

4) Applications in physics.
The X-ray data from the star Cyg X1 is a good object to be investigated in the theory of stochastic process.

The Feynman path integrals.

Problems of measurement in Quantum dynamics.

Molecular biology.
etc.

§**0.2.** Lévy's *stochastic infinitesima equation* for a stochastic process $X(t)$ is expressed in the form

$$\delta X(t) = \Phi(X(s), s \leq t, Y(t), t, dt),$$

where $\delta X(t)$ stands for the variation of $X(t)$ for the infinitesimal time interval $[t, t+dt)$, the $\Phi$ is a sure functional and the $Y(t)$ is the *innovation.* Intuitively speaking, the innovation is a system such that the $Y(t)$ contains the same information as that newly gained by the $X(t)$ during the infinitesimal time interval $[t, t+dt)$. If such an equation is obtained, then the pair $(\Phi, Y(t))$ can completely characterize the probabilistic structure of the given process $X(t)$. Npte that, the $Y(t)$ is, sometimes, taken to be a vector valued generalized stochasic process.

As a generalization of the stochastic infinitesimal equation for $X(t)$, one can introduce a *stochastic variational equation* for random field $X(C)$ parameterized by an ovaloid $C$:

$$\delta X(C) = \Phi(X(C'), C' < C, Y(s), s \in C, C, \delta C),$$

where $C' < C$ means that $C'$ is in the inside of $C$. The system $\{Y(s), s \in C\}$ is the innovation which is understood in the similar sense to the case of $X(t)$.

The two equations above have only a formal significance, however we can give rigorous meaning to the equations with some additional assumptions and the interpretations to the notations introduced there (see, e.g. [9]).

The results obtained at present are, of course, far from the general theory, however one is given a guideline of the approach to those random complex evolutional systems in line with the innovation theory and hence with the white noise theory.

## §1. Gaussian systems.

**§1.1.** First we discuss a Gaussian process $X(t), t \in T$, where $T$ is an interval of $R^1$, say $[0, \infty)$. Assume that it is separable and has no remote past. Then, the innovation can be constructed explicitly in this case. The original idea came from P. Lévy (The third Berkeley Synposium paper; see [13]). Under the assumption that the process has unit multiplicity and other mild conditions, a Gaussian process $X(t)$ has the innovation $\dot{B}(t)$ which is a white noise such that $X(t)$ is expressed as the Wiener integral of the form

$$X(t) = \int_0^t F(t,u)\dot{B}(u)du, \tag{1}$$

This is the so-called *canonical representation.* It might seem to be rather elementary, however such an easy understanding is, in a sense, not quite correct. The profound structure sitting behind this formula would lead us to a deep insight that is applicable to a general class of Gaussian processes and to non Gaussian case, too.

Take a Brownian motion $B(t)$ and a kernel function $G(t,u)$ of Volterra type. Define a Gaussian process $X(t)$ by

$$X(t) = \int_0^t G(t,u)\dot{B}(u)du.$$

Now we assume that $G(t,u)$ is a smooth function on the domain $0 \le u \le t < \infty$ and $G(t,t)$ never vanishes. Then we have

**Theorem 1.** The variation $\delta X(t)$ of the process $X(t)$ is defined and is given by

$$\delta X(t) = G(t,t)\dot{B}(t)dt + dt\int_0^t G_t(t,u)\dot{B}(u)du,$$

where $G_t(t,u) = \frac{\partial}{\partial t}G(t,u)$. The $\dot{B}(t)$ is the innovation of $X(t)$ if and only if G(t,u) is the canonical kernel.

Proof. The formula for the variation of $X(t)$ is easily obtained. If $G$ is not a canonical kernel, then the sigma field $\mathbf{B}_t(X)$ is strictly smaller than $\mathbf{B}(\dot{B})$, in particular the $\dot{B}(t)$ is not really a function of $X(s), s \le t+0$.

Now follow important notes. By the smoothness assumption on the kernel $G(t,u)$ the integral is defined path-wise, so that the formula on the variational equation for $X(t)$ give us a white noise equaivalent to $\dot{B}(t)$ (Accardi and Si Si). The equiavalence means the same innovation up to sign.

Another note is that if, in particular, $G(t,u)$ is of the form $f(t)g(u)$, then $X(t)$ is a Markov process and there is always given a canonical representation. Hence $\dot{B}(t)$ is the innovation.

**Remark.** In the variational equation, the two terms in the right hand side are of different order as $dt$ tend to zero, so that two terms may be discrimiated. But in reality the problem like that is not so simple and even not our present concern.

As a result of having obtained the innovation, we can define the partial derivative denoted by $\partial_t$ and expressed in the form

$$\partial_t = \frac{\partial}{\partial \dot{B}(t)}.$$

It is given by the knowledge of the original process $X(s), s \leq t$. Hence the canonical kernel is obtained by

$$F(t,u) = \partial_u X(t), u < t.$$

§**1.2.** Gaussian random fields.

To fix the idea we consider a Gaussian random field $X(C)$ parameterized by a smooth convex contour in $R^2$ that runs through a certain class **C** which is topologized by the usual method using the Euclidean metric. Denote by $W(u), u \in R^2$, a two dimensional parameter white noise. Let $(C)$ denote the domain enclosed by the contour $C$.

Given a Gaussian random field $X(C)$ and assume that it is exressed as a stochastic integral of the form:

$$X(C) = \int_{(C)} F(C,u)W(u)du,$$

where $F(C,u)$ be a kernel function which is locally square integrable in $u$. For convenience we assume that $F(C,u)$ is smooth in $(C,u)$. The integral is a *causal* representation of the $X(C)$. The canonical property can be defined as a generalization to a random field as in the case of a Gaussian process.

The stochastic variational equation for this $X(C)$ is of the form

$$\delta X(C) = \int_C F(C,s)\delta n(s)W(s)ds + \int_{(C)} \delta F(C,u)W(u)du.$$

In a similar manner to the case of a process $X(t)$, but somewhat complicated manner, we can form the innovation $\{W(s), s \in C\}$.

Example. A variational equation of Langevin type.

Given a stochastic variational equation

$$\delta X(C) = -X(C)\int_C k\delta n(s)ds + X_0 \int_C v(s)\partial_s^* \delta n(s)ds, C \in \mathbf{C},$$

where **C** is taken to be a class of concentric circles, $v$ is a given continuous function and $\partial_s^*$ is the adjoint operator of the differential operator $\partial_s$.

Applying the equation the so-called $S$-transform, which is an infinite dimensional analogue of the Lapalce transform, we can solve the transformed equation by appealing to the classical theory of functional analysis. Then, applying the inverse transform $S^{-1}$, the solution is given:

$$X(C) = X_0 \int_{(C)} \exp[-k\rho(C,u)]\partial_u^* v(u)du,$$

where $\rho$ is the Euclidean distance.

Now one may ask the integrability condition of a given stochastic variational equation. This question has been discussed by Si Si [15].

Anther question concerning how to obtain the innovation from a random field may be discussed by refering to the literature [9].

**§2. General innovation.**

Returning to the innovation $Y(t)$ of a process $X(t)$ one can see that, in favourable cases, there is an additive process $Z(t)$ such that its derivative $\dot{Z}(t)$ is equal to the $Y(t)$, since the collection $\{Y(t)\}$ is an independent system. There is tacitly assumed that, in the system, there is no random function singular in $t$.

There is the Lévy decomposition of an additive process. If $Z(t)$ has stationary independent increments, then except trivial component the $Z(t)$ involves a compound Poisson process $X_1(t)$ and a Brownian motion $B(t)$ up to constant:

$$Z(t) = X_1(t) + \sigma B(t).$$

With this remark in mind we proceed to the Poisson case.

**§2.1.** After Brownian motion comes another kind of elemental additive process which is to be the Poisson process denoted by $P(t), t \geq 0$. Taking its time derivative $\dot{P}(t)$ we have a *Poisson white noise*. It is a generalized stationary stochastic process with independent value at every point. For convenience we may assume that $t$ runs through the whole real line. In fact, it is easy to define such a noise. The characteristic functional of the centered Poisson white noise is of the form

$$C_P(\xi) = \exp[\int_{-\infty}^{\infty} (e^{i\xi(t)} - 1 - i\xi(t))dt],$$

where $\xi \in E$.

There is the associated measure space $(E^*, \mu_P)$, and the Hilbert space $L^2(E^*, \mu_P) = (L^2)_P$ is defined.

Many results of the analysis on $(L^2)_P$ have been obtained, however most of them have been studied by analogy with the Gaussian case or its modifications, as far as the construction of the space of generalized functionals. Here we only note that the $(L^2)_P$ admits the direct sum decomposition of the form

$$(L^2)_P = \bigoplus_n H_{P,n}.$$

The subspace is formed by the Poisson Charlier polynomials.

However, there might occur a misunderstanding regarding the functionals of Poisson noise, even in the case of linear functional. The following example would illustrate this fact (see [8]).

Let a stochastic process $X(t)$ be given by an integral

$$X(t) = \int_0^t F(t,u)\dot{P}(u)du.$$

It seems to be simply a linear functional of $P(t)$, however there are two ways of understanding the meaning of the integral; one is defined

i) in the Hilbert space by taking $\dot{P}(t)dt$ to be a random measure.

Another way is to define the integral

ii) for each sample function of $P(t)$ (the path-wise integral). This can be done if the kernel is a smooth function of $u$ over the interval $[0,t]$.

Assume that $F(t,t)$ never vanishes and that it is not a canonical kernel, that is, it is not a kernel function of an invertible integral operator. Then, we can claim that for the integral in the first sense $X(t)$ has less information compared to $P(t)$. Because there is a linear function of $P(s), s \leq t$ which is orthogonal to $X(s), s \leq t$. On the other hand, if $X(t)$ is defined in the second sense, then we can prove

**Proposition.** Under the assumptions stated above, if the $X(t)$ above is defined sample function-wise, we have the following equality for sigma-fields:

$$\mathbf{B}_t(X) = \mathbf{B}_t(P), t \geq 0.$$

Proof. By assumption it is eqasy to see that $X(t)$ and $P(t)$ share the jump points, which means the information is fully transfered from $P(t)$ to $X(t)$.This proves the equality

The above argument tells us that we are led to introduce a space $(\mathbf{P})$ of random variables that come from separable stochastic processes for which existence of variance is not expected. This sounds to be a vague statement, however we can rigorously defined by using a Lebesgue space without atoms, and others. There the topology is defined by either the almost sure convergent or the convergence in probability, and there is no need to think of mean square topology. On the space $(\mathbf{P})$ filtering and prediction for strictly stationary process can naturally be discussed. For further idea we may refer to the literatures [17] and [18], where one can see further profound idea of N. Wiener.

It is almost straightforward to come to an introduction of a multi-parameter Poisson white noise, denoted by $\{V(u)\}$, which is a generalization of $\{\dot{P}(t)\}$.

**Theorem 2.** Let a random field $X(C)$ parameterized by a contour $C$ be given by a stochastic integral

$$X(C) = \int_{(C)} G(C,u)V(u)du,$$

where the kernel $G(C,u)$ is continuous in $(C,u)$. Assume that $G(C,s)$ never vanishes on $C$ for every $C$. Then, the $V(u)$ is the innovation.
Proof. The variation $\delta X(C)$ exists and it involves the term

$$\int_C G(C,s)\delta n(s)V(s)ds,$$

where $\{\delta n(s)\}$ determines the variation $\delta C$ of $C$. Here is used the same technique as in the case of [9], so that the values $V(s), s \in C$, are determined by taking various $\delta C$'s. This shows that the $V(s)$ is obtained by the $X(C)$ according to the infinitesimal change of $C$. Hence $V(s)$ is the innovation.

Here is an important remark. In the Poisson case one can see a significant difference on getting the innovation from the case of a representation of a Gaussian process. However, if one is permitted to use some nonlinear operations acting on sample functions, it is possible to form the innovation from a non-canonical representation of a Gaussian process ( Si Si [16]), although the proof needs a profound property of a Brownian motion (see P. Lévy [11, Chapt. VI]).

§**2.2.** Compound Poisson process.

As soon as we come to a compound Poisson process, which is a more general innovation, the second order moment may not exist, so that we have to come to the space (**P**). The Lévy decomposition of an additive process, with which we are now concerned, is expressed in he form

$$Z(t) = \int (uP_{du}(t) - \frac{tu}{1+u^2}dn(u)) + \sigma B(t),$$

where $P_{du}(t)$ is a random measure of the set of Poisson processes, and where $dn(u)$ is the Lévy measure such that

$$\int \frac{u^2}{1+u^2}dn(u) < \infty.$$

The decomposition of a Compound Poisson process into the individual elemental Poisson processes with different jumps can be carried out in the space (**P**) with the use of the quasi-convergence (see [11, Chapt.V]) . We are now ready to discuss the analysis acting on sample functions of a compound Poisson process.

A generalization of the Proposition in the last subsection to the case of compound Poisson white noise is not difficult in a formal way without paying much attention. However, we wish to pause at this moment to consider carefully about how to find a jump point of $Z(t)$ with the height $u$ designated in adavance. This question is heavily depending on the computability or measurement problem. Further questions related to this problem shall be discussed in the separate paper.

## §3. Concluding remarks.

A Brownian motion and each component of the compound Poisson process seem to be *elemental*. Indeed, this is true in a sense. On the other hand, there is another aspect. Indeed, we know that the inverse function of the Maximum of a Brownian motion is a stable process, which is a compound Poisson process ( see. [11, Chapt. VI]). A Poisson process comes from a Browniann motion! Certainly not by the $L^2$ method.

Also, in terms of the probability distribution, it is shown in [2] that some genetalized (Gaussian) white noise functional has the same distribution as that of a Poisson white noise. There arises a question on how to find concrete operations (variational calculus may be involved there) acting on the sample functions of $\dot{B}(t)$'s to have a Poisson white noise. We need some more examples to propose a problem to give a good interpretation to such phenomena.

In Section 1.1, we have noted that non-canonical representation of a Gaussian process may give an innovation equivalent to the original white noise. An interpretation to this fact by using the infinite dimensional rotation group will be reported later.

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