Two-Valued Weak Kleene Logics

Abstract

In the literature, Weak Kleene logics are usually taken as three-valued logics. However, Suszko has challenged the main idea of many-valued logic claiming that every logic can be presented in a two-valued fashion. In this paper, we provide two-valued semantics for the Weak Kleene logics and for a number of four-valued subsystems of them. We do the same for the so-called Logics of Nonsense, which are extensions of the Weak Kleene logics with unary operators that allow looking at them as Logics of Formal Inconsistency (LFIs) and Logics of Formal Underterminedness (LFUs). Our aim with this work, rather than arguing for Suszko's thesis, is to show that two-valued presentations of these peculiar logics enlighten the non-standard behavior of their logical connectives. More specifically, the two-valued presentations of paraconsistent logics illustrate and clarify the disjunctive flavor of the conjunction, and dually, the two-valued presentations of paracomplete subsystems of Weak Kleene logics reveal the conjunctive flavor of the disjunction.

1 Background and aim

This paper has three thematic backgrounds: Weak Kleene logics, Logics of Nonsense and Suszko's Thesis. By *Weak Kleene logics* we refer to two different matrix logics, i.e. (Paracomplete) Weak Kleene logic, symbolized as $\mathbf{K_3^w}$, and Paraconsistent Weak Kleene logic, symbolized as **PWK**. These logics are associated to the 3-element Weak Kleene algebra **WK** (due to Kleene in [21], and recently analyzed in [4]) built using the 3-valued *weak* truth-tables from Kleene as follows

$$\mathbf{WK} = \langle \{\mathbf{t}, \mathbf{u}, \mathbf{f}\}, \{f^{\neg}_{\mathbf{WK}}, f^{\wedge}_{\mathbf{WK}}, f^{\vee}_{\mathbf{WK}}\} \rangle$$

where the functions $f^{\neg}_{\mathbf{WK}}, f^{\wedge}_{\mathbf{WK}}, f^{\vee}_{\mathbf{WK}}$ are

	$f^{\neg}_{\mathbf{WK}}$
t	**f**
u	**u**
f	**t**

$f^{\wedge}_{\mathbf{WK}}$	**t**	**u**	**f**
t	**t**	**u**	**f**
u	**u**	**u**	**u**
f	**f**	**u**	**f**

$f^{\vee}_{\mathbf{WK}}$	**t**	**u**	**f**
t	**t**	**u**	**t**
u	**u**	**u**	**u**
f	**t**	**u**	**f**

What is interesting of this algebra and of the logics defined using it (as we shall see next) is that they exhibit a peculiar semantic behavior: they have a truth-value (v.g. the intermediate truth-value **u**) that is assigned to a compound formula whenever it is assigned to one of its components. This justifies referring to these logics as infectious, contaminating, propagating or absorbent systems (see e.g. [16], [20], [35] for these denominations), given they have a value that behaves in this particular way.

These two different logics are induced by taking the set of designated values—i.e. the set of values preserved from premises to conclusion—*not* to include the intermediate value, in the case of $\mathbf{K_3^w}$, and to *include* the intermediate value, in the case of **PWK**.[1] If we define a logic to be *paracomplete* whenever the Law of Excluded Middle is invalid in it, and a logic to be *paraconsistent* whenever the Principle of Explosion is invalid in it, it is easy to see that whereas $\mathbf{K_3^w}$ is a paracomplete logic, **PWK** is a paraconsistent logic.

Some readers, though, might object referring to the latter as *Paraconsistent* Weak Kleene logic, for Kleene devised his logics conceiving the intermediate value to be a truth-value *gap* or a representative of an *indeterminate* semantics status. But paraconsistent logics are usually identified with the availability of truth-value *gluts* or truth-values representing an *inconsistent* semantic status. Thus, even if the intermediate value from the 3-element Weak Kleene algebra is considered as a designated value, the resulting logic should not be called Paraconsistent Weak Kleene, for it does not honor Kleene's motivation for entertaining the underlying algebra.

To this we reply by saying that, although we think the best way to interpret **PWK** is by taking the intermediate value to be a truth-value *glut* (as will be clear given the two-valued semantics we will offer for it in Section 4), nothing said in this paper depends on calling the resulting system 'Paraconsistent Weak Kleene'. Nevertheless, we will stick to the former denomination, for it is by now the standard thing to do in the literature about this systems (cf. [10], [9], [4], [35]). As part of the present investigation, we will take into consideration four-valued subsystems of these logics, which also exhibit the sort of infectious semantic behavior previously discussed. In particular, we will consider the four-valued logics $\mathbf{S_{\mathtt{fde}}}, \mathbf{dS_{\mathtt{fde}}}, \mathbf{S^w_{\mathtt{fde}}}, \mathbf{dS^w_{\mathtt{fde}}}$.[2]

By *Logics of Nonsense* we refer to a collection of many-valued systems, developed in the last century by Bochvar [3] and Halldén [19] (later discussed by Segerberg [31] and others), in order to design logics where reasoning with nonsensical propositions is possible. This is done with the help of matrix logics where formulae can be attributed nonsensical or meaningless semantic values. Notably—as it is well known—the logics $\mathbf{K_3^w}$ and **PWK** correspond to the $\{\neg, \wedge, \vee\}$-fragment of, respectively, Bochvar's and Halldén's Logics of Nonsense.

Furthermore, the distinctive ingredient of the Logics of Nonsense is that they are equipped with one, or many operations that take every value (including the nonsensical value **u**) to either **t** or **f**, thereby 'classicizing' the formulae to which they are applied. For example, Bochvar has an 'external assertion' operator which represents the idea of being true within the language, that takes all values except **t** to **f**, and naturally takes **t** to **t** (cf. [3]). Halldén, on his own, has a meaningfulness operator which represents the idea of being either true or false within the language, that takes all values except **u** to **t**, and takes **u** to **f** (cf. [19]).

Finally, by *Suszko's Thesis* (cf. [33]) we refer to the claim that every Tarskian *many-valued* (i.e. matrix) logic can be provided with a two-valued or *bivalent* semantics. The further application of this idea to a given many-valued logic, to render the corresponding two-valued system, is referred to as Suszko's Reduction.

The aim of this paper is to study Weak Kleene logics and Logics of Nonsense under the lens of Suszko's Thesis, and analyze the resulting systems, specially regarding the defined

[1] For a precise definition of these systems, see Section 2.

[2] All of these logics are defined in Section 3.2. These systems are referred to, in [35], as the logics $\mathbf{L_{be}}$, $\mathbf{L_{nb'}}$, $\mathbf{L_{b'e}}$ and $\mathbf{L_{eb'}}$, respectively.

connectives. There are several reasons for having a special interest in the application of Suszko's Reduction to these logics. The first is that—to the best of our knowledge—no analysis of logics presenting an infectious behavior of the aforementioned sort was carried out in the literature. The second is that Logics of Nonsense in particular were recognized in [6] and [11] as Logics of Formal Inconsistency and Logics of Formal Undeterminedness (LFIs and LFUs, hereafter). Usually, LFIs and LFUs are presented in terms of bivaluations or two-valued semantics, but Logics of Nonsense are commonly treated as essentially many-valued. Thus, the present investigation represents an attempt to bring Logics of Nonsense closer to the standard formalism entertained when dealing with LFIs and LFUs.

Finally, the main reason for providing suitable two-valued semantics to Weak Kleene logics is to enlighten some properties of the systems that, we think, are somewhat hidden in their many-valued presentations. In other words, although [5] offer an algorithm for obtaining two-valued semantics for a wide range of logics (including the treated here), our aim here is not to apply this general method, but to provide some specific two-valued semantics that we think are natural for the logics, and also to analyze the peculiarity of the systems.[3] In this sense, we are particularly interested in studying the behavior of the connectives in the two-valued systems and its relation with the concepts of paraconsistency and paracompletness.

To this extent, the paper is structured as follows. In Section 2 we present some preliminaries, related with technical definitions that we assume for the rest of the paper. Next, in Section 3 we introduce the infectious logics we will work with from a technical point of view. In particular, in Section 3.1 we present the three-valued Weak Kleene logics $\mathbf{K_3^w}$ and $\mathbf{PWK}$ and in Section 3.2, the four four-valued subsystems of these logics $\mathbf{S}_{\mathtt{fde}}, \mathbf{dS}_{\mathtt{fde}}, \mathbf{S}^{\mathtt{w}}_{\mathtt{fde}}, \mathbf{dS}^{\mathtt{w}}_{\mathtt{fde}}$. In Section 4 we will discuss Suszko's thesis and some of its philosophical consequences. After this, we will proceed in Section 5 to reduce the many-valued logics above mentioned. Specifically, in Section 5.1 we introduce two-valued semantics for the three-valued logics. In Section 5.2, we do the same for the four-valued subsystems: $\mathbf{S}_{\mathtt{fde}}, \mathbf{dS}_{\mathtt{fde}}, \mathbf{S}^{\mathtt{w}}_{\mathtt{fde}}$ and $\mathbf{dS}^{\mathtt{w}}_{\mathtt{fde}}$. Thorough this section, we not only present the technical reduction of the logics, but also we discuss the specific behavior of the connectives in the two-valued systems. Finally, in Section 6, as mentioned before, we consider Logics of Nonsense from a technical point of view, i.e. as resulting from adding a unary operator to the three-valued Weak Kleene logics and considering them as LFIs and LFUs. Later, in Section 6.1, as was done with the other logics, we provide a two-valued semantic for them. We close the article in Section 7, with a number of concluding remarks.

[3]As pointed out by an anonymous reviewer, two-valued semantics for the Weak Kleene logics have been presented in [27] and more recently in [36]. The main difference between these works and our is in the way in which such semantics are presented. In the case of Szmuc and Omori's papers a two-valued presentation is arrived at by implementing a particular sort of *relational semantics* called plurivalent logics after Graham Priest's paper [29], while in our case a two-valued presentation is arrived at through *functional semantics* in the spirit of bivaluations. There is, though, the possibility of translating the bivaluations presented here to a relational semantics, in the vein of what is pointed out in [36, fn. 15], i.e. a sort of Dunn-semantics where truth and falsity conditions are independently provided for each connective. However, even in this case, the kind of two-valued relational semantics that one would arrive at would be of a radically different type than those discussed in [27] and [36]. The reason for this is that truth and falsity conditions are not provided independently in the context of plurivalent semantics, but are systematically induced for all the connectives—as explained in [20]—by the operations of the power algebra of the two-element Boolean algebra.

2 Technicalities

Definition 2.1. Let Prop be a denumerable set of propositional variables and let $\Sigma = \bigcup_{n\in\mathbb{N}} \Sigma_n$ be a propositional language, such that each Σ_n is a set of connectives, for which $\Sigma_n \cap \Sigma_m = \emptyset$, if $n \neq m$. Σ_n is the set of n-ary connectives. In particular, the elements of Σ_0 are usually called constants. $\mathbf{FOR}(\Sigma)$ is the absolutely free algebra of formulae generated by Σ over Prop, whose universe we denote by $FOR(\Sigma)$.

Definition 2.2. A *Tarskian consequence relation* over a propositional language Σ is a relation $\vdash \subseteq \wp(FOR(\Sigma)) \times FOR(\Sigma)$ obeying the following conditions for all $A \in FOR(\Sigma)$ and for all $\Gamma, \Delta \subseteq FOR(\Sigma)$:

1. $\Gamma \vdash A$ if $A \in \Gamma$ (Reflexivity)
2. If $\Gamma \vdash A$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash A$ (Monotonicity)
3. If $\Delta \vdash A$ and $\Gamma \vdash B$ for every $B \in \Delta$, then $\Gamma \vdash A$ (Cut)

Additionally, a (Tarskian) consequence relation $\vdash$ is substitution-invariant whenever if $\Gamma \vdash A$, and σ is a substitution on $FOR(\Sigma)$, then $\{\sigma(B) \mid B \in \Gamma\} \vdash \sigma(A)$.

Definition 2.3. A *Tarskian logic* over a propositional language Σ is an ordered pair $\langle FOR(\Sigma), \vdash\rangle$, where $\vdash$ is a substitution-invariant Tarskian consequence relation.

Definition 2.4. For Σ a propositional language, a Σ-*matrix* is a structure $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$, such that $\langle \mathcal{V}, \mathcal{O}\rangle$ is an algebra of the same similarity type as Σ, with universe $\mathcal{V}$ and a set of operations $\mathcal{O}$, and $\mathcal{D} \subset \mathcal{V}$.

Notice, in the first place, that the set $\mathcal{O}$ includes for every n-ary connective $\diamond$ in the language Σ, a corresponding n-ary truth-function $f^{\diamond}_{\mathcal{M}} : \mathcal{V}^n \longrightarrow \mathcal{V}$. With regard to these, when context allows it, we will sometimes identify the connectives themselves (which are linguistic items), with their corresponding truth-functions in a given matrix. In the second place, notice that typically, when dealing with non-classical logics, the set $\mathcal{V}$ is taken to be a superset of $\{\mathbf{t}, \mathbf{f}\}$.

Definition 2.5. For $\mathcal{M}$ an Σ-matrix an $\mathcal{M}$-valuation v is an homomorphism from $FOR(\Sigma)$ to $\mathcal{V}$ respecting that for all $f^{\diamond}_{\mathcal{M}} \in \mathcal{O}$:

$$v(f^{\diamond}_{\mathcal{M}}(\mathbf{v}_1, \ldots, \mathbf{v}_n)) = f^{\diamond}_{\mathcal{M}}(v(\mathbf{v}_1), \ldots, v(\mathbf{v}_n))$$

for which we denote by $v[\Gamma]$ the set $\{v(B) \mid B \in \Gamma\}$, i.e. the image of v under Γ. When $\mathcal{V} = \{\mathbf{t}, \mathbf{f}\}$ we call the valuation Boolean or, alternatively, we refer to it as a *bivaluation*. The set of all $\mathcal{M}$-valuations is denoted by $V^{\mathcal{M}}$.

Definition 2.6. A logical matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$ induces a substitution-invariant consequence relation $\vDash_{\mathcal{M}}$ by letting

$$\Gamma \vDash_{\mathcal{M}} A \iff \text{for every valuation } v \in V^{\mathcal{M}}, \text{ if } v[\Gamma] \subseteq \mathcal{D}, \text{ then } v(A) \in \mathcal{D}$$

Whence, the pair $\langle FOR(\Sigma), \vDash_{\mathcal{M}}\rangle$, is a substitution-invariant Tarskian consequence relation, to which we might refer as the *matrix logic* $\mathbf{L}$ induced by or associated to $\mathcal{M}$.

Moreover, when some logic $\mathbf{L}$ is induced by a single matrix $\mathcal{M}$, we may interchangeably refer to $\vDash_{\mathcal{M}}$ as $\vDash_{\mathbf{L}}$ and replace talk of $\mathcal{M}$ by talk of $\mathbf{L}$ and viceversa.

Now, from the immense set of matrix logics, in this paper we decided to focus in a proper and peculiar subset, to which the Weak Kleene logics belong: that of the *infectious* logics. Intuitively, infectious logics are matrix logics where a compound formula receives a given truth-value if some of its components receives that truth-value, first.

Definition 2.7. We say that a matrix logic $\mathbf{L} = \langle FOR(\Sigma), \vDash_{\mathcal{M}} \rangle$ is *infectious* if and only if there is an element $\mathbf{x} \in \mathcal{V}$ such that for every n-ary operator $\diamond$ and for all $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathcal{V}$

$$\text{if } \mathbf{x} \in \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}, \text{ then } f^{\diamond}_{\mathcal{M}}(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \mathbf{x}$$

As advertised, from these infectious logics, we will be furthermore interested in *two* particular cases: the 3-valued (Paracomplete) Weak Kleene and Paraconsistent Weak Kleene logics. Therefore, we proceed to present these systems in their corresponding matrix settings.

3 The Logics

3.1 Weak Kleene Logics

Definition 3.1. The *internal* propositional language Σ^I is defined by:

$$\Sigma^I_0 = \emptyset \qquad \Sigma^I_1 = \{\neg\} \qquad \Sigma^I_2 = \{\wedge, \vee\} \qquad \Sigma^I_n = \emptyset, \text{ for each } n > 2$$

Definition 3.2. For $X \in \{\mathbf{K^w_3}, \mathbf{PWK}\}$, a 3-valued X-matrix is the following Σ^I-matrix, where where $\langle \{\mathbf{t}, \mathbf{u}, \mathbf{f}\}, \{f^{\neg}_{\mathbf{WK}}, f^{\wedge}_{\mathbf{WK}}, f^{\vee}_{\mathbf{WK}}\} \rangle$ is the 3-element Weak Kleene algebra

$$\mathcal{M}_{\mathbf{K^w_3}} = \langle \{\mathbf{t}, \mathbf{u}, \mathbf{f}\}, \{\mathbf{t}\}, \{f^{\neg}_{\mathbf{WK}}, f^{\wedge}_{\mathbf{WK}}, f^{\vee}_{\mathbf{WK}}\} \rangle$$
$$\mathcal{M}_{\mathbf{PWK}} = \langle \{\mathbf{t}, \mathbf{u}, \mathbf{f}\}, \{\mathbf{t}, \mathbf{u}\}, \{f^{\neg}_{\mathbf{WK}}, f^{\wedge}_{\mathbf{WK}}, f^{\vee}_{\mathbf{WK}}\} \rangle$$

Definition 3.3. For $X \in \{\mathbf{K^w_3}, \mathbf{PWK}\}$, the 3-valued logic $X = \langle FOR(\Sigma^I), \vDash_X \rangle$ is defined as follows, letting V^X be the set of $\mathcal{M}_X$-valuations

$$\Gamma \vDash_{\mathbf{K^w_3}} A \iff \text{for every valuation } v \in V^{\mathbf{K^w_3}}, \text{ if } v[\Gamma] \subseteq \{\mathbf{t}\}, \text{ then } v(A) \in \{\mathbf{t}\}$$

$$\Gamma \vDash_{\mathbf{PWK}} A \iff \text{for every valuation } v \in V^{\mathbf{PWK}}, \text{ if } v[\Gamma] \subseteq \{\mathbf{t}, \mathbf{u}\}, \text{ then } v(A) \in \{\mathbf{t}, \mathbf{u}\}$$

Along with $\mathbf{K^w_3}$ being a paracomplete and $\mathbf{PWK}$ being a paraconsistent logic, i.e. with it being the case that

$$B \nvDash_{\mathbf{K^w_3}} A \vee \neg A \qquad A \wedge \neg A \nvDash_{\mathbf{PWK}} B$$

it is also the case that these logics are subclassical in yet another more profound sense: they invalidate some characteristic classically valid inferences regarding, respectively, disjunction and conjunction. Namely,

$$A \nvDash_{\mathbf{K^w_3}} A \vee B \qquad A \wedge B \nvDash_{\mathbf{PWK}} B$$

which makes it fair pointing out that these logics count with rather *weak* disjunctions and conjunctions. In Section 5, we will discuss some conceptual or philosophical reasons for such failures, which will also be incarnated in our clauses for disjunction and conjunction in the two-valued presentations for $\mathbf{K^w_3}$ and $\mathbf{PWK}$ that we will advance later on.

3.2 Four-valued subsystems of Weak Kleene Logics

Some of the four-valued subsystems of Weak Kleene Logics that we are about to present—to be precise, those that are subsystems of Paracomplete Weak Kleene—were introduced many times in the literature, independently by many authors and at very different given times. More recently, they were discussed (together with many more systems) in [35].

Definition 3.4. The 4-element algebra $\mathbf{SWK_4}$ is defined as

$$\mathbf{SWK_4} = \langle \{\mathbf{t}, \mathbf{i}, \mathbf{u}, \mathbf{f}\}, \{f^{\neg}_{\mathbf{SWK_4}}, f^{\wedge}_{\mathbf{SWK_4}}, f^{\vee}_{\mathbf{SWK_4}}\}\rangle$$

where the functions $f^{\neg}_{\mathbf{SWK_4}}, f^{\wedge}_{\mathbf{SWK_4}}, f^{\vee}_{\mathbf{SWK_4}}$ are

	$f^{\neg}_{\mathbf{SWK_4}}$
$\mathbf{t}$	$\mathbf{f}$
$\mathbf{i}$	$\mathbf{i}$
$\mathbf{u}$	$\mathbf{u}$
$\mathbf{f}$	$\mathbf{t}$

$f^{\wedge}_{\mathbf{SWK_4}}$	$\mathbf{t}$	$\mathbf{i}$	$\mathbf{u}$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{i}$	$\mathbf{u}$	$\mathbf{f}$
$\mathbf{i}$	$\mathbf{i}$	$\mathbf{i}$	$\mathbf{u}$	$\mathbf{f}$
$\mathbf{u}$	$\mathbf{u}$	$\mathbf{u}$	$\mathbf{u}$	$\mathbf{u}$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{u}$	$\mathbf{f}$

$f^{\vee}_{\mathbf{SWK_4}}$	$\mathbf{t}$	$\mathbf{i}$	$\mathbf{u}$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{u}$	$\mathbf{t}$
$\mathbf{i}$	$\mathbf{t}$	$\mathbf{i}$	$\mathbf{u}$	$\mathbf{i}$
$\mathbf{u}$	$\mathbf{u}$	$\mathbf{u}$	$\mathbf{u}$	$\mathbf{u}$
$\mathbf{f}$	$\mathbf{t}$	$\mathbf{i}$	$\mathbf{u}$	$\mathbf{f}$

Definition 3.5. For $X \in \{\mathbf{S_{\texttt{fde}}}, \mathbf{dS_{\texttt{fde}}}\}$, a 4-valued X-matrix is the following Σ^I-matrix, where $\langle \{\mathbf{t}, \mathbf{i}, \mathbf{u}, \mathbf{f}\}, \{f^{\neg}_{\mathbf{SWK_4}}, f^{\wedge}_{\mathbf{SWK_4}}, f^{\vee}_{\mathbf{SWK_4}}\}\rangle$ is the 4-element algebra $\mathbf{SWK_4}$

$$\mathcal{M}_{\mathbf{S_{\texttt{fde}}}} = \langle \{\mathbf{t}, \mathbf{i}, \mathbf{u}, \mathbf{f}\}, \{\mathbf{t}, \mathbf{i}\}, \{f^{\neg}_{\mathbf{SWK_4}}, f^{\wedge}_{\mathbf{SWK_4}}, f^{\vee}_{\mathbf{SWK_4}}\}\rangle$$
$$\mathcal{M}_{\mathbf{dS_{\texttt{fde}}}} = \langle \{\mathbf{t}, \mathbf{i}, \mathbf{u}, \mathbf{f}\}, \{\mathbf{t}, \mathbf{u}\}, \{f^{\neg}_{\mathbf{SWK_4}}, f^{\wedge}_{\mathbf{SWK_4}}, f^{\vee}_{\mathbf{SWK_4}}\}\rangle$$

Definition 3.6. For $X \in \{\mathbf{S_{\texttt{fde}}}, \mathbf{dS_{\texttt{fde}}}\}$, the 4-valued logic $X = \langle FOR(\Sigma^I), \vDash_X\rangle$ is defined as follows, letting V^X be the set of $\mathcal{M}_X$-valuations

$$\Gamma \vDash_{\mathbf{S_{\texttt{fde}}}} A \iff \text{for every valuation } v \in V^{\mathbf{S_{\texttt{fde}}}}, \text{ if } v[\Gamma] \subseteq \{\mathbf{t}, \mathbf{i}\}, \text{ then } v(A) \in \{\mathbf{t}, \mathbf{i}\}$$

$$\Gamma \vDash_{\mathbf{dS_{\texttt{fde}}}} A \iff \text{for every valuation } v \in V^{\mathbf{dS_{\texttt{fde}}}}, \text{ if } v[\Gamma] \subseteq \{\mathbf{t}, \mathbf{u}\}, \text{ then } v(A) \in \{\mathbf{t}, \mathbf{u}\}$$

In particular, the logic $\mathbf{S_{\texttt{fde}}}$ was first discussed by Harry Deutsch in [15] as the first-degree fragment of his intensional logic $\mathbf{S}$, designed with the aim of modelling analytic inferences. This logic was later rediscovered by Melvin Fitting in [17] as a four-valued generalization of Paracomplete Weak Kleene Logic, motivated by Fitting's epistemic understanding of truth-value gaps and truth-value gluts as modelling cases in which qualified experts expressed no opinion (neither positive nor negative), and where they expressed an inconsistent opinion (both positive and negative) towards a certain issue. It was also rediscovered by Carlos Oller, who proposed it in [26] as a logic that solved some of the "paradoxes" of Parry's analytic implication, presented in [28]. Finally, it was recently reconsidered by Graham Priest in [29] as the logic induced by Priest's generalized plurivalent semantics, which allow sentences to receive more than one truth-value of a given set and, in this generalized setting, allow also sentences to receive no truth-value of a given set. If, along the previous lines, we think of $\mathbf{S_{\texttt{fde}}}$ as a logic counting with both truth-value gluts and truth-value gaps, such that the latter are infectious, while the former are not, then validity in this logic can be standardly understood as truth-preservation from premises to conclusion.

The logic $\mathbf{dS_{\texttt{fde}}}$ has not received such an extensive discussion in the specialized literature, although some works analyze it under the lens of both Fitting epistemic semantics and Priest's plurivalent logics, i.e. [35] and [27], respectively. Drawing the analogy with the previous case, we can think of $\mathbf{dS}_{\mathtt{fde}}$ as a logic having both truth-value gluts and truth-value gaps, such that the former are infectious, while the latter are not. In this vein, validity in this logic can also be standardly understood as truth-preservation from premises to conclusion. This is, in fact, the reading assumed throughout [14], were these systems are put to use in the analysis of semantic paradoxes.

Moreover, in [35], two additional subsystems of three-valued Weak Kleene Logics were discussed, the systems $\mathbf{S}^{\mathbf{w}}_{\mathtt{fde}}$ and $\mathbf{dS}^{\mathbf{w}}_{\mathtt{fde}}$.

Definition 3.7. The 4-element algebra $\mathbf{WK_4}$ is defined as

$$\mathbf{WK_4} = \langle \{\mathbf{t}, \mathbf{u}_1, \mathbf{u}_2, \mathbf{f}\}, \{f^{\neg}_{\mathbf{WK_4}}, f^{\wedge}_{\mathbf{WK_4}}, f^{\vee}_{\mathbf{WK_4}}\}\rangle$$

where the functions $f^{\neg}_{\mathbf{WK_4}}, f^{\wedge}_{\mathbf{WK_4}}, f^{\vee}_{\mathbf{WK_4}}$ are

	$f^{\neg}_{\mathbf{WK_4}}$
$\mathbf{t}$	$\mathbf{f}$
$\mathbf{u}_1$	$\mathbf{u}_1$
$\mathbf{u}_2$	$\mathbf{u}_2$
$\mathbf{f}$	$\mathbf{t}$

$f^{\wedge}_{\mathbf{WK_4}}$	$\mathbf{t}$	$\mathbf{u}_1$	$\mathbf{u}_2$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{u}_1$	$\mathbf{u}_2$	$\mathbf{f}$
$\mathbf{u}_1$	$\mathbf{u}_1$	$\mathbf{u}_1$	$\mathbf{u}_2$	$\mathbf{u}_1$
$\mathbf{u}_2$	$\mathbf{u}_2$	$\mathbf{u}_2$	$\mathbf{u}_2$	$\mathbf{u}_2$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{u}_1$	$\mathbf{u}_2$	$\mathbf{f}$

$f^{\vee}_{\mathbf{WK4}}$	$\mathbf{t}$	$\mathbf{u}_1$	$\mathbf{u}_2$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{u}_1$	$\mathbf{u}_2$	$\mathbf{t}$
$\mathbf{u}_1$	$\mathbf{u}_1$	$\mathbf{u}_1$	$\mathbf{u}_2$	$\mathbf{u}_1$
$\mathbf{u}_2$	$\mathbf{u}_2$	$\mathbf{u}_2$	$\mathbf{u}_2$	$\mathbf{u}_2$
$\mathbf{f}$	$\mathbf{t}$	$\mathbf{u}_1$	$\mathbf{u}_2$	$\mathbf{f}$

Definition 3.8. For $X \in \{\mathbf{S}^{\mathbf{w}}_{\mathtt{fde}}, \mathbf{dS}^{\mathbf{w}}_{\mathtt{fde}}\}$, a 4-valued X-matrix is the following Σ^I-matrix, where $\langle \{\mathbf{t}, \mathbf{u}_1, \mathbf{u}_2, \mathbf{f}\}, \{f^{\neg}_{\mathbf{WK_4}}, f^{\wedge}_{\mathbf{WK_4}}, f^{\vee}_{\mathbf{WK_4}}\}\rangle$ is the 4-element algebra $\mathbf{WK_4}$

$$\mathcal{M}_{\mathbf{S}^{\mathbf{w}}_{\mathtt{fde}}} = \langle \{\mathbf{t}, \mathbf{u}_1, \mathbf{u}_2, \mathbf{f}\}, \{\mathbf{t}, \mathbf{u}_1\}, \{f^{\neg}_{\mathbf{WK_4}}, f^{\wedge}_{\mathbf{WK_4}}, f^{\vee}_{\mathbf{WK_4}}\}\rangle$$
$$\mathcal{M}_{\mathbf{dS}^{\mathbf{w}}_{\mathtt{fde}}} = \langle \{\mathbf{t}, \mathbf{u}_1, \mathbf{u}_2, \mathbf{f}\}, \{\mathbf{t}, \mathbf{u}_2\}, \{f^{\neg}_{\mathbf{WK_4}}, f^{\wedge}_{\mathbf{WK_4}}, f^{\vee}_{\mathbf{WK_4}}\}\rangle$$

Definition 3.9. For $X \in \{\mathbf{S}^{\mathbf{w}}_{\mathtt{fde}}, \mathbf{dS}^{\mathbf{w}}_{\mathtt{fde}}\}$, the 4-valued logic $X = \langle FOR(\Sigma^I), \vDash_X\rangle$ is defined as follows, letting V^X be the set of $\mathcal{M}_X$-valuations

$$\Gamma \vDash_{\mathbf{S}^{\mathbf{w}}_{\mathtt{fde}}} A \iff \text{for every valuation } v \in V^{\mathbf{S}^{\mathbf{w}}_{\mathtt{fde}}}, \text{ if } v[\Gamma] \subseteq \{\mathbf{t}, \mathbf{u}_1\}, \text{ then } v(A) \in \{\mathbf{t}, \mathbf{u}_1\}$$

$$\Gamma \vDash_{\mathbf{dS}^{\mathbf{w}}_{\mathtt{fde}}} A \iff \text{for every valuation } v \in V^{\mathbf{dS}^{\mathbf{w}}_{\mathtt{fde}}}, \text{ if } v[\Gamma] \subseteq \{\mathbf{t}, \mathbf{u}_2\}, \text{ then } v(A) \in \{\mathbf{t}, \mathbf{u}_2\}$$

If validity is taken to be truth-preservation, then these logics can be interpreted as systems with infectious truth-value gaps and infectious truth-value gluts, such that in the case of $\mathbf{S}^{\mathbf{w}}_{\mathtt{fde}}$ gaps are "more infectious" than gluts, and in the case of $\mathbf{dS}^{\mathbf{w}}_{\mathtt{fde}}$ gluts are "more infectious" than gaps. This is, again, the route taken in [14].

To close this preliminary section, we provide a few examples of the inferential failures provoked by the peculiar behavior of the logical connectives in these Weak Kleene logics.

	$\mathbf{K}^{\mathbf{w}}_{\mathbf{3}}$	**PWK**	$\mathbf{S}_{\mathtt{fde}}$	$\mathbf{dS}_{\mathtt{fde}}$	$\mathbf{S}^{\mathbf{w}}_{\mathtt{fde}}$	$\mathbf{dS}^{\mathbf{w}}_{\mathtt{fde}}$
$B \vDash A \vee \neg A$	×	✓	×	×	×	×
$A \vDash A \vee B$	×	✓	×	✓	×	×
$A \wedge \neg A \vDash (A \wedge \neg A) \vee B$	✓	✓	×	✓	×	✓
$A \wedge \neg A \vDash B$	✓	×	×	×	×	×
$A \wedge B \vDash A$	✓	×	✓	×	×	×
$(A \vee \neg A) \wedge B \vDash A \vee \neg A$	✓	✓	✓	×	✓	×

4 Suszko's Thesis and Suszko's Reduction

In the last section we introduced a number of Weak Kleene logics characterized in terms of many-valued semantics. So, in this context, one could ask whether or not it is possible to give two-valued semantics for these logics, or if they are *irreducibly* many-valued.

More generally, one might be interested to know whether or not it is possible to give two-valued semantics for *any* many-valued logic whatsoever. The Polish logician Roman Suszko thought the answer to this last question is, indeed, positive. Thus, in [34] he proposed the so-called *Suszko's Thesis* stating, as Carnielli, Caleiro, Coniglio and Marcos say in [5], that "there are but two logical values, true and false". This thesis implies, as Tsuji puts it in [37], that "many-valued logics do not exist at all", meaning that they are not *genuinely* many-valued. In fact, in his original paper, Suszko argues against Łukasiewicz[4] claiming that he

> is the the chief perpetrator of a magnificent conceptual deceit lasting out in mathematical logic to the present day ([34, p. 377])

Roughly speaking, Suszko's idea is to distinguish between algebraic values and logical values. In this sense, when a logic is characterized by a many-valued matrix these values are playing a merely *algebraic* role. On the other hand, once we turn our attention to philosophical or ontological notions, there are only two *logical* values: truth and falsity, represented by the set of designated and undesignated values, respectively. Thus, he claims that algebraic valuations and logical valuations "are functions of quite different conceptual nature" ([34, p. 378]), given logical valuations relate formulas with truth or falsity, while algebraic valuations represent reference assignments.

Furthermore, this thesis is sustained with the so-called *Suszko Reduction*: every Tarskian many-valued logic can be characterized with a bivalent semantics. Intuitively speaking, the main idea of the reduction consists in dividing the set of semantic values between designated and non-designated ones, and so identifying the concept of designated with truth and non-designated with falsity.

Following this idea, Suszko claims that three-valued logics such as Łukasiewicz three-valued logic $\mathbf{Ł}_3$ have just two *logical* values, but can be characterized by means of 3-element algebras, i.e. structures with three *algebraic* elements. Although Suszko showed in [33] how to build a 2-valued semantic for $\mathbf{Ł}_3$, it is not the aim of this section to give a detailed proof of this reduction, but just an intuitive idea. In order to do so, however, we need to introduce the logic $\mathbf{Ł}_3$.

Definition 4.1. The propositional language $\Sigma^{\rightarrow}$ is defined by:

$$\Sigma^{\rightarrow}_0 = \emptyset \qquad \Sigma^{\rightarrow}_1 = \{\neg\} \qquad \Sigma^{\rightarrow}_2 = \{\wedge, \rightarrow\} \qquad \Sigma^{\rightarrow}_n = \emptyset, \text{ for each } n > 2$$

Definition 4.2. A 3-valued $\mathbf{Ł}_3$-matrix is the following $\Sigma^{\rightarrow}$-matrix

$$\mathcal{M}_{\mathbf{Ł}_3} = \langle \{\mathbf{t}, \mathbf{u}, \mathbf{f}\}, \{\mathbf{t}\}, \{f^{\neg}_{\mathbf{Ł}_3}, f^{\wedge}_{\mathbf{Ł}_3}, f^{\rightarrow}_{\mathbf{Ł}_3}\}\rangle$$

where the functions $f^{\neg}_{\mathbf{Ł}_3}, f^{\wedge}_{\mathbf{Ł}_3}, f^{\rightarrow}_{\mathbf{Ł}_3}$ are defined as follows:

[4] Let's recall that Łukasiewicz worked on many-valued logics, and, among others, proposed the very well known system $\mathbf{Ł}_3$ (cf. [22]), on which more below.

	$f^{\neg}_{\mathbf{\L}_3}$
t	**f**
u	**u**
f	**t**

$f^{\wedge}_{\mathbf{\L}_3}$	**t**	**u**	**f**
t	**t**	**u**	**f**
u	**u**	**u**	**f**
f	**f**	**f**	**f**

$f^{\rightarrow}_{\mathbf{\L}_3}$	**t**	**u**	**f**
t	**t**	**u**	**f**
u	**t**	**t**	**u**
f	**t**	**t**	**t**

Definition 4.3. The 3-valued logic $\mathbf{\L}_3 = \langle FOR(\mathcal{L}), \vDash_{\mathbf{\L}_3}\rangle$ is defined as follows, letting $V^{\mathbf{\L}_3}$ be the set of $\mathcal{M}_{\mathbf{\L}_3}$-valuations

$$\Gamma \vDash_{\mathbf{\L}_3} A \iff \text{for every valuation } v \in V^{\mathbf{\L}_3}, \text{ if } v[\Gamma] \subseteq \{\mathbf{t}\}, \text{ then } v(A) \in \{\mathbf{t}\}$$

So, once the logic is defined, let us informally present how to give a two-valued semantics for it.[5] The main idea is to divide the set of semantic values into designated and non-designated ones. This is done by means of a function t_v, transforming e.g. the 3-valued valuations $v \in V^{\mathbf{\L}_3}$ into bivaluations:

$$t_v(B) = \begin{cases} \mathbf{t} & \text{if } v(B) = \mathbf{t} \\ \mathbf{f} & \text{otherwise} \end{cases}$$

In the particular case of the logic $\mathbf{\L}_3$, the function $t_v : FOR(\Sigma^{\rightarrow}) \longrightarrow \{\mathbf{t}, \mathbf{f}\}$, can be defined by $t_v(B) = v(\neg(B \rightarrow \neg B))$, generalizing this in the intuitive way to $t_v[\Gamma]$ for $\Gamma \subseteq FOR(\Sigma^{\rightarrow})$. Thus, by looking at the tables it is straightforward to check that $\neg(B \rightarrow \neg B)$ always receives a classical value in every $v \in V^{\mathbf{\L}_3}$. Finally, it is easy to confirm that

$$\Gamma \vDash_{\mathbf{\L}_3} A \iff \text{for every valuation } v \in V^{\mathbf{\L}_3}, \text{ if } t_v[\Gamma] \subseteq \{\mathbf{t}\}, \text{ then } t_v(A) \in \{\mathbf{t}\}$$

In other words, as Suszko claims, $\mathbf{\L}_3$ "may be seen as a two-valued logic" [33, p. 87]. In the next section, we will see in details how this reduction can also be carried out for Weak Kleene logics and the Logics of Nonsense.[6]

Finally, let us remark that Suszko's Thesis and the corresponding reduction have a quite non-effective or non-algorithmic side to them, for as Caleiro, Carnielli, Coniglio and Marcos remark

> Suszko's Reduction does not give you any hint, in general, on *how* a 2-valued semantics could be determined by anything like a *finite recursive set of clauses*, even for the case of logics with finite-valued truth-functional semantics. [5, p. 4, our emphasis]

which motivates the authors to provide, in the cited work, an effective technique to carry out the reduction, for logics that are sufficiently expressive, i.e. that have means to "separate" truth-values appropriately.

[5]Here, we are not following the original notation found in [33], but an adaptation to more contemporary terminology.

[6]Let us mention that Malinowski ([23] and [24]) challenged this approach and proposes the so-called *q-consequence* relation. In these works, he shows that there exist q-consequences lacking two-valued semantics and being essentially three-valued. Later on Frankowski ([18]) and Wansing and Shramko ([32]) advanced their proposals of p-consequence relations and k-dimensional consequence relations, respectively, by means of which logics are said to be also genuinely many-valued. Interesting as these alternatives are, here we will focus on Suszko's proposal and we will put aside the arguments spelled out by Malinowski and others.

In what follows, though, we will not be pursuing a reduction for Weak Kleene logics and Logics of Nonsense based on the algorithmic method presented by Caleiro, Carnielli, Coniglio and Marcos, since we will base our reduction on the intuitive and direct reading of the connectives. However, we would like to highlight that, from a technical point of view, an application of such a method to the logics in questions would lead, unfailingly, to an equivalent outcome.

5 Suszko's Reduction applied

In this section we will provide two-valued semantics for the logics presented in Section 3 and we will discuss the properties of the obtained systems, focusing on the non-standard features of their connectives. So, first we will focus on Weak Kleene logics and then we will turn our attention to Logics of Nonsense, in the next section.

In order to define two-valued versions of the aforementioned logics, it will be important to consider bivaluations constrained by a number of clauses. These we enumerate in what follows. For what is worth, most of the denominations are taken from, or inspired by the ones appearing in, the reference text [6]. In the case of clause **(vNeg)'** is the converse of clause **(vNeg)** appearing in [6], but is not mentioned throughout that text. Finally, clauses **(vOr*)** and **(vAnd*)**, as well as the clauses **(vOr**)** and **(vAnd**)**, refer to the non-standard clauses for disjunction and conjunction in Weak Kleene Logics that are original to the present essay.

5.1 Two-valued Weak Kleene Logics

Let us consider (Paracomplete) Weak Kleene logic first. With regard to this system we have already seen, in previous sections, that its disjunction works in a pretty odd or strange way, by invalidating e.g. the inference usually regarded as $\vee$-Introduction: $A \vDash A \vee B$.

The most straightforward explanation for this is due to Bochvar himself. Bochvar considered that sentences could be one of *true*, *false* and *neither-true-nor-false* and—more importantly—thought that neither-true-nor-false sentences were *nonsensical*, thereby taking any sentence which had a neither-true-nor-false component to be classified in that way, too. In addition to that, he took valid inferences to be characterized by truth-preservation; in other words, an inference was taken to be valid by Bochvar if and only if it had a true conclusion whenever it had true premises.

Thus, it is e.g. Ferguson's understanding that disjunction in paracomplete infectious logics—such as Weak Kleene logic—has a *conjunctive* flavor. By this he means that, in Bochvar's case, a disjunction is true if and only if at least one of its disjuncts is true *and* none of its disjuncts is neither-true-nor-false, i.e. an admittedly conjunctive requirement that is stronger than the classical truth-condition for disjunction.

Now, if we take truth-value gaps (i.e. neither-true-nor-false-sentences) to be represented in a two-valued setting by sentences such that A and $\neg A$ are false, the Bochvar-Ferguson reading of disjunction in paracomplete infectious logics such as Weak Kleene becomes the following: a disjunction is true if and only if at least one of its disjuncts is true *and* each of its disjuncts is such that either it or its negation is true. It can be easily observed that this is clearly incarnated in the nonstandard clause **(vOr*)** for disjunction that we propose below.

Definition 5.1. A bivaluation $v : FOR(\Sigma^I) \longrightarrow \{\mathbf{t}, \mathbf{f}\}$ satisfying the following clauses is a ${}^2_*\mathbf{K_3^w}$ valuation:

$$\textbf{(vNeg)'} \qquad v(\neg A) = \mathbf{t} \quad \Longrightarrow \quad v(A) = \mathbf{f}$$

$$\textbf{(vAnd)} \qquad v(A \wedge B) = \mathbf{t} \quad \Longleftrightarrow \quad v(A) = \mathbf{t} \text{ and } v(B) = \mathbf{t}$$

$$\textbf{(vOr}^*\textbf{)} \qquad v(A \vee B) = \mathbf{t} \quad \Longleftrightarrow \quad \begin{cases} v(A) = \mathbf{t} \text{ or } v(B) = \mathbf{t}, & \text{and} \\ v(A) = \mathbf{t} \text{ or } v(\neg A) = \mathbf{t}, & \text{and} \\ v(B) = \mathbf{t} \text{ or } v(\neg B) = \mathbf{t} \end{cases}$$

The set of all such valuations is $V^{{}^2_*\mathbf{K_3^w}}$.

Definition 5.2. Let the two-valued logic ${}^2_*\mathbf{K_3^w} = \langle FOR(\Sigma^I), \vDash_{{}^2_*\mathbf{K_3^w}} \rangle$ be defined such that

$$\Gamma \vDash_{{}^2_*\mathbf{K_3^w}} A \Longleftrightarrow \text{for every valuation } v \in V^{{}^2_*\mathbf{K_3^w}}, \text{ if } v[\Gamma] \subseteq \{\mathbf{t}\}, \text{ then } v(A) \in \{\mathbf{t}\}$$

Let us consider Paraconsistent Weak Kleene logic now. With regard to this system we have already seen, in previous sections, that its conjunction works in a pretty odd or strange way too, by invalidating e.g. the inference usually regarded as $\wedge$-Elimination: $A \wedge B \vDash A$.

The most straightforward explanation for this is due to Halldén himself. Halldén considered that sentences could be one of *true*, *false* and *neither-true-nor-false*, just like Bochvar, and—more importantly—thought that neither-true-nor-false sentences were *nonsensical*, thereby taking any sentence which had a neither-true-nor-false component to be classified in that way, too. Furthermore (here lies the difference with Bochvar) he took valid inferences to be characterized by non-falsity-preservation; in other words, an inference was taken to be valid by Halldén if and only if it had a non-false conclusion whenever it had non-false premises.

Some philosophers think, though, that *analetheism*, i.e. the idea of non-falsity preservation as a motivation for paraconsistent logics (as is the case with Halldén's motivation for his logic of nonsense, of which Paraconsistent Weak Kleene is a fragment) is ill-conceived, because it is in conflict with the idea that truth is the aim of assertion—for more on this, see the recent debate in [2] and [1]. In line with this, then, we might ask if it is possible to construct an interpretation of Paraconsistent Weak Kleene that treats the non-classical infectious value as *both-true-and-false*. The answer is that this is actually possible and, moreover, easy to do—as Roberto Ciuni shows in [9].

Thus, it is e.g. Ciuni's understanding in [9] that conjunction in paraconsistent infectious logics—such as Paraconsistent Weak Kleene logic—has a *disjunctive* flavor. By this he means that a conjunction is true if and only if both of its conjuncts are true *or* some of its disjuncts is both-true-nor-false, i.e. an admittedly disjunctive requirement that is weaker than the classical truth-condition for conjunction.

Now, if we take truth-value gluts (i.e. both-true-and-false-sentences) to be represented in a two-valued setting by sentences such that A and $\neg A$ are true, the Ciuni's reading of conjunction in paraconsistent infectious logics such as Paraconsistent Weak Kleene becomes the following: a conjunction is true if and only if both of its conjuncts are true *or* some of its conjuncts is such that both it and its negation are true. It can be easily observed that this is clearly incarnated in the nonstandard clause **(vAnd*)** for conjunction that we propose below.

Definition 5.3. A bivaluation $v : FOR(\Sigma^I) \longrightarrow \{\mathbf{t}, \mathbf{f}\}$ satisfying the following clauses is a ${}^2_*\mathbf{PWK}$ valuation:

$$\textbf{(vNeg)} \qquad v(\neg A) = \mathbf{f} \quad \Longrightarrow \quad v(A) = \mathbf{t}$$

$$\textbf{(vOr)} \qquad v(A \vee B) = \mathbf{t} \quad \Longleftrightarrow \quad v(A) = \mathbf{t} \text{ or } v(B) = \mathbf{t}$$

$$\textbf{(vAnd}^*\textbf{)} \qquad v(A \wedge B) = \mathbf{t} \quad \Longleftrightarrow \quad \begin{cases} v(A) = \mathbf{t} \text{ and } v(B) = \mathbf{t}, & \text{or} \\ v(A) = \mathbf{t} \text{ and } v(\neg A) = \mathbf{t}, & \text{or} \\ v(B) = \mathbf{t} \text{ and } v(\neg B) = \mathbf{t} \end{cases}$$

The set of all such valuations is $V^{{}^2_*\mathbf{PWK}}$.

Definition 5.4. Let the two-valued logic ${}^2_*\mathbf{PWK} = \langle FOR(\Sigma^I), \vDash_{{}^2_*\mathbf{PWK}} \rangle$ be defined such that

$$\Gamma \vDash_{{}^2_*\mathbf{PWK}} A \Longleftrightarrow \text{for every valuation } v \in V^{{}^2_*\mathbf{PWK}}, \text{ if } v[\Gamma] \subseteq \{\mathbf{t}\}, \text{ then } v(A) \in \{\mathbf{t}\}$$

Are these, then, the two-valued versions of the Weak Kleene logics that we are aiming at? Not yet. As is noticed by Lucas Rosenblatt in [30], valuations constructed in these or similar ways do not respect Involutivity for negation, or any of the De Morgan properties for conjunction and disjunction.

Fact 5.5. *There are bivaluations v in $V^{{}^2_*\mathbf{K_3^w}}$ and $V^{{}^2_*\mathbf{PWK}}$ such that*

$$\begin{aligned} v(\neg\neg A) &\neq v(A) \\ v(\neg(A \wedge B)) &\neq v(\neg A \vee \neg B) \\ v(\neg(A \vee B)) &\neq v(\neg A \wedge \neg B) \end{aligned}$$

Proof. We prove the three cases separately.

For $v(\neg\neg A) \neq v(A)$:

- Let $v \in V^{{}^2_*\mathbf{K_3^w}}$ and let $v(A) = \mathbf{f}$. This allows $v(\neg A) = \mathbf{f}$ and, finally $v(\neg\neg A) = \mathbf{t}$.
- Let $v \in V^{{}^2_*\mathbf{PWK}}$ and let $v(A) = \mathbf{t}$. This allows $v(\neg A) = \mathbf{t}$ and, finally $v(\neg\neg A) = \mathbf{f}$.

For $v(\neg(A \wedge B)) \neq v(\neg A \vee \neg B)$:

- Let $v \in V^{{}^2_*\mathbf{K_3^w}}$ and let $v(A) = \mathbf{f}$, $v(B) = \mathbf{t}$. This allows $v(\neg A) = \mathbf{t}$, $v(\neg B) = \mathbf{f}$, $v(A \wedge B) = \mathbf{f}$ and, finally by **(vNeg)'**, $v(\neg(A \wedge B)) = \mathbf{f}$. In addition, it can be established by **(vOr*)**, that $v(\neg A \vee \neg B) = \mathbf{t}$.
- Let $v \in V^{{}^2_*\mathbf{PWK}}$ and let $v(A) = \mathbf{f}$, $v(B) = \mathbf{t}$. This allows to establish by **(vNeg)** that $v(\neg A) = \mathbf{t}$ and $v(\neg B) = \mathbf{t}$. Moreover it can be established by **(vAnd*)** that $v(A \wedge B) = \mathbf{t}$ and, finally by **(vNeg)'**, that $v(\neg(A \wedge B)) = \mathbf{f}$. In addition, it can be shown by **(vOr)** that $v(\neg A \vee \neg B) = \mathbf{t}$.

For $v(\neg(A \vee B)) \neq v(\neg A \wedge \neg B)$:

- Let $v \in V_*^{2\mathbf{K_3^w}}$ and let $v(A) = \mathbf{f}$, $v(B) = \mathbf{t}$. This allows to establish by **(vNeg)'** that $v(\neg A) = \mathbf{f}$ and $v(\neg B) = \mathbf{f}$. Moreover it can be established by **(vOr*)** that $v(A \vee B) = \mathbf{f}$ and, finally by **(vNeg)'**, that $v(\neg(A \vee B)) = \mathbf{t}$. In addition, it can be shown by **(vAnd)** that $v(\neg A \wedge \neg B) = \mathbf{f}$.

- Let $v \in V_*^{2\mathbf{PWK}}$ and let $v(A) = \mathbf{f}$, $v(B) = \mathbf{t}$. This allows $v(\neg A) = \mathbf{t}$, $v(\neg B) = \mathbf{f}$, $v(A \vee B) = \mathbf{t}$ and, finally by **(vNeg)**, that $v(\neg(A \vee B)) = \mathbf{t}$. In addition, it can be shown by **(vAnd*)** that $v(\neg A \wedge \neg B) = \mathbf{f}$. □

This allows us to establish that the previously defined two-valued logics are indeed *proper subsystems* of the 3-valued target systems.

Fact 5.6. ${}^2_*\mathbf{K_3^w} \subsetneq \mathbf{K_3^w}$

Proof. Assume that $\Gamma \nvDash_{\mathbf{K_3^w}} A$ and let a $\mathbf{K_3^w}$-valuation v such that $v[\Gamma] \subseteq \{\mathbf{t}\}$ and $v(A) \notin \{\mathbf{t}\}$ witness this fact. Construct a valuation v^* as follows:

- If $v(\phi) = \mathbf{u}$, then $v^*(\phi) = \mathbf{f}$
- If $v(\phi) \neq \mathbf{u}$, then $v^*(\phi) = v(\phi)$

for every propositional variable $\phi \in \mathsf{Prop}$. By a straightforward induction it can be shown that v^* is a ${}^2_*\mathbf{K_3^w}$-valuation such that $v^*[\Gamma] \subseteq \{\mathbf{t}\}$ and $v^*(A) \notin \{\mathbf{t}\}$, witnessing the fact that $\Gamma \nvDash_{{}^2_*\mathbf{K_3^w}} A$. Thus, ${}^2_*\mathbf{K_3^w} \subseteq \mathbf{K_3^w}$.

Now, that ${}^2_*\mathbf{K_3^w}$ is a *proper* subsystem of $\mathbf{K_3^w}$ can be established by, first, noticing that[7] the inference forms of Double Negation and the De Morgan laws, i.e.

$$\neg\neg A \quad \Dashv\vDash \quad A$$

$$\neg(A \wedge B) \quad \Dashv\vDash \quad \neg A \vee \neg B$$

$$\neg(A \vee B) \quad \Dashv\vDash \quad \neg A \wedge \neg B$$

hold in $\mathbf{K_3^w}$, whereas they *do not hold* in ${}^2_*\mathbf{K_3^w}$, as can be shown by using the valuations cited in Fact 5.5 above. □

Fact 5.7. ${}^2_*\mathbf{PWK} \subsetneq \mathbf{PWK}$

Proof. Similar to the proof of Fact 5.6. □

Thus, given the above, it is reasonable to think that restricting the above set of two-valued valuations to those which respect the following De Morgan properties will suffice to render two-valued versions of the Weak Kleene Logics.

(vNeg)” $\quad v(\neg\neg A) = v(A)$

(vDM$_\wedge$) $\quad v(\neg(A \wedge B)) = v(\neg A \vee \neg B)$

(vDM$_\vee$) $\quad v(\neg(A \vee B)) = v(\neg A \wedge \neg B)$

[7] We express the conjoined fact that $C \vDash D$ and $D \vDash C$ symbolizing it as $C \Dashv\vDash D$.

That this is, indeed, the case, is verified by the proofs below.

Definition 5.8. We denote by $V^{{}^2\mathbf{K_3^w}}$ and $V^{{}^2\mathbf{PWK}}$ the proper subset of, respectively, $V_*^{{}^2\mathbf{K_3^w}}$ and $V_*^{{}^2\mathbf{PWK}}$ valuations, that satisfy—additionally—the De Morgan clauses **(vNeg)”**, **(vDM$_\wedge$)** and **(vDM$_\vee$)**.

Definition 5.9. Let the two-valued logic ${}^2\mathbf{K_3^w} = \langle FOR(\Sigma^I), \vDash_{{}^2\mathbf{K_3^w}}\rangle$ be defined such that

$$\Gamma \vDash_{{}^2\mathbf{K_3^w}} A \iff \text{for every valuation } v \in V^{{}^2\mathbf{K_3^w}}, \text{ if } v[\Gamma] \subseteq \{\mathbf{t}\}, \text{ then } v(A) \in \{\mathbf{t}\}$$

Definition 5.10. Let the two-valued logic ${}^2\mathbf{PWK} = \langle FOR(\Sigma^I), \vDash_{{}^2\mathbf{PWK}}\rangle$ be defined such that

$$\Gamma \vDash_{{}^2\mathbf{PWK}} A \iff \text{for every valuation } v \in V^{{}^2\mathbf{PWK}}, \text{ if } v[\Gamma] \subseteq \{\mathbf{t}\}, \text{ then } v(A) \in \{\mathbf{t}\}$$

Fact 5.11. ${}^2\mathbf{K_3^w} = \mathbf{K_3^w}$

Proof. The proof of the left to right direction is essentially as in the proof of Fact 5.6. The proof of the right to left direction goes as follows. Assume $\Gamma \nvDash_{{}^2\mathbf{K_3^w}} A$. Thus, there is a ${}^2\mathbf{K_3^w}$-valuation v such that $v[\Gamma] \subseteq \{\mathbf{t}\}$ and $v(A) \notin \{\mathbf{t}\}$ witness this fact. Construct a valuation v^* as follows, letting ϕ be any *atomic* formula:

- If $v(\phi) = v(\neg\phi)$, then $v^*(\phi) = \mathbf{u}$
- If $v(\phi) \neq v(\neg\phi)$, then $v^*(\phi) = v(\phi)$

Again, by a straightforward induction it can be observed that v^* is a $\mathbf{K_3^w}$-valuation such that for *every formulae* B:

- If $v(B) = \mathbf{t}$, then $v^*(B) = \mathbf{t}$
- If $v(B) = \mathbf{f}$, then either $v^*(B) = \mathbf{u}$ or $v^*(B) = \mathbf{f}$

Moreover, it is a $\mathbf{K_3^w}$-valuation such that $v^*[\Gamma] \subseteq \{\mathbf{t}\}$ and $v^*(A) \notin \{\mathbf{t}\}$, thereby witnessing the fact that $\Gamma \nvDash_{\mathbf{K_3^w}} A$. □

Fact 5.12. ${}^2\mathbf{PWK} = \mathbf{PWK}$

Proof. Similar to the proof of Fact 5.11, with the only difference that in the above proof we need to change the clauses for the construction of v^* in the obvious way, and we get that:

- If $v(B) = \mathbf{f}$, then $v^*(B) = \mathbf{f}$
- If $v(B) = \mathbf{t}$, then either $v^*(B) = \mathbf{u}$ or $v^*(B) = \mathbf{t}$ □

5.2 Two-valued subsystems of Weak Kleene Logics

In this subsection, we will provide two-valued semantics for the four-valued subsystems of Weak Kleene Logics above presented: $\mathbf{S}_{\mathtt{fde}}, \mathbf{dS}_{\mathtt{fde}}, \mathbf{S}^{\mathbf{w}}_{\mathtt{fde}}$ and $\mathbf{dS}^{\mathbf{w}}_{\mathtt{fde}}$. We begin with the first two.

Definition 5.13. A bivaluation $v : FOR(\Sigma^I) \longrightarrow \{\mathbf{t}, \mathbf{f}\}$ satisfying the clauses **(vOr*****)**, **(vAnd)**, **(vDM**$_\wedge$**)**, **(vDM**$_\vee$**)** and **(vNeg)"** is a $^2\mathbf{S}_{\mathtt{fde}}$ valuation.

The set of all such valuations is $V^{^2\mathbf{S}_{\mathtt{fde}}}$.

Definition 5.14. Let the two-valued logic $^2\mathbf{S}_{\mathtt{fde}} = \langle FOR(\Sigma^I), \vDash_{^2\mathbf{S}_{\mathtt{fde}}} \rangle$ be defined such that

$$\Gamma \vDash_{^2\mathbf{S}_{\mathtt{fde}}} A \iff \text{for every valuation } v \in V^{^2\mathbf{S}_{\mathtt{fde}}}, \text{ if } v[\Gamma] \subseteq \{\mathbf{t}\}, \text{ then } v(A) \in \{\mathbf{t}\}$$

Definition 5.15. A bivaluation $v : FOR(\Sigma^I) \longrightarrow \{\mathbf{t}, \mathbf{f}\}$ satisfying the clauses **(vOr)**, **(vAnd*****)**, **(vDM**$_\wedge$**)**, **(vDM**$_\vee$**)** and **(vNeg)"** is a $^2\mathbf{dS}_{\mathtt{fde}}$ valuation.

The set of all such valuations is $V^{^2\mathbf{dS}_{\mathtt{fde}}}$.

Definition 5.16. Let the two-valued logic $^2\mathbf{dS}_{\mathtt{fde}} = \langle FOR(\Sigma^I), \vDash_{^2\mathbf{dS}_{\mathtt{fde}}} \rangle$ be defined such that

$$\Gamma \vDash_{^2\mathbf{dS}_{\mathtt{fde}}} A \iff \text{for every valuation } v \in V^{^2\mathbf{dS}_{\mathtt{fde}}}, \text{ if } v[\Gamma] \subseteq \{\mathbf{t}\}, \text{ then } v(A) \in \{\mathbf{t}\}$$

It's worth mentioning that the clauses for the valuations of $^2\mathbf{S}_{\mathtt{fde}}$ are a proper subset of the clauses given for $^2\mathbf{K}^{\mathbf{w}}_{\mathbf{3}}$. In fact, from a two-valued perspective, the only difference between these two logics has to do with negation: whereas the valuations of $^2\mathbf{K}^{\mathbf{w}}_{\mathbf{3}}$ are restricted by the clause **(vNeg)**, $^2\mathbf{S}_{\mathtt{fde}}$ lacks such a restriction. This is due to the fact that sentences in $\mathbf{S}_{\mathtt{fde}}$ are allowed to be such that both them and their negations are true or, alternatively, designated. This shall be modeled in the two-valued semantics for it, i.e. in the system $^2\mathbf{S}_{\mathtt{fde}}$, by lifting the constraints on negated formulae imposed by clause **(vNeg)**.

Dually, the clauses for the valuations of $^2\mathbf{dS}_{\mathtt{fde}}$ are a proper subset of the clauses given for $^2\mathbf{PWK}$, and, from a two-valued perspective, whereas the valuations of $^2\mathbf{PWK}$ are restricted by the clause **(vNeg)'**, $^2\mathbf{dS}_{\mathtt{fde}}$ lacks a similar restriction. Again, this is due to the fact that sentences in $\mathbf{dS}_{\mathtt{fde}}$ are allowed to be such that both them and their negations are false or, alternatively, undesignated. This shall be modeled in the two-valued semantics for it, i.e. in the system $^2\mathbf{dS}_{\mathtt{fde}}$, by lifting the constraints on negated formulae imposed by clause **(vNeg)'**.

Therefore, all what we claimed about the behaviour of disjunction in the two-valued presentation of $\mathbf{K}^{\mathbf{w}}_{\mathbf{3}}$, and about conjunction in the two-valued presentation of $\mathbf{PWK}$ can be directly extrapolated to the properties of these connectives in $^2\mathbf{S}_{\mathtt{fde}}$ and in $^2\mathbf{dS}_{\mathtt{fde}}$, respectively.

As expressed in the following two facts, the two-valued logics we just built are equivalent to the systems induced by the corresponding four-valued presentation.

Fact 5.17. $^2\mathbf{S}_{\mathtt{fde}} = \mathbf{S}_{\mathtt{fde}}$.

Proof. To prove the left to right direction, suppose that $\Gamma \nvDash_{\mathbf{S}_{\mathtt{fde}}} A$. Therefore, there must be a $\mathbf{S}_{\mathtt{fde}}$-valuation, v, such that $v[\Gamma] \subseteq \{\mathbf{t}, \mathbf{i}\}$, but $v(A) \notin \{\mathbf{t}, \mathbf{i}\}$, witnessing this fact. So, let's build a valuation v^*, such that for each propositional letter ϕ:

- If $v(\phi) \in \{\mathbf{t}, \mathbf{i}\}$ then $v^*(\phi) = \mathbf{t}$.
- If $v(\phi) \notin \{\mathbf{t}, \mathbf{i}\}$ then $v^*(\phi) = \mathbf{f}$.

By a straightforward induction it can be shown that v^* is a ${}^2\mathbf{S}_{\mathtt{fde}}$-valuation, such that $v^*[\Gamma] \subseteq \{\mathbf{t}\}$, but $v^*(A) \notin \{\mathbf{t}\}$, and therefore $\Gamma \nvDash_{{}^2\mathbf{S}_{\mathtt{fde}}} A$.

To prove the right to left direction, suppose that $\Gamma \nvDash_{{}^2\mathbf{S}_{\mathtt{fde}}} A$. Therefore, there must be a ${}^2\mathbf{S}_{\mathtt{fde}}$-valuation, v, such that $v[\Gamma] \subseteq \{\mathbf{t}\}$, but $v(A) \notin \{\mathbf{t}\}$, witnessing this fact. Thus, let's build a valuation v^*, such that for each propositional letter ϕ:

- If $v(\phi) = \mathbf{t}$ and $v(\phi) = v(\neq \phi)$ then $v^*(\phi) = \mathbf{i}$.
- If $v(\phi) = \mathbf{f}$ and $v(\phi) = v(\neq \phi)$ then $v^*(\phi) = \mathbf{u}$.
- Otherwise, $v(\phi) = v^*(\phi)$.

Again, by a typical induction, it's straightforward to check that for every formula B,

- If $v(B) = \mathbf{t}$ then $v^*(B) \in \{\mathbf{i}, \mathbf{t}\}$.
- If $v(B) = \mathbf{f}$ and $v^*(B) \in \{\mathbf{u}, \mathbf{f}\}$.

It's worth noting that the only problematic case in the induction is related with disjunction, since the infectious value $\mathbf{u}$ is not designated. In other words, given two formulae B and C, it's not sufficient that $v^*(B) \in \{\mathbf{i}, \mathbf{t}\}$ or $v^*(C)\{\mathbf{i}, \mathbf{t}\}$ in order to guarantee that $v^*(B \vee C) \in \{\mathbf{i}, \mathbf{t}\}$, since if, for instance, $v^*(B) = \mathbf{u}$ then $v(B \vee C) = \mathbf{u}$, no matter if $v^*(C) \in \{\mathbf{i}, \mathbf{t}\}$. This is solved by clause **(vOr*)**, which explicitly gets rid of this case.

Therefore, it can be easily shown that v^* is a $\mathbf{S}_{\mathtt{fde}}$-valuation, such that $v^*[\Gamma] \subseteq \{\mathbf{i}, \mathbf{t}\}$, but $v^*(A) \notin \{\mathbf{i}, \mathbf{t}\}$, and thus $\Gamma \nvDash_{\mathbf{S}_{\mathtt{fde}}} A$.

$\square$

Fact 5.18. ${}^2\mathbf{dS}_{\mathtt{fde}} = \mathbf{dS}_{\mathtt{fde}}$.

Proof. The proof is similar to the proof of Fact 5.17. The main difference is that, in order to proof $\mathbf{dS}_{\mathtt{fde}} \subseteq\ {}^2\mathbf{dS}_{\mathtt{fde}}$, the problematic case in the induction is related with conjunction, and not with disjunction. However, in the same way as in the previous proof that was solved by clause **(vOr*)**, in this case it's solved by clause **(vAnd*)**. $\square$

So far, both of the two-valued versions ${}^2\mathbf{S}_{\mathtt{fde}}$ and ${}^2\mathbf{dS}_{\mathtt{fde}}$ of the logics $\mathbf{S}_{\mathtt{fde}}$ and $\mathbf{dS}_{\mathtt{fde}}$ were built just deleting one of the clauses of ${}^2\mathbf{K}_3^{\mathbf{w}}$ and ${}^2\mathbf{PWK}$, respectively. However, things are more complicated in the case of the logics $\mathbf{S}^{\mathbf{w}}_{\mathtt{fde}}$ and $\mathbf{dS}^{\mathbf{w}}_{\mathtt{fde}}$.

As we will see next, we need to make further restrictions in the clauses of the connectives. This is not surprising at all, since both of them are much weaker subsystems of the usual Weak Kleene Logics. We provide next the technical definitions and later discuss the extent to which they are philosophically justified.

Definition 5.19. A bivaluation $v : FOR(\Sigma^I) \longrightarrow \{\mathbf{t}, \mathbf{f}\}$ satisfying **(vOr*)**, **(vDM$_\wedge$)**, **(vDM$_\vee$)**, **(vNeg)"** plus the following clause is a ${}^2\mathbf{S}^{\mathbf{w}}_{\mathtt{fde}}$ valuation:

$$
\textbf{(vAnd}^{**}\textbf{)} \quad v(A \wedge B) = \mathbf{t} \quad \Longleftrightarrow \quad \begin{cases} (v(A) = \mathbf{t} \text{ and } v(B) = \mathbf{t}, & \text{or} \\ v(A) = \mathbf{t} \text{ and } v(\neg A) = \mathbf{t}, & \text{or} \\ v(B) = \mathbf{t} \text{ and } v(\neg B) = \mathbf{t}) & \\ & \\ \qquad\quad \text{and} & \\ & \\ (v(A) = \mathbf{t} \text{ or } v(\neg A) = \mathbf{t}, & \text{and} \\ v(B) = \mathbf{t} \text{ or } v(\neg B) = \mathbf{t}) & \end{cases}
$$

The set of all such valuations is $V^{{}^2\mathbf{S}^{\mathbf{w}}_{\mathtt{fde}}}$.

Definition 5.20. Let the two-valued logic ${}^2\mathbf{S}^{\mathbf{w}}_{\mathtt{fde}} = \langle FOR(\Sigma^I), \vDash_{{}^2\mathbf{S}^{\mathbf{w}}_{\mathtt{fde}}} \rangle$ be defined such that

$$
\Gamma \vDash_{{}^2\mathbf{S}^{\mathbf{w}}_{\mathtt{fde}}} A \Longleftrightarrow \text{for every valuation } v \in V^{{}^2\mathbf{S}^{\mathbf{w}}_{\mathtt{fde}}}, \text{ if } v[\Gamma] \subseteq \{\mathbf{t}\}, \text{ then } v(A) \in \{\mathbf{t}\}
$$

Definition 5.21. A bivaluation $v : FOR(\Sigma^I) \longrightarrow \{\mathbf{t}, \mathbf{f}\}$ satisfying the following clauses **(vAnd)***, **(vDM$_\wedge$)**, **(vDM$_\vee$)**, **(vNeg)"** plus the following clause is a ${}^2\mathbf{dS}^{\mathbf{w}}_{\mathtt{fde}}$ valuation.

$$
\textbf{(vOr}^{**}\textbf{)} \quad v(A \vee B) = \mathbf{t} \quad \Longleftrightarrow \quad \begin{cases} (v(A) = \mathbf{t} \text{ or } v(B) = \mathbf{t}, & \text{and} \\ v(A) = \mathbf{t} \text{ or } v(\neg A) = \mathbf{t}, & \text{and} \\ v(B) = \mathbf{t} \text{ or } v(\neg B) = \mathbf{t}) & \\ & \\ \qquad\quad \text{or} & \\ & \\ (v(A) = \mathbf{t} \text{ and } v(\neg A) = \mathbf{t}, & \text{or} \\ v(B) = \mathbf{t} \text{ and } v(\neg B) = \mathbf{t}) & \end{cases}
$$

The set of all such valuations is $V^{{}^2\mathbf{dS}^{\mathbf{w}}_{\mathtt{fde}}}$.

Definition 5.22. Let the two-valued logic ${}^2\mathbf{dS}^{\mathbf{w}}_{\mathtt{fde}} = \langle FOR(\Sigma^I), \vDash_{{}^2\mathbf{dS}^{\mathbf{w}}_{\mathtt{fde}}} \rangle$ be defined such that

$$
\Gamma \vDash_{{}^2\mathbf{dS}^{\mathbf{w}}_{\mathtt{fde}}} A \Longleftrightarrow \text{for every valuation } v \in V^{{}^2\mathbf{dS}^{\mathbf{w}}_{\mathtt{fde}}}, \text{ if } v[\Gamma] \subseteq \{\mathbf{t}\}, \text{ then } v(A) \in \{\mathbf{t}\}
$$

The rather cumbersome clauses **(vAnd**)** and **(vOr**)** are intended to model in a two-valued setting the peculiar behavior of conjunction and disjunction in the logics $\mathbf{S}^{\mathbf{w}}_{\mathtt{fde}}$ and $\mathbf{dS}^{\mathbf{w}}_{\mathtt{fde}}$, respectively. To see this, we should keep in mind the interpretation of $\mathbf{S}^{\mathbf{w}}_{\mathtt{fde}}$ and $\mathbf{dS}^{\mathbf{w}}_{\mathtt{fde}}$ as logics counting with infectious truth-value gluts and infectious truth-value gaps, such that gaps are more infectious than gluts in $\mathbf{S}^{\mathbf{w}}_{\mathtt{fde}}$, while gluts are more infectious than gaps in $\mathbf{dS}^{\mathbf{w}}_{\mathtt{fde}}$.

Thus given this reading, on the one hand, clause **(vAnd**)** of ${}^2\mathbf{S}^{\mathbf{w}}_{\mathtt{fde}}$ represents the fact that a conjunction is true in $\mathbf{S}^{\mathbf{w}}_{\mathtt{fde}}$ if either both conjuncts are true (the classical case), or one of the conjuncts is a truth-value glut (the **PWK** case) but the remaining conjunct is not a truth-value gap. The corresponding two-valued representation of this, taking truth-value gluts to be true sentences whose negations are also true, and taking truth-value gaps to be false sentences whose negations are also false, is clearly the target clause **(vAnd**)**.

On the other hand, given this reading, clause **(vOr**)** of $^2\mathbf{dS}^{\mathbf{w}}_{\mathtt{fde}}$ represents the fact that a disjunction is true in $\mathbf{dS}^{\mathbf{w}}_{\mathtt{fde}}$ if either disjunct is true (the classical case) and none of the disjuncts is a truth-value gap (the $\mathbf{K}^{\mathbf{w}}_{\mathbf{3}}$ case), or alternatively if one of the disjuncts is a truth-value glut. The corresponding two-valued representation of this, is clearly the target clause **(vOr**)**.

Again, as expected, in the following facts we show the equivalence between the two-valued logics and the corresponding four-valued presentation.

Fact 5.23. $^2\mathbf{S}^{\mathbf{w}}_{\mathtt{fde}} = \mathbf{S}^{\mathbf{w}}_{\mathtt{fde}}$.

Proof. To prove the left to right direction, suppose that $\Gamma \nvDash_{\mathbf{S}^{\mathbf{w}}_{\mathtt{fde}}} A$. Hence, there must be a $\mathbf{S}^{\mathbf{w}}_{\mathtt{fde}}$-valuation, v, such that $v[\Gamma] \subseteq \{\mathbf{t}, \mathbf{u}_1\}$, but $v(A) \notin \{\mathbf{t}, \mathbf{u}_1\}$, witnessing this fact. So, let's build a valuation v^*, such that for each propositional letter ϕ:

- If $v(\phi) \in \{\mathbf{t}, \mathbf{u}_1\}$ then $v^*(\phi) = \mathbf{t}$.
- If $v(\phi) \notin \{\mathbf{t}, \mathbf{u}_1\}$ then $v^*(\phi) = \mathbf{f}$.

By a straightforward induction it can be shown that v^* is a $^2\mathbf{S}^{\mathbf{w}}_{\mathtt{fde}}$-valuation, such that $v^*[\Gamma] \subseteq \{\mathbf{t}\}$, but $v^*(A) \notin \{\mathbf{t}\}$, and therefore $\Gamma \nvDash_{^2\mathbf{S}^{\mathbf{w}}_{\mathtt{fde}}} A$.

To prove the right to left direction, suppose that $\Gamma \nvDash_{^2\mathbf{S}^{\mathbf{w}}_{\mathtt{fde}}} A$. Therefore, there must be a $^2\mathbf{S}^{\mathbf{w}}_{\mathtt{fde}}$-valuation, v, such that $v[\Gamma] \subseteq \{\mathbf{t}\}$, but $v(A) \notin \{\mathbf{t}\}$, witnessing this fact. Thus, let's build a valuation v^*, such that for each propositional letter ϕ:

- If $v(\phi) = \mathbf{t}$ and $v(\phi) = v(\neq \phi)$ then $v^*(\phi) = \mathbf{u}_1$.
- If $v(\phi) = \mathbf{f}$ and $v(\phi) = v(\neq \phi)$ then $v^*(\phi) = \mathbf{u}_2$.
- Otherwise, $v(\phi) = v^*(\phi)$.

Again, by a typical induction, it's easy to check that for every formula B,

- If $v(B) = \mathbf{t}$ then $v^*(B) \in \{\mathbf{u}_1, \mathbf{t}\}$.
- If $v(B) = \mathbf{f}$ and $v^*(B) \in \{\mathbf{u}_2, \mathbf{f}\}$.

It's worth pointing out that there are two problematic cases in the induction: the first one related with disjunction and the other, with conjunction. In both cases, this is caused by the fact that the infectious value is not designated. In the case of disjunction, the situation is exactly the same as in the proof of $\mathbf{S}_{\mathtt{fde}} \subseteq^2 \mathbf{S}_{\mathtt{fde}}$ (see Fact 5.17). In the case of conjunction, given two formulae B and C, it's not sufficient that $v^*(B) \in \{\mathbf{t}, \mathbf{u}_1\}$ and $v^*(C) \in \{\mathbf{t}, \mathbf{u}_1\}$ in order to guarantee that $v^*(B \wedge C) \in \{\mathbf{t}, \mathbf{u}_1\}$ (clause **(vAnd)**). Moreover, it's easy to check that similarly the clause **(vAnd)*** is not sufficient either. Thus, it's straightforward looking at the truth tables that the more restrictive clause **(vAnd)**** does the job.

Therefore, it can be easily shown that v^* is a $\mathbf{S}^{\mathbf{w}}_{\mathtt{fde}}$-valuation, such that $v^*[\Gamma] \subseteq \{\mathbf{u}_1, \mathbf{t}\}$, but $v^*(A) \notin \{\mathbf{u}_1, \mathbf{t}\}$, and thus $\Gamma \nvDash_{\mathbf{S}^{\mathbf{w}}_{\mathtt{fde}}} A$. $\square$

Fact 5.24. $^2\mathbf{dS}^{\mathbf{w}}_{\mathtt{fde}} = \mathbf{dS}^{\mathbf{w}}_{\mathtt{fde}}$.

Proof. The proof is similar to the proof of Fact 5.23. $\square$

6 LFIs and LFUs based on Weak Kleene logics

As we remarked, the Weak Kleene logics can be shown to be the $\{\neg, \wedge, \vee\}$- or the *internal* fragments of the the Logics of Nonsense. Therefore, we proceed to present all of these systems in their corresponding matrix settings.

In what follows we will provide a slightly modified presentation of Bochvar, Halldén and Segerberg's Logics of Nonsense. On the one hand, it is a notational variation because, in Bochvar's case, we substituted the unary external assertion operator $\mathbf{A}A$ with the unary connective $\circ A$; in Halldén's case we substituted the unary meaningfulness operator $+A$ with the unary connective $\circ A$; and in Segerberg's case we substituted the unary external assertion operator $\mathbf{T}A$ for the unary connective $\circ A$. On the other hand, the following is an abbreviated presentation of these logics because, in Bochvar's case, the external negation operator $\sim A$ is not introduced as primitive, but is definable as $\circ\neg A$. These differences in the presentation do not imply, however, any substantial modification whatsoever of the systems that we now introduce.

Definition 6.1. The *external* propositional language Σ^E is defined by:

$$\Sigma^E_i = \Sigma^I_i, \text{ for } i = 0 \text{ or } i \geq 2 \qquad \Sigma^E_1 = \Sigma^I_1 \cup \{\circ\}$$

Definition 6.2. Let the functions $f^{\circ}_{\mathbf{B}_3}$ and $f^{\circ}_{\mathbf{H}_3}$ defined on the set $\{\mathbf{t}, \mathbf{u}, \mathbf{f}\}$ be as follows

	$f^{\circ}_{\mathbf{B}_3}$
$\mathbf{t}$	$\mathbf{t}$
$\mathbf{u}$	$\mathbf{f}$
$\mathbf{f}$	$\mathbf{f}$

	$f^{\circ}_{\mathbf{H}_3}$
$\mathbf{t}$	$\mathbf{t}$
$\mathbf{u}$	$\mathbf{f}$
$\mathbf{f}$	$\mathbf{t}$

These functions are referred to as the external assertion operation and the meaningfulness operator, respectively, in [3] and [19].

Definition 6.3. For $X \in \{\mathbf{B}_3, \mathbf{H}_3, \mathbf{S}_3\}$, a 3-valued X-matrix is the following Σ^E-matrix, where $\langle\{\mathbf{t}, \mathbf{u}, \mathbf{f}\}, \{f^{\neg}_{\mathbf{WK}}, f^{\wedge}_{\mathbf{WK}}, f^{\vee}_{\mathbf{WK}}\}\rangle$ is the 3-element Weak Kleene algebra, and moreover $f^{\circ}_{\mathbf{B}_3}$ and $f^{\circ}_{\mathbf{H}_3}$ are as in Definition 6.2, and $f^{\circ}_{\mathbf{S}_3} = f^{\circ}_{\mathbf{B}_3}$.

$$\mathcal{M}_{\mathbf{B}_3} = \langle\{\mathbf{t}, \mathbf{u}, \mathbf{f}\}, \{\mathbf{t}\}, \{f^{\neg}_{\mathbf{WK}}, f^{\circ}_{\mathbf{B}_3}, f^{\wedge}_{\mathbf{WK}}, f^{\vee}_{\mathbf{WK}}\}\rangle$$

$$\mathcal{M}_{\mathbf{H}_3} = \langle\{\mathbf{t}, \mathbf{u}, \mathbf{f}\}, \{\mathbf{t}, \mathbf{u}\}, \{f^{\neg}_{\mathbf{WK}}, f^{\circ}_{\mathbf{H}_3}, f^{\wedge}_{\mathbf{WK}}, f^{\vee}_{\mathbf{WK}}\}\rangle$$

$$\mathcal{M}_{\mathbf{S}_3} = \langle\{\mathbf{t}, \mathbf{u}, \mathbf{f}\}, \{\mathbf{t}, \mathbf{u}\}, \{f^{\neg}_{\mathbf{WK}}, f^{\circ}_{\mathbf{S}_3}, f^{\wedge}_{\mathbf{WK}}, f^{\vee}_{\mathbf{WK}}\}\rangle$$

Definition 6.4. For $X \in \{\mathbf{B}_3, \mathbf{H}_3, \mathbf{S}_3\}$, the 3-valued logic $X = \langle FOR(\Sigma^E), \vDash_X\rangle$ is defined as follows, letting V^X be the set of $\mathcal{M}_X$-valuations

$$\Gamma \vDash_{\mathbf{B}_3} A \iff \text{for every valuation } v \in V^{\mathbf{B}_3}, \text{ if } v[\Gamma] \subseteq \{\mathbf{t}\}, \text{ then } v(A) \in \{\mathbf{t}\}$$

$$\Gamma \vDash_{\mathbf{H}_3} A \iff \text{for every valuation } v \in V^{\mathbf{H}_3}, \text{ if } v[\Gamma] \subseteq \{\mathbf{t}, \mathbf{u}\}, \text{ then } v(A) \in \{\mathbf{t}, \mathbf{u}\}$$

$$\Gamma \vDash_{\mathbf{S}_3} A \iff \text{for every valuation } v \in V^{\mathbf{S}_3}, \text{ if } v[\Gamma] \subseteq \{\mathbf{t}, \mathbf{u}\}, \text{ then } v(A) \in \{\mathbf{t}, \mathbf{u}\}$$

Remark. The logic $\mathbf{K_3^w}$ is the $\{\neg, \wedge, \vee\}$- or the *internal* fragment of the logic $\mathbf{B}_3$, whereas the logic **PWK** is the $\{\neg, \wedge, \vee\}$- or the *internal* fragment of the logics $\mathbf{H}_3$ and $\mathbf{S}_3$.

As noticed in [11] $\mathbf{B}_3$ is a Logic of Formal Undeterminedness (LFU, for short), whereas $\mathbf{H}_3$ and $\mathbf{S}_3$ are Logics of Formal Inconsistency (LFI, for short). LFUs and LFIs are, respectively, non-classical logics paracomplete and paraconsistent logics that are endowed with recovery operators that count with linguistic devices to mark those pieces of language which can be used to infer classically. The tradition of such well-behavedness operators goes back to the works of Newton da Costa and his C-systems in [13], [12], but it later sprung as a field of study in itself, focusing on logics with primitive recovery operators in the works [8], [7], [25], [6] of e.g. Carnielli, Marcos and Coniglio, among others. These family of logics can be defined as follows.

Definition 6.5 ([7])**.** A logic $\mathbf{L}$ is a Logic of Formal Inconsistency if and only if there is some possibly empty set of formulae $\circ(\alpha)$ depending on α such that the following conditions are met:

There are some some $\Gamma \subseteq FOR(\Sigma)$ and $\alpha, \beta \in FOR(\Sigma)$ such that:

1. $\Gamma, \alpha, \neg\alpha \nvDash \beta$
2. $\Gamma, \circ(\alpha), \alpha \nvDash \beta$
3. $\Gamma, \circ(\alpha), \neg\alpha \nvDash \beta$

And for all $\Gamma \subseteq FOR(\Sigma)$ and $\alpha, \beta \in FOR(\Sigma)$:

4. $\Gamma, \circ(\alpha), \alpha, \neg\alpha \vDash \beta$

Definition 6.6 ([11])**.** A logic $\mathbf{L}$ is a Logic of Formal Undeterminedness if and only if there is a possibly empty set of formulae $\star(\alpha)$ depending on α, such that the following conditions are met.

There are some some $\Gamma \subseteq FOR(\Sigma)$ and $\alpha \in FOR(\Sigma)$ such that:

1. $\Gamma \nvDash \alpha, \neg\alpha$
2. $\Gamma \nvDash \star(\alpha), \alpha$
3. $\Gamma \nvDash \star(\alpha), \neg\alpha$

And for all $\Gamma \subseteq FOR(\Sigma)$ and $\alpha \in FOR(\Sigma)$:

4. $\Gamma \vDash \star(\alpha), \alpha, \neg\alpha$

Finally, let us highlight that the logics $\mathbf{K_3^w}$ and **PWK** are indeed infectious logics, while the logics $\mathbf{B}_3$, $\mathbf{H}_3$ and $\mathbf{S}_3$ are not infectious logics. What makes the latter be non-infectious is, precisely, the fact that they are equipped with the so-called external connectives that "classicize" every formula they are applied to, thereby taking the alleged infectious value to some other different value and disrupting its infectious behavior. Thus, it is right to say that $\mathbf{B}_3$, $\mathbf{H}_3$ and $\mathbf{S}_3$ are, respectively, an LFU and two LFIs defined in *extensions* of infectious logics.[8]

[8]For more on LFUs and LFIs defined in extensions of infectious logics, see [35].

6.1 Two-valued Logics of Nonsense

As in the previous section, we take most of the terminology deployed for defining two-valued Logics of Nonsense from [6]. In this case the clause **(vConsCiw)** is the combination of clauses **(vCons)** and **(vCiw)**, while **(vConsCiw*)** is original to this work.

Definition 6.7. A bivaluation $v : FOR(\Sigma^E) \longrightarrow \{\mathbf{t}, \mathbf{f}\}$ satisfying the clauses **(vNeg)'**, **(vAnd)**, **(vOr*)**, **(vNeg)"**, **(vDM$_\wedge$)**, **(vDM$_\vee$)** and— additionally—the following clause, is a $^2\mathbf{B}_3$ valuation:

(vConsCiw*) $v(\circ A) = \mathbf{t} \Longleftrightarrow v(A) = \mathbf{t}$

The set of all such valuations is $V^{^2\mathbf{B}_3}$.

Definition 6.8. Let the two-valued logic $^2\mathbf{B}_3 = \langle FOR(\Sigma^E), \vDash_{^2\mathbf{B}_3} \rangle$ be defined such that

$$\Gamma \vDash_{^2\mathbf{B}_3} A \Longleftrightarrow \text{for every valuation } v \in V^{^2\mathbf{B}_3}, \text{ if } v[\Gamma] \subseteq \{\mathbf{t}\}, \text{ then } v(A) \in \{\mathbf{t}\}$$

Notice that the clause **(vConsCiw*)** makes sense in a two-valued presentation of Bochvar's logic $\mathbf{B}_3$, since Bochvar's interpretation of his external assertion operator, here represented with the symbol $\circ$, has it that $\circ A$ is true if and only if A is true, and is false otherwise.

Definition 6.9. A bivaluation $v : FOR(\Sigma^E) \longrightarrow \{\mathbf{t}, \mathbf{f}\}$ satisfying satisfying the clauses **(vNeg)**, **(vAnd*)**, **(vOr)**, **(vNeg)"**, **(vDM$_\wedge$)**, **(vDM$_\vee$)** and— additionally—the following clause, is a $^2\mathbf{H}_3$ a valuation.

(vConsCiw) $v(\circ A) = \mathbf{t} \Longleftrightarrow v(A) \neq v(\neg A)$

The set of all such valuations is $V^{^2\mathbf{H}_3}$.

Definition 6.10. Let the two-valued logic $^2\mathbf{H}_3 = \langle FOR(\Sigma^E), \vDash_{^2\mathbf{H}_3} \rangle$ be defined such that

$$\Gamma \vDash_{^2\mathbf{H}_3} A \Longleftrightarrow \text{for every valuation } v \in V^{^2\mathbf{H}_3}, \text{ if } v[\Gamma] \subseteq \{\mathbf{t}\}, \text{ then } v(A) \in \{\mathbf{t}\}$$

Notice, moreover, that the clause **(vConsCiw)** makes sense in a two-valued presentation of Halldén's logic $\mathbf{H}_3$, since his meaningfulness operator, here represented with the symbol $\circ$, has it that $\circ A$ is true if and only if A is either true or false, in other words, if it is not nonsensical or meaningless. But given, in our two-valued presentation of Halldén's logic, nonsensical sentences are the only sentences which allow for $v(A)$ to be identical to $v(\neg A)$, our target clause does exactly the job we need it to perform.

Definition 6.11. A bivaluation $v : FOR(\Sigma^E) \longrightarrow \{\mathbf{t}, \mathbf{f}\}$ satisfying satisfying the clauses **(vNeg)**, **(vAnd*)**, **(vOr)**, **(vNeg)"**, **(vDM$_\wedge$)**, **(vDM$_\vee$)** and **(vConsCiw)** is a $^2\mathbf{S}_3$ a valuation. The set of all such valuations is $V^{^2\mathbf{S}_3}$.

Definition 6.12. Let the two-valued logic $^2\mathbf{S}_3 = \langle FOR(\Sigma^E), \vDash_{^2\mathbf{S}_3} \rangle$ be defined such that

$$\Gamma \vDash_{^2\mathbf{S}_3} A \Longleftrightarrow \text{for every valuation } v \in V^{^2\mathbf{S}_3}, \text{ if } v[\Gamma] \subseteq \{\mathbf{t}\}, \text{ then } v(A) \in \{\mathbf{t}\}$$

Fact 6.13. $^2\mathbf{B}_3 = \mathbf{B}_3$

Proof. The proof is similar to the proof of the Fact 5.11. From left to right, assume that $\Gamma \nvDash_{\mathbf{B}_3} A$ and let a $\mathbf{B}_3$-valuation v such that $v[\Gamma] \subseteq \{\mathbf{t}\}$ and $v(A) \notin \{\mathbf{t}\}$ witness this fact. Construct a valuation v^* as follows:

- If $v(\phi) = \mathbf{u}$, then $v^*(\phi) = \mathbf{f}$
- If $v(\phi) \neq \mathbf{u}$, then $v^*(\phi) = v(\phi)$

with ϕ any atomic formula. By a straightforward induction it can be shown that v^* is a ${}^2\mathbf{B}_3$-valuation such that $v^*[\Gamma] \subseteq \{\mathbf{t}\}$ and $v^*(A) \notin \{\mathbf{t}\}$, witnessing the fact that $\Gamma \nvDash_{{}^2\mathbf{B}_3} A$. Among other things, it's worth noting that the presence of the operator $\circ$ doesn't affect this induction, since $f^{\circ}_{\mathbf{B}_3}(\mathbf{u}) = f^{\circ}_{\mathbf{B}_3}(\mathbf{f}) = \mathbf{f}$. Thus, ${}^2\mathbf{B}_3 \subseteq \mathbf{B}_3$.

From right to left, assume $\Gamma \nvDash_{{}^2\mathbf{B}_3} A$. Thus, there is a ${}^2\mathbf{B}_3$-valuation v such that $v[\Gamma] \subseteq \{\mathbf{t}\}$ and $v(A) \notin \{\mathbf{t}\}$ witnesses this fact. Construct a valuation v^* as follows, letting ϕ be any *atomic* formula:

- If $v(\phi) = v(\neg\phi)$, then $v^*(\phi) = \mathbf{u}$
- If $v(\phi) \neq v(\neg\phi)$, then $v^*(\phi) = v(\phi)$

Again, by a straightforward induction it can be observed that v^* is a $\mathbf{B}_3$-valuation such that for *every formulae* B:

- If $v(B) = \mathbf{t}$, then $v^*(B) = \mathbf{t}$
- If $v(B) = \mathbf{f}$, then either $v^*(B) = \mathbf{u}$ or $v^*(B) = \mathbf{f}$

Notice that the induction is not interfered by the presence of $\circ$ since for every valuation $v^* \in^2 \mathbf{B}_3$, $v(\circ A) = \mathbf{t}$ if and only if $v(A) \neq v(\neg A)$. Therefore, it is a $\mathbf{B}_3$-valuation such that $v^*[\Gamma] \subseteq \{\mathbf{t}\}$ and $v^*(A) \notin \{\mathbf{t}\}$, thereby witnessing the fact that $\Gamma \nvDash_{\mathbf{B}_3} A$. $\square$

Fact 6.14. ${}^2\mathbf{H}_3 = \mathbf{H}_3$

Proof. Similar to the proof of Fact 6.13, with the only difference that in the above proof we need to change the clauses for the construction of v^*, as in the proof of Fact 5.12. $\square$

Fact 6.15. ${}^2\mathbf{S}_3 = \mathbf{S}_3$

Proof. Similar to the proof of Facts 6.13 and 6.14. $\square$

7 Conclusion

In this paper, we have provided two-valued semantics to some many-valued logics, and we have analyzed the behaviour of the connectives defined in the resulting systems. In this sense, firstly, we have focused on Weak Kleene logics ($\mathbf{K_3^w}$ and $\mathbf{PWK}$) and on some four-valued infectious subsystems of these logics ($\mathbf{S}_{\mathtt{fde}}, \mathbf{dS}_{\mathtt{fde}}, \mathbf{S}^{\mathbf{w}}_{\mathtt{fde}}$ and $\mathbf{dS}^{\mathbf{w}}_{\mathtt{fde}}$). Next, we have shown how to provide two-valued semantics to the Logics of Nonsense ($\mathbf{B}_3$, $\mathbf{H}_3$ and $\mathbf{S}_3$), which are proper LFIs and LFUs defined in extensions of the Weak Kleene logics. All of these systems are usually characterized via 3-element or 4-element algebras and they are often considered as essentially many-valued logics. Providing a different perspective on this issue, we offered alternative two-valued semantics for them and showed the equivalence between these semantics and the original ones. With these reductions we think we clarify the non-standard behavior of the logical connectives in Weak Kleene systems. In particular, two-valued semantics are suitable to enlighten the properties of paraconsistency and paracompleteness related with the oddity in the definition of the conjunction and the disjunction, respectively. Finally, our aim here was not to argue for the two-valued presentations of the systems. On the other hand, we think that both approaches are complemented and each one is interesting by itself.

References

[1] B. Armour-Garb and G. Priest. Analetheism: a pyrrhic victory. *Analysis*, 65(286):167–173, 2005.

[2] J. Beall and D. Ripley. Analetheism and dialetheism. *Analysis*, 64(281):30–35, 2004.

[3] D. A. Bochvar. On a three-valued logical calculus and its application to the analysis of the paradoxes of the classical extended functional calculus. *History and Philosophy of Logic*, 2(1-2):87–112, 1981. Translated by M. Bergmann.

[4] S. Bonzio, J. Gil-Férez, F. Paoli, and L. Peruzzi. On Paraconsistent Weak Kleene Logic: Axiomatisation and Algebraic Analysis. *Studia Logica*, 105(2):253–297, 2017.

[5] C. Caleiro, W. Carnielli, M. E. Coniglio, and J. Marcos. Two's company: "The humbug of many logical values". In J.-Y. Beziau, editor, *Logica Universalis*, pages 169–189. Birkhäuser, Basel, 2005.

[6] W. Carnielli and M. Coniglio. *Paraconsistent Logic: Consistency, Contradiction and Negation*. Springer, 2016.

[7] W. Carnielli, M. Coniglio, and J. Marcos. Logics of Formal Inconsistency. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, volume 14, pages 1–93. Springer, Dodrecht, 2007.

[8] W. Carnielli, J. Marcos, and S. De Amo. Formal inconsistency and evolutionary databases. *Logic and Logical Philosophy*, 8:115–152, 2004.

[9] R. Ciuni. Conjunction in Paraconsistent Weak Kleene Logic. In P. Arazim and M. Dančák, editors, *The Logica Yearbook 2014*, pages 61–76. College Publications, London, 2015.

[10] R. Ciuni and M. Carrara. Characterizing Logical Consequence in Paraconsistent Weak Kleene. In L. Felline, A. Ledda, F. Paoli, and E. Rossanese, editors, *New Directions in Logic and the Philosophy of Science*, pages 165–176. College Publications, London, 2016.

[11] M. I. Corbalán. Conectivos de Restauração Local. Master's thesis, Universidade Estadual de Campinas, Instituto de Filosofia e Ciências Humanas, 2012.

[12] N. da Costa and E. Alves. A semantical analysis of the calculi C_n. *Notre Dame Journal of Formal Logic*, 18(4):621–630, 1977.

[13] N. da Costa and A. Arruda. Une sémantique pour le calcul C_1. *Comptes Rendus de l'Académie des Sciences de Paris*, 284:279–282, 1976.

[14] B. Da Ré, F. Pailos, and D. Szmuc. Theories of truth based on four-valued infectious logics. *Logic Journal of the IGPL*, 2017. *Forthcoming.*

[15] H. Deutsch. Paraconsistent analytic implication. *Journal of Philosophical Logic*, 13(1):1–11, 1984.

[16] T. M. Ferguson. Faulty Belnap computers and subsystems of FDE. *Journal of Logic and Computation*, 2014.

[17] M. Fitting. Bilattices are nice things. In T. Bolander, V. Hendricks, and S. A. Pedersen, editors, *Self-Reference*, pages 53–78. CSLI Publications, 2006.

[18] S. Frankowski. Formalization of a plausible inference. *Bulletin of the Section of Logic*, 33(1):41–52, 2004.

[19] S. Halldén. *The Logic of Nonsense*. Uppsala Universitets Årsskrift, Uppsala, 1949.

[20] L. Humberstone. Power Matrices and Dunn–Belnap Semantics: Reflections on a Remark of Graham Priest. *Australasian Journal of Logic*, 11(1):14–45, 2014.

[21] S. C. Kleene. *Introduction to Metamathematics*. North-Holland, Amsterdam, 1952.

[22] J. Łukasiewicz. On three-valued logic. *Ruch Filozoficzny*, 5, 1920. English translation in Borkowski, L. (ed.) *Jan Lukasiewicz: Selected Works*, Amsterdam: North Holland, 1970.

[23] G. Malinowski. Q-consequence operation. *Reports on Mathematical Logic*, 24(1):49–59, 1990.

[24] G. Malinowski. Inferential many-valuedness. In J. Wolenski, editor, *Philosophical Logic in Poland*, pages 75–84. Kluwer Academic Publishers, 1994.

[25] J. Marcos. Nearly every normal modal logic is paranormal. *Logique et Analyse*, 48(189-192):279–300, 2005.

[26] C. A. Oller. Paraconsistency and analyticity. *Logic and Logical Philosophy*, 7:91–99, 1999.

[27] H. Omori and D. Szmuc. Conjunction and disjunction in infectious logics. In A. Baltag, J. Seligman, and T. Yamada, editors, *Logic, Rationality, and Interaction: 6th International Workshop (LORI 2017)*, pages 268–283, Berlin, 2017. Springer.

[28] W. T. Parry. Ein axiomensystem für eine neue art von implikation (analytische implikation). *Ergebnisse eines mathematischen Kolloquiums*, 4:5–6, 1933.

[29] G. Priest. Plurivalent logics. *Australasian Journal of Logic*, 11(1):2–13, 2014.

[30] L. Rosenblatt. Two-valued logics for transparent truth theory. *Australasian Journal of Logic*, 12:44–66, 2015.

[31] K. Segerberg. A contribution to nonsense-logics. *Theoria*, 31(3):199–217, 1965.

[32] Y. Shramko and H. Wansing. *Truth and falsehood: An inquiry into generalized logical values*. Springer, 2011.

[33] R. Suszko. Remarks on Łukasiewicz's three-valued logic. *Bulletin of the Section of Logic*, 4(3):87–90, 1975.

[34] R. Suszko. The Fregean Axiom and Polish Mathematical Logic in the 1920s. *Studia Logica*, 36(4):376–380, 1977.

[35] D. Szmuc. Defining LFIs and LFUs in extensions of infectious logics. *Journal of Applied Non-Classical Logics*, 26(4):286–314, 2016.

[36] D. Szmuc and H. Omori. A Note on Goddard and Routley's Significance Logic. *Australasian Journal of Logic*, 15(2):431–448, 2018.

[37] M. Tsuji. Many-valued logics and Suszko's thesis revisited. *Studia Logica*, 60(2):299–309, 1998.