Relative entropy for coherent states from Araki formula

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We make a rigorous computation of the relative entropy between the vacuum state and a coherent state for a free scalar in the framework of algebraic description of quantum field theory (AQFT). We study the case of the Rindler wedge. Previous calculations including path integral methods and computations from the lattice give a result for such relative entropy which involves integrals of expectation values of the energy-momentum stress tensor along the considered region. However, the stress tensor is in general nonunique. That means that if we start with some stress tensor, then we can "improve" it adding a conserved term without modifying the Poincaré charges. On the other hand, the presence of such an improving term affects the naive expectation for the relative entropy by a nonvanishing boundary contribution along the entangling surface. In other words, this means that there is an ambiguity in the usual formula for the relative entropy coming from the nonuniqueness of the stress tensor. The main motivation of this work is to solve this puzzle. We first show that all choices of stress tensor except the canonical one are not allowed by positivity and monotonicity of the relative entropy. Then we fully compute the relative entropy between the vacuum and a coherent state in the framework of AQFT using the Araki formula and the techniques of modular theory. After all, both results coincide and give the usual expression for the relative entropy calculated with the canonical stress tensor.

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I. INTRODUCTION

The algebraic description of quantum field theory (AQFT) focuses on the local algebras of operators generated by fields in regions of the space rather than the field operators themselves. This gives a "basis independent" formulation which does not depend on the particular fields used for the description of the theory. Statistical properties of the state in these local algebras have been the subject of much recent interest in different areas of physics ranging from holography to condensed matter. Given one or more states and algebras, several entropic quantities can be defined which give natural measures of the statistics of fluctuations. In a certain sense, these assignations of numbers to algebras in AQFT is analogous to the study of correlators in the approach based on field operators.

In actual computations in specific models, it is customary and useful to assume a cutoff model, such as a lattice, and proceed to the computation taking the continuum limit as a final step. In general, we expect that the quantity computed belongs to the continuous theory as far as the result does not depend on the regularization. In the cutoff model, given a global pure state $\Phi \in \mathcal{H}$ one can consider the reduced density matrix ρ_Φ^R in a space region R of a lattice and compute its von Neumann (vN) entropy

$$S_\Phi^R = -\mathrm{tr}\rho_\Phi^R \log \rho_\Phi^R. \tag{1.1}$$

This is divergent and not well defined in the continuum due to a large amount of entanglement of UV modes between both sides of the region boundary. However, given two states Ψ and Ω we can also compute the relative entropy

$$S_R(\Phi|\Omega) = \mathrm{tr}\rho_\Phi^R(\log \rho_\Phi^R - \log \rho_\Omega^R), \tag{1.2}$$

which is much better behaved than the entropy (see e.g., [1–14] for actual calculations). In fact, the relative entropy has an expression directly in the continuous theory for type III algebras in terms of the Araki formula [15]. This shows it is free from ambiguities. Relative entropy is an important quantity in quantum information that measures distinguishability between states. It is always positive and increasing

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for fixed states under increasing algebras. It has recently been very useful in holography to understand the bulk-boundary map [16–19] and in the proof of the quantum null energy condition [20].

Another object that has a nice continuum limit is the following one parameter group of unitaries,

$$(\rho_\Omega^R)^{is} \otimes (\rho_\Omega^{R'})^{-is}, \tag{1.3}$$

where R' is the complement of R and we are assuming there is a decomposition of the full operator algebra as a tensor product of the algebras in R and R'. This one-parametric group is called the modular group. The generator,

$$K_\Omega = -K_R \otimes 1 + 1 \otimes K_{R'}, \qquad K_R = -\log \rho_\Omega^R, \tag{1.4}$$

is called the modular Hamiltonian. A well-known case where the modular Hamiltonian can be computed exactly is the case when R is the Rindler wedge corresponding to a spatial slice $x^1 > 0$ at $x^0 = 0$, and the state is the vacuum. In this case, $K_\Omega = 2\pi K_1$ with K_1 being the boost generator. In terms of the energy density operator we can write

$$K_\Omega = 2\pi \int d^{d-1}x x^1 T_{00}(x). \tag{1.5}$$

Returning to the relative entropy, it is useful to write (1.2) as

$$S_R(\Phi|\Omega) = \Delta\langle K_R\rangle - \Delta S_R, \tag{1.6}$$

where

$$\Delta\langle K_R\rangle = \mathrm{tr}\rho_\Phi^R K_R - \mathrm{tr}\rho_\Omega^R K_R, \tag{1.7}$$

$$\Delta S = S_\Phi^R - S_\Omega^R. \tag{1.8}$$

Written in this way, the relative entropy is the variation in expectation value of an operator minus the variation in the entropy between the two states. The positivity of relative entropy means that $\Delta\langle K_R\rangle \geq \Delta S_R$. In this form, when R is the Rindler wedge, this inequality has been related to Bekenstein's bound on entropy [21].

Even if the relative entropy is well defined in the continuum, a mathematically rigorous definition of the continuum limit of the two terms in (1.6) has not been worked out in the literature yet. One difficulty is that the operator K_R is only half of the modular Hamiltonian (1.4). Even if the modular Hamiltonian has a good operator limit in the continuum, its half part K_R is at most a sesquilinear form. If we focus for simplicity on the case of the half space and where Ω is the vacuum, we could induce from (1.5) that

$$K_R = 2\pi \int_{x^1>0} d^{d-1}x x^1 T_{00}(x). \tag{1.9}$$

This is not a well-defined operator in Hilbert space because its fluctuation $\langle\Omega, K_R^2\Omega\rangle$ diverges. However, expectation values as in $\Delta\langle K_R\rangle$ can still be computed. Another more important issue is that the act of cutting the modular Hamiltonian in two pieces generate ambiguities. We are allowed for example to add field operators localized at the boundary such that K_R has still the same localization and commutation relations with operators inside R. Another view of the same problem is that hidden in expression (1.9), there is an ambiguity related to the nonuniqueness of the stress tensor. For example, for the free Hermitian scalar field, starting from the canonical stress tensor

$$T_{\mu\nu}^{\mathrm{can}} =: \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}(\partial_\sigma\phi\partial^\sigma\phi - m^2\phi^2):, \tag{1.10}$$

we can add an "improving term" to obtain a new stress tensor,

$$T_{\mu\nu} = T_{\mu\nu}^{\mathrm{can}} + \frac{\lambda}{2\pi}(\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2):\phi^2:. \tag{1.11}$$

The Poincaré generators obtained from (1.11) equal the ones obtained from (1.10), since both expressions differ in a boundary term which vanishes when the integration region is the whole space. However, the expression (1.9) for K_R involves an integration in a semi-infinite region, and hence the presence of an improving term adds a nonzero extra boundary term to the result,

$$K_R \to K_R + \lambda \int_{x^1=0} d^{d-2}x :\phi^2(x):. \tag{1.12}$$

This is essentially the only boundary term we can add with the correct dimensions and that does not require a dimensionful coefficient with negative dimensions. This can have nonzero expectation values for certain states and makes the definition of $\Delta\langle K_R\rangle$ ambiguous.

Since the relative entropy is well defined, this ambiguity must be compensated by another one in the definition of ΔS_R in (1.6). This is the subtraction of two divergent quantities and again we do not have a mathematically rigorous definition in the continuum. We can make this definition unambiguous in a natural way by using a particular regularization of entropy that has been proposed in the literature [22,23]. The idea is to associate the entropy (for a pure state) with half the mutual information $I(R_\epsilon^+, R_\epsilon^-)$ between two nonintersecting regions on both sides of the boundary of R. The regions $R_\epsilon^\pm$ are displaced a distance ϵ from the boundary of R. For the case of the Rindler wedge we can take R_ϵ^+ formed by points with $x^1 > \epsilon$ and R_ϵ^- formed by points with $x^1 < -\epsilon$. The mutual information is also a relative entropy and is well defined in the continuum. Then, a well-defined ΔS_R is given by

$$\Delta S_R = \frac{1}{2}\lim_{\epsilon\to 0}(I_\Phi(R_\epsilon^+, R_\epsilon^-) - I_\Omega(R_\epsilon^+, R_\epsilon^-)). \tag{1.13}$$

When it is computed in the lattice, it coincides with the usual ΔS_R.

Defining ΔS_R rigorously through (1.13), then $\Delta\langle K_R\rangle$ is also well defined through

$$\Delta\langle K_R\rangle = S_R(\Phi|\Omega) + \Delta S_R. \tag{1.14}$$

Then the question that motivates this paper is whether this definition agrees with the expectation value of (1.9). In such a case, boundary terms in this expression should be automatically fixed. In particular, we should be able to study which value of the improvement term is the correct one for a scalar field in (1.11).

In order to (partially) settle this issue, in this paper we analyze the relative entropy between a coherent state for a free scalar field and the vacuum in the Rindler wedge. Coherent states are states formed out by acting on the vacuum with a unitary operator that is the exponential of the smeared field, i.e.,

$$\Phi = \mathrm{e}^{i\int d^{d-1}x[\varphi(\bar{x})f_\varphi(\bar{x})+\pi(\bar{x})f_\pi(\bar{x})]}\Omega, \tag{1.15}$$

where $\varphi(\bar{x}) \coloneqq \phi(0,\bar{x})$ and $\pi(\bar{x}) \coloneqq \partial_0\phi(0,\bar{x})$. For the purpose of the definition (1.13), we can represent the same state with a different vector $\tilde{\Phi} = UU'\Omega$, where U is a unitary belonging to the region R and U' is a unitary belonging to its complementary region R'. Indeed, we can replace each of the smooth functions $f_\varphi(\bar{x})$ and $f_\pi(\bar{x})$ in (1.15) by the sum of two new smooth functions,

$$f_\varphi \to f_{\varphi,R} + f_{\varphi,R'}, \qquad f_\pi \to f_{\pi,R} + f_{\pi,R'}, \tag{1.16}$$

such that $f_{\varphi,R}, f_{\pi,R}$ vanish inside R' and $f_{\varphi,R'}, f_{\pi,R'}$ vanish inside R. We must also require that $f_{\varphi,R} \equiv f_\varphi$ inside R_ϵ^+ and $f_{\varphi,R'} \equiv f_\varphi$ inside R_ϵ^- (idem for π). Under this assumptions, the new state $\tilde{\Phi} = UU'\Omega$, defined through

$$U = \mathrm{e}^{i\int d^{d-1}x[\varphi(\bar{x})f_{\varphi,R}(\bar{x})+\pi(\bar{x})f_{\pi,R}(\bar{x})]} \quad \text{and}$$
$$U' = \mathrm{e}^{i\int d^{d-1}x[\varphi(\bar{x})f_{\varphi,R'}(\bar{x})+\pi(\bar{x})f_{\pi,R'}(\bar{x})]}, \tag{1.17}$$

represents the same state as Φ in the algebra of the region $R_\epsilon^+ \cup R_\epsilon^-$. In fact, the above computation can be done because of the presence of the finite corridor of width 2ϵ. Moreover, we have that the operator U (respectively, U') acts, by adjoint action, as an automorphism of the algebra of the region R_ϵ^+ (respectively, R_ϵ^-), and as the identity transformation over the algebra of the region R_ϵ^- (respectively, R_ϵ^+). Such automorphisms do not change the mutual information, and with our definition (1.13), we automatically have $\Delta S_R = 0$ for these states.

Hence, the question simplifies to see whether for coherent states

$$S_R(\Phi|\Omega) = 2\pi\int_{x^1>0} d^{d-1}x x^1\langle\Phi, T_{00}(\bar{x})\Phi\rangle, \tag{1.18}$$

and which is the right improvement term. Notice that coherent states can change the expectation value of $:\phi^2:$.

In Sec. II, assuming that (1.18) is correct for some improvement, we show that the only possibility is the canonical stress tensor, i.e., $\lambda = 0$. We show this by imposing bounds which come from the positivity and monotonicity of the relative entropy.

In the rest of the paper, we actually compute the relative entropy using the Araki formula and show the result (1.18) is correct for the canonical stress tensor. We note that, while this paper was being prepared, a similar calculation by R. Longo has appeared in the literature [13]. A simpler case where the unitary has support inside the wedge has previously appeared in [24]. Our paper differs from the one by Longo in motivation, scope, and several details, while there is an overlap in the main technical ideas.

To make this article as self-contained as possible, in Sec. III we briefly review the algebraic formulation of the free scalar field. Because of a forthcoming necessity, we consider two different approaches. The first one is the usual approach where we define the net of algebras associated to spacetime regions. The second one consists in defining the local algebras associated to spatial sets belonging to a common Cauchy surface. We also explain how these two approaches are related. In Sec. IV we review the basic concepts of the modular theory of von Neumann algebras. In particular, we introduce the modular operator used to derive the modular Hamiltonian and the modular flow. We also discuss the theorems of Tomita-Takesaki and Bisognano-Whichmann. And finally, we introduce the relative modular operator used in the definition of the relative entropy for general von Neumann algebras. The reader who is familiar with these concepts may skip these sections and go directly to V, where we explicitly compute the proposed relative entropy. We study separately the (trivial) case when the coherent state belongs to the wedge algebra, and the more interesting (and also more difficult) case when the coherent state has a nonvanishing density along the entangling surface. In this section, we also first study some general aspects concerning the relative entropy for coherent states which applies to any region. We provide a complete mathematical rigorous proof of all the results. For a better reading of the article, the proof of some theorems and some tedious but straightforward calculations were placed into the Appendixes.

II. BOUNDARY TERMS IN THE RELATIVE ENTROPY

According to the discussion above, there is an ambiguity on the expression (1.18) for the relative entropy of a coherent state coming from the different possible choices of an improving term for the stress energy-momentum tensor. According to (1.12), the relative entropy could be written as the usual contribution with the canonical stress tensor plus a boundary term coming from the improving

$$S_R(\Phi|\Omega) = \lambda \int_{x_1=0} d^{d-2}x \langle \Phi, \phi^2(\bar{x})\Phi\rangle + 2\pi \int_{x_1>0} d^{d-1}x x^1 \langle \Phi, T_{00}^{can}(\bar{x})\Phi\rangle. \quad (2.1)$$

In this section we assume this formula is correct and show that the only consistent choice is $\lambda = 0$.

A general coherent state can be written as in (1.15) with $f_\varphi, f_\pi \in \mathcal{S}(\mathbb{R}^{d-1}, \mathbb{R})$.[1] In this case, a straightforward computation from (2.1) gives

$$S_R(\Phi|\Omega) = \lambda \int_{x_1=0} d^{d-2}x f_\pi(\bar{x})^2 + 2\pi \int_{x_1>0} d^{d-1}x \frac{1}{2}(f_\varphi(\bar{x})^2 + (\nabla f_\pi(\bar{x}))^2 + m^2 f_\pi(\bar{x})^2). \quad (2.2)$$

Regardless of what should be the true value for λ, if we want (2.1) and (2.2) to represent real expressions for a relative entropy, they must satisfy all the properties known for a relative entropy. In particular we concentrate on the positivity

$$S_R(\Phi|\Omega) \geq 0, \quad (2.3)$$

and the monotonicity, that for the case of wedges implies

$$S_R(\Phi|\Omega)|_{\mathcal{W}_y} \geq S_R(\Phi|\Omega)|_{\mathcal{W}_{y'}}, \quad \text{for any } y' \geq y, \quad (2.4)$$

where $S_R(\Phi|\Omega)|_{\mathcal{W}_y}$ is the relative entropy for the states Ψ, Ω but associated to the algebra of the translated Rindler wedge $\mathcal{W}_y := \{x \in \mathbb{R}^d : x^1 - y > |x^0|\}$. In fact, $\mathcal{W}_y$ is obtained applying a translation of amount y, in the x^1 positive direction, to the original Rindler wedge $\mathcal{W}$. From now on, we denote $S_R(y) := S_R(\Phi|\Omega)|_{\mathcal{W}_y}$.

Therefore, the strategy we adopt is to choose conveniently functions f_φ and f_π and impose (2.3) and (2.4) on (2.2) in order to bound the allowed values for λ. In fact, we show that from positivity we obtain $\lambda \geq 0$ and from the monotonicity we obtain $\lambda \leq 0$, an hence it must be

$$\lambda = 0. \quad (2.5)$$

Then we conclude that, if we assume that (1.18) is the correct result for the relative entropy, such an expression holds for the canonical stress-energy-momentum tensor (1.10).

Before we start, we make two simplifications. The first one, which is obvious, is to take $f_\varphi \equiv 0$ and denote $f := f_\pi$. The second one is to work in $d = 1 + 1$ dimensions. The general result for any dimensions could be obtained easily from the former.

A. Lower bound from positivity

We start with the expression

$$S_R(\Phi|\Omega) = \lambda f(0)^2 + \pi \int_0^{+\infty} dx x (f'(x)^2 + m^2 f(x)^2), \quad (2.6)$$

where f is a real-valued function belonging to $\mathcal{S}(\mathbb{R})$. Then, the positivity of the relative entropy means that

$$0 \leq \lambda f(0)^2 + \pi \int_0^{+\infty} dx x f'(x)^2 + \pi m^2 \int_0^{+\infty} dx x f(x)^2. \quad (2.7)$$

By scaling the function $f(x) \to f(x/\beta)$ the first two terms of the right-hand side are constant while the last one gets multiplies by β^2. Hence, we can make the last term as small as we want and simply take $m = 0$ in the following. Taking f such that $f(0) \neq 0$ we get

$$0 \leq \lambda + \pi \frac{\int_0^{+\infty} dx x f'(x)^2}{f(0)^2}. \quad (2.8)$$

Now, we introduce a convenient family of real functions $(f_a)_{a>0} \in \mathcal{S}(\mathbb{R})$ given by

$$f_a(x) := \log\left(\frac{x}{L} + a\right) e^{-\frac{x}{L}}, \quad x \geq 0, \quad (2.9)$$

and where $L > 0$ is a dimensionful fixed constant.[2] A straightforward computation shows that the integral in Eq. (2.8) behaves as

$$\int_0^{+\infty} dx x f_a'(x)^2 = -\log(a) + \mathcal{O}(1), \qquad a \gtrsim 0. \quad (2.10)$$

Then replacing (2.10) into (2.8) we get

$$0 \leq \lambda - \frac{L^2}{4}\pi \frac{\log(a) + \mathcal{O}(1)}{\log^2(a)}. \quad (2.11)$$

Finally, taking the limit $a \to 0^+$ we get the desired result

$$\lambda \geq 0. \quad (2.12)$$

[1] $\mathcal{S}(\mathbb{R}^n, \mathbb{R})$ denotes the Schwartz space of real, smooth and exponentially decreasing functions at infinity.

[2] The functions f_a are smoothly extended to the whole real line. Such an extension is guaranteed by a theorem due to Seeley [25].

B. Upper bound from monotonicity

We start with the expressions

$$S_R(0)=\lambda f(0)^2+\pi\int_0^{+\infty}dxx(f'(x)^2+m^2f(x)^2), \tag{2.13}$$

$$S_R(y)=\lambda f(y)^2+\pi\int_y^{+\infty}dx(x-y)(f'(x)^2+m^2f(x)^2), \tag{2.14}$$

where f is a real-valued function belonging to $\mathcal{S}(\mathbb{R})$. We can eliminate the mass terms by scaling as in the previous section. The monotonicity $S_R(0)\geq S_R(y)$ for $y\geq 0$ reads

$$\lambda(f(y)^2-f(0)^2)\leq\pi\int_0^y dxxf'(x)^2+\pi y\int_y^{+\infty}dxf'(x)^2. \tag{2.15}$$

Now, we introduce a convenient family of functions parametrized with the constants $\alpha\in(0,\frac{1}{2}),\delta\in(0,1),y>0,\epsilon>0$ given by

$$f_{\alpha,\delta,y,\epsilon}(x)\coloneqq g_{\alpha,\delta,y}(x)\Theta_{y,\epsilon}(x),\quad \text{for } x\geq 0, \tag{2.16}$$

where

$$g_{\alpha,\delta,y}(x)\coloneqq\left(\frac{x}{y}(1-\delta)+\delta\right)^{\alpha}, \tag{2.17}$$

and $\Theta_{y,\epsilon}$ is a smooth step function with the condition

$$\Theta_{y,\epsilon}(x)=\begin{cases}1 & x\leq y,\\ 0 & x\geq y+\epsilon.\end{cases} \tag{2.18}$$

We introduce such a step function to ensure that $f_{\alpha,\delta,y,\epsilon}\in\mathcal{S}(\mathbb{R})$ for any values of $(\alpha,\delta,y,\epsilon)$ in the set specified above. The functions $f_{\alpha,\delta,y,\epsilon}$ are smoothly extended to the whole real line. In particular, we use

$$\Theta_{y,\epsilon}(x)\coloneqq\left[1+\exp\left(-\frac{2\epsilon(x-y-\frac{\epsilon}{2})}{(x-y-\frac{\epsilon}{2})^2-\frac{\epsilon^2}{4}}\right)\right]^{-1},\quad \text{if } y<x<y+\epsilon, \tag{2.19}$$

which has the useful property $\max_{x\in\mathbb{R}}|\Theta'_{y,\epsilon}(x)|=\frac{2}{\epsilon}$. From now on, we do not write the cumbersome subindices of the above functions. For the different terms of (2.15) we have that

$$f(y)^2-f(0)^2=1-\delta^{2\alpha}, \tag{2.20}$$

$$\pi\int_0^y dxxf'(x)^2=\pi\frac{\alpha}{2}\left(\frac{2\alpha\delta-\delta^{2\alpha}}{1-2\alpha}+1\right), \tag{2.21}$$

$$\pi y\int_y^{+\infty}dxf'(x)^2 \leq \pi y\int_y^{y+\epsilon}dx|g'(x)^2\Theta(x)^2|+\pi y\int_y^{y+\epsilon}dx|g(x)^2\Theta'(x)^2| \tag{2.22}$$

$$+\pi y\int_y^{y+\epsilon}dx|2g'(x)g(x)\Theta(x)\Theta'(x)|. \tag{2.23}$$

We deal with each term of (2.23) separately,

$$\pi y\int_y^{y+\epsilon}dx|g'(x)^2\Theta(x)^2| \leq\pi y\int_y^{y+\epsilon}dxg'(x)^2 =\frac{\pi\alpha^2(1-\delta)}{1-2\alpha}\left[1-\left(1+\frac{(1-\delta)\epsilon}{y}\right)^{2\alpha-1}\right]\underset{\epsilon\to+\infty}{\to}\frac{\pi\alpha^2(1-\delta)}{1-2\alpha}, \tag{2.24}$$

$$\pi y\int_y^{y+\epsilon}dx|2g'(x)g(x)\Theta(x)\Theta'(x)| \leq\pi y\frac{2}{\epsilon}\int_y^{y+\epsilon}dx2g'(x)g(x)=2\pi\frac{y}{\epsilon}[g(y+\epsilon)^2-g(y)^2] =\frac{2\pi y}{\epsilon}\left[\left(1+\frac{(1-\delta)\epsilon}{y}\right)^{2\alpha}-1\right]\underset{\epsilon\to+\infty}{\to}0, \tag{2.25}$$

$$\pi y\int_y^{y+\epsilon}dx|g(x)^2\Theta'(x)^2| \leq\pi y\frac{4}{\epsilon^2}\int_y^{y+\epsilon}dxg(x)^2 =\frac{4\pi y^2}{(1+2\alpha)(1-\delta)\epsilon^2}\left(\left(1+\frac{(1-\delta)\epsilon}{y}\right)^{2\alpha+1}-1\right)\underset{\epsilon\to+\infty}{\to}0, \tag{2.26}$$

where in the last steps of each computation we take the limit $\epsilon\to+\infty$. It is valid to take this limit in the inequality since it must hold for all $\epsilon>0$. Replacing these partial results on (2.15) we arrive at

$$\lambda(1-\delta^{2\alpha})\leq\pi\frac{\alpha}{2}\left(\frac{2\alpha\delta-\delta^{2\alpha}}{1-2\alpha}+1\right)+\frac{\pi\alpha^2(1-\delta)}{1-2\alpha}. \tag{2.27}$$

Then, taking the limit $\delta\to 0^+$ we get

$$\lambda\leq\pi\frac{\alpha}{2}+\pi\frac{\alpha^2}{1-2\alpha}, \tag{2.28}$$

and finally, taking $\alpha\to 0^+$ we arrive at the desired result

$$\lambda\leq 0. \tag{2.29}$$

III. ALGEBRAIC THEORY OF THE FREE HERMITIAN SCALAR FIELD

A. Axioms of AQFT

In the AQFT, we associate for each region of the spacetime a C^* or von Neumann algebra which encodes the algebraic relations between the quantum fields. Such an assignment must satisfy a set of axioms that encode the physical conditions in the algebraic framework. Unless the specific set of axioms considered could depend on the underlying theory (especially on the spacetime considered), the assumptions listed below are very standard for the treatment of QFT's on Minkowski spacetime.

To start we call a double cone to any open region $\mathcal{O} \subset \mathbb{R}^d$ of Minkowski spacetime defined by the intersection of the future open null cone of some point $x \in \mathbb{R}^d$ with the past open null cone of other point $y \in \mathbb{R}^d$.[3] In AQFT, we start with a C^*-algebra $\mathfrak{A}$, called the *quasilocal algebra*, and we assign to each (nonempty) double cone $\mathcal{O} \subset \mathbb{R}^d$ a C^*-subalgebra $\mathfrak{A}(\mathcal{O}) \subset \mathfrak{A}$, which are called the *local algebra*s. This collection (net[4]) of local algebras must satisfy the following:

(1) Generating property: $\mathfrak{A} = \overline{\bigcup_{\mathcal{O}} \mathfrak{A}(\mathcal{O})}^{\|\cdot\|}$, where the union runs over the set of all double cones.
(2) Isotony: for any pair of double cones $\mathcal{O}_1 \subset \mathcal{O}_2$, then $\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$.
(3) Causality: if $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated (i.e., $\mathcal{O}_1 \sim \mathcal{O}_2$), then $[\mathfrak{A}(\mathcal{O}_1), \mathfrak{A}(\mathcal{O}_2)] = \{0\}$.
(4) Poincaré covariance: there is a (norm) continuous linear representation α_g of $\mathcal{P}_+^\uparrow$ in $\mathfrak{U}$, such that $\alpha_g(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(g\mathcal{O})$ for any open bounded region $\mathcal{O}$ and all $g \in \mathcal{P}_+^\uparrow$, where the action of $g \in \mathcal{P}_+^\uparrow$ over a region $\mathcal{O}$ is given by $g\mathcal{O} \coloneqq \{\Lambda x + a : x \in \mathcal{O}\}$.
(5) Vacuum: there is a pure state ω in $\mathfrak{A}$ invariant under all α_g. Then, in its GNS representation $(\pi, \mathcal{H}, \Omega)$ the linear representation α_g is implemented by a positive energy unitary representation of $\mathcal{P}_+^\uparrow$ in $\mathcal{H}$ in the sense that $U(g)\pi(A)U(g)^* = \pi(\alpha_g(A))$ for all $A \in \mathfrak{A}$ and all $g \in \mathcal{P}_+^\uparrow$. Positive energy means that the representation is strongly continuous and the infinitesimal generators P^μ of the translation subgroup (i.e. $U(0,a) = e^{iP^\mu a_\mu}$) have their spectral projections on the *closed forward light cone* $\bar{V}_+ \coloneqq \{p \in \mathbb{R}^d : p \cdot p > 0 \text{ and } p^0 > 0\}$.

For a general open region (possibly unbounded) $\mathcal{O} \subset \mathbb{R}^d$, we define $\mathfrak{A}(\mathcal{O}) \coloneqq \overline{\bigcup_{\tilde{\mathcal{O}} \subset \mathcal{O}} \mathfrak{A}(\tilde{\mathcal{O}})}^{\|\cdot\|}$ where the union runs over the set of all double cones $\tilde{\mathcal{O}} \subset \mathcal{O}$.

When we want to study states which are constructed by local perturbations around the vacuum state ω, we often work directly by the collection of concrete C^*-algebras $\pi(\mathfrak{A}(\mathcal{O})) \subset \mathcal{B}(\mathcal{H})$ acting on the vacuum Hilbert space $\mathcal{H}$. For technical reasons, we usually work with the net of von Neumann algebras $\mathcal{R}(\mathcal{O}) \coloneqq \pi(\mathfrak{A}(\mathcal{O}))''$, where $''$ denotes the double commutant which coincides with the weak closure. Moreover, when we want to construct a concrete example of a QFT satisfying the axioms above, it is, in general, easier to construct a net of von Neumann algebras $\mathcal{O} \to \mathcal{R}(\mathcal{O})$ acting on a given Hilbert space.

One immediate consequence of the axioms is the Reeh-Schlieder theorem. Before we state it we need to introduce some definitions. For any open region $\mathcal{O} \subset \mathbb{R}^d$, we define its (open) *spacelike complement* as

$$\mathcal{O}' \coloneqq \mathrm{Int}\{x \in \mathbb{R}^d : (x-y)^2 < 0, \quad \forall\ y \in \mathcal{O}\}. \tag{3.1}$$

Let $\mathcal{R} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. We say that a vector $\Phi \in \mathcal{H}$ is cyclic if $\overline{\mathcal{R}\Phi} = \mathcal{H}$, and separating if $A\Phi = 0$ with $A \in \mathcal{R}$ implies $A = 0$.

Theorem 3.1: (*Reeh-Schlieder* [26]) In any QFT satisfying the axioms 1. to 5. above, the vacuum vector Ω is cyclic for any algebra $\pi(\mathfrak{A}(\mathcal{O}))''$ corresponding to any (nonempty) open region. Moreover, if $\mathcal{O}'$ is also open and nonempty, then Ω is also separating for $\pi(\mathfrak{A}(\mathcal{O}))''$.

In the following subsections, we concretely define the net of algebras associated to a free Hermitian scalar field satisfying the axioms listed above.

B. Local algebras for spacetime regions

The algebraic theory of the real scalar field is defined as a net of von Neumann algebras acting in the Fock Hilbert space $\mathcal{H}$. This space is constructed as the (symmetric) tensor product of the one-particle Hilbert space. To describe it properly, we introduce the following three vector spaces.

The space of test functions. The space of test functions is the Schwartz space $\mathcal{S}(\mathbb{R}^d, \mathbb{R})$ of real, smooth and exponentially decreasing functions at infinity. This space carries naturally a representation of the restricted Poincaré group $\mathcal{P}_+^\uparrow$ given by $f \mapsto f_{(\Lambda,a)}$, with $f_{(\Lambda,a)}(x) \coloneqq f(\Lambda(x-a))$ for any $(\Lambda, a) \in \mathcal{P}_+^\uparrow$.

The one-particle Hilbert space. The Hilbert space $\mathfrak{H}$ of one-particle states of mass $m > 0$ and zero spin is made up of the square-integrable functions on the mass shell hyperboloid $H_m \coloneqq \{p \in \mathbb{R}^d : p^2 = m^2, p^0 > 0\}$ with the Poincaré invariant measure $d\mu(p) \coloneqq \Theta(p^0) \times \delta(p^2 - m^2) d^d p$. It can be realized as

$$\mathfrak{H} = L^2\left(\mathbb{R}^{d-1}, \frac{d^{d-1}p}{2\omega(\bar{p})}\right), \tag{3.2}$$

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle_{\mathfrak{H}} = \int_{\mathbb{R}^{d-1}} \frac{d^{d-1}p}{2\omega(\bar{p})} \boldsymbol{f}(\bar{p})^* \boldsymbol{g}(\bar{p}), \tag{3.3}$$

[3] In particular, if y is not in the timelike future of x, then $\mathcal{O} = \emptyset$.
[4] Mathematically, due to axiom 1, the collection of local algebras forms a net indexed by the set of double cones. The set of double cones forms a direct set when it is ordered by the usual set inclusion.

where $p^0 = \sqrt{\bar{p}^2 + m^2} =: \omega(\bar{p})$. Such a space carries a unitary representation of $\mathcal{P}_+^\uparrow$ given by $(u(\Lambda, a)\boldsymbol{f})(p) = \mathrm{e}^{ip\cdot a}\boldsymbol{f}(\Lambda^{-1}p)$ for any $\boldsymbol{f} \in \mathfrak{H}$ and $(\Lambda, a) \in \mathcal{P}_+^\uparrow$.

The Fock Hilbert space. The Fock Hilbert space $\mathcal{H}$ is the direct sum of the symmetric tensor powers of the one-particle Hilbert space $\mathfrak{H}$, i.e.,

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathfrak{H}^{\otimes n,\mathrm{sym}}. \tag{3.4}$$

For each $\boldsymbol{h} \in \mathfrak{H}$, the creation and annihilation operators $A^*(\boldsymbol{h})$ and $A(\boldsymbol{h})$ act over $\mathcal{H}$ as usual. The Fock space naturally inherits from $\mathfrak{H}$ a unitary representation of $\mathcal{P}_+^\uparrow$ which is denoted by $U(\Lambda, a)$. According to that there is a unique (up to a phase) $\mathcal{P}_+^\uparrow$-invariant vector denoted by $\Omega = \mathbf{1} \in \mathfrak{H}^{\otimes 0}$, which is called the *vacuum vector*. For each $\boldsymbol{h} \in \mathfrak{H}$, the normalized vector

$$\mathrm{e}^{\boldsymbol{h}} := \mathrm{e}^{-\frac{1}{2}\|\boldsymbol{h}\|^2_{\mathfrak{H}}} \sum_{n=0}^{\infty} \frac{\boldsymbol{h}^{\otimes n}}{\sqrt{n!}} \in \mathcal{H},$$

is called a *coherent vector*, and it satisfies the relations $\mathrm{e}^{\mathbf{0}} = \Omega$ and $\langle \Omega, \mathrm{e}^{\boldsymbol{h}} \rangle_{\mathcal{H}} = \mathrm{e}^{-\frac{1}{2}\|\boldsymbol{h}\|^2_{\mathfrak{H}}}$.

It is a very well-known fact that the structure of a free QFT is completely determined by the underlying one-particle quantum theory. More concretely, the assignment $\mathcal{O} \to \mathcal{R}(\mathcal{O})$ is determined by the composition of two different maps,

$$\mathcal{O} \subset \mathbb{R}^d \to K(\mathcal{O}) \subset \mathfrak{H}, \tag{3.5}$$

$$K \subset \mathfrak{H} \to \mathcal{R}(K) \subset \mathcal{B}(\mathcal{H}), \tag{3.6}$$

which are called first and second quantization maps, respectively. We treat each map separately.

1. First quantization map

Given any open region $\mathcal{O} \subset \mathbb{R}^d$, we remember that $\mathcal{O}'$ denotes its spacelike complement, and then we define its *causal completation* as $\mathcal{O}''$.[5] A region $\mathcal{O} \subset \mathbb{R}^d$ is called *causally complete* if $\mathcal{O} \equiv \mathcal{O}''$. In particular, any double cone is causally complete. From now on, we work with causally complete regions.

For any closed linear subspace $K \subset \mathfrak{H}$ we define its the *symplectic complemen*t as

$$K' := \{h \in \mathfrak{H} : \mathrm{Im}\langle h, k \rangle_{\mathfrak{H}} = 0, \quad \text{for all } k \in K\}. \tag{3.7}$$

Now, we consider the following real dense embedding $E : \mathcal{S}(\mathbb{R}^d, \mathbb{R}) \to \mathfrak{H}$,

[5]It is always true that $\mathcal{O} \subset \mathcal{O}''$.

$$(Ef)(\bar{p}) := \sqrt{2\pi}\hat{f}|_{H_m}(\bar{p}) = \sqrt{2\pi}\hat{f}(\omega(\bar{p}), \bar{p}), \tag{3.8}$$

where $\hat{f}(p) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(x)\mathrm{e}^{ip\cdot x} d^d x$ is the usual Fourier transform. Such embedding is Poincaré covariant, i.e., $E(f_{(\Lambda,a)}) = u(\Lambda, a)E(f)$. From now on, we naturally identified functions on $\mathcal{S}(\mathbb{R}^d, \mathbb{R})$ with vectors on $\mathfrak{H}$ through the above embedding.

The first quantization map is assignment $\mathcal{O} \subset \mathbb{R}^d \to K(\mathcal{O}) \subset \mathfrak{H}$, where $K(\mathcal{O})$ is a real closed linear subspace. It is defined by

$$\begin{aligned} &\mathcal{O} \subset \mathbb{R}^d \to K(\mathcal{O}) \\ &\quad := \overline{\{E(f) : f \in \mathcal{S}(\mathbb{R}^d, \mathbb{R}) \text{ and } \mathrm{supp}(f) \subset \mathcal{O}\}} \subset \mathfrak{H}. \end{aligned} \tag{3.9}$$

It is not difficult to see that this satisfies the duality $K(\mathcal{O}') = K(\mathcal{O})'$.

2. Second quantization map

We define the embedding $W : \mathfrak{H} \to \mathcal{B}(\mathcal{H})$,

$$W(\boldsymbol{h}) := \mathrm{e}^{i(A(\boldsymbol{h}) + A^*(\boldsymbol{h}))}. \tag{3.10}$$

The operators $W(\boldsymbol{h})$ are called *Weyl unitaries*. These operators satisfy the canonical commutation relations (CCR) in Segal's form [27]

$$W(\boldsymbol{h}_1)W(\boldsymbol{h}_2) = \mathrm{e}^{-i\mathrm{Im}\langle \boldsymbol{h}_1, \boldsymbol{h}_2 \rangle_{\mathfrak{H}}} W(\boldsymbol{h}_1 + \boldsymbol{h}_2), \tag{3.11}$$

$$W(\boldsymbol{h})^* = W(-\boldsymbol{h}). \tag{3.12}$$

A Poincaré unitary $U(\Lambda, a)$ acts covariantly on a Weyl operator according to

$$U(\Lambda, a)W(\boldsymbol{h})U(\Lambda, a)^* = W(u(\Lambda, a)\boldsymbol{h}), \tag{3.13}$$

$$W(\boldsymbol{h})\Omega = \mathrm{e}^{i\boldsymbol{h}}. \tag{3.14}$$

The second quantization map is an assignment $K \subset \mathfrak{H} \to \mathcal{R}(K) \subset \mathcal{B}(\mathcal{H})$, from the set of real closed linear subspace of $\mathfrak{H}$ to the set of von Neumann subalgebras of $\mathcal{B}(\mathcal{H})$. It is defined as

$$K \in \mathfrak{H} \to \mathcal{R}(K) := \{W(\boldsymbol{k}) : \boldsymbol{k} \in K\}'' \subset \mathcal{B}(\mathcal{H}). \tag{3.15}$$

This map satisfies the duality $\mathcal{R}(K') = \mathcal{R}(K)'$.

3. Net of local algebras

According to the above discussion, the net of local algebras $\mathcal{O} \subset \mathbb{R}^d \to \mathcal{R}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H})$ of the free Hermitian scalar field is defined as the composition of the first and second quantization maps, i.e.,

$$\mathcal{R}(\mathcal{O}) \coloneqq \mathcal{R}(K(\mathcal{O})). \tag{3.16}$$

This net satisfies all the axioms listed above, including the Haag duality. For $f \in \mathcal{S}(\mathbb{R}^d, \mathbb{R})$, the field operator $\phi(f)$ is defined through the relation

$$W(E(f)) = \mathrm{e}^{i\phi(f)} \eqqcolon W(f).$$

C. Local algebras at a fixed time

In this subsection, we discuss the theory of the von Neumann algebras for the real scalar free field at a fixed time $x^0 = 0$. Naively speaking, they are the local algebras generated by the field operator at a fixed time $\varphi(\bar{x})$ and its canonical conjugate momentum field $\pi(\bar{x})$. This theory is very useful for the computation of the relative entropy in Sec. V.

We can decompose $\mathfrak{H} = \mathfrak{H}_\varphi \oplus_{\mathbb{R}} \mathfrak{H}_\pi$ into two $\mathbb{R}$-linear closed subspaces,

$$\mathfrak{H}_\varphi \coloneqq \{\boldsymbol{h} \in \mathfrak{H} : \boldsymbol{h}(\bar{p}) = \boldsymbol{h}(-\bar{p})^* \text{a.e.}\}, \tag{3.17}$$

$$\mathfrak{H}_\pi \coloneqq \{\boldsymbol{h} \in \mathfrak{H} : \boldsymbol{h}(\bar{p}) = -\boldsymbol{h}(-\bar{p})^* \text{a.e.}\}. \tag{3.18}$$

Each $\boldsymbol{h} \in \mathfrak{H}$ can be uniquely written as $\boldsymbol{h} = \boldsymbol{h}_\varphi + \boldsymbol{h}_\pi$, where

$$\boldsymbol{h}_\varphi(\bar{p}) = \frac{\boldsymbol{h}(\bar{p}) + \boldsymbol{h}(-\bar{p})^*}{2} \quad \text{and} \quad \boldsymbol{h}_\pi(\bar{p}) = \frac{\boldsymbol{h}(\bar{p}) - \boldsymbol{h}(-\bar{p})^*}{2}. \tag{3.19}$$

We also have the useful relations

$$\mathrm{Im}\langle \boldsymbol{h}_\varphi, \boldsymbol{h}'_\varphi\rangle = \mathrm{Im}\langle \boldsymbol{h}_\pi, \boldsymbol{h}'_\pi\rangle = \mathrm{Re}\langle \boldsymbol{h}_\varphi, \boldsymbol{h}'_\pi\rangle = 0, \tag{3.20}$$

for all $\boldsymbol{h}_\varphi, \boldsymbol{h}'_\varphi \in \mathfrak{H}_\varphi$ and $\boldsymbol{h}_\pi, \boldsymbol{h}'_\pi \in \mathfrak{H}_\pi$.

Now, we consider the following real dense embeddings $E_{\varphi,\pi} : \mathcal{S}(\mathbb{R}^{d-1}, \mathbb{R}) \to \mathfrak{H}_{\varphi,\pi}$,

$$(E_\varphi f)(\bar{p}) \coloneqq \hat{f}(\bar{p}) \quad \text{and} \quad (E_\pi f)(\bar{p}) \coloneqq i\omega(\bar{p})\hat{f}(\bar{p}), \tag{3.21}$$

where $\hat{f}(\bar{p}) \coloneqq (2\pi)^{-\frac{d-1}{2}} \int_{\mathbb{R}^{d-1}} f(\bar{x}) \mathrm{e}^{-i\bar{p}\cdot\bar{x}} d^{d-1}x$. From now on, we naturally identify functions on $\mathcal{S}(\mathbb{R}^{d-1}, \mathbb{R})$ with vectors on $\mathfrak{H}_\varphi, \mathfrak{H}_\pi$ through these embeddings. The map E_φ (respectively, E_π) is actually defined on a bigger class of test functions, namely $H^{-\frac{1}{2}}(\mathbb{R}^{d-1}, \mathbb{R})$ [respectively, $H^{\frac{1}{2}}(\mathbb{R}^{d-1}, \mathbb{R})$], i.e.,

$$E_\varphi : H^{-\frac{1}{2}}(\mathbb{R}^{d-1}, \mathbb{R}) \to \mathfrak{H}_\varphi \quad \text{and} \quad E_\pi : H^{\frac{1}{2}}(\mathbb{R}^{d-1}, \mathbb{R}) \to \mathfrak{H}_\pi, \tag{3.22}$$

where $H^\alpha(\mathbb{R}^{d-1}, \mathbb{R})$ is the real Sobolev space of order α (see Appendix A 1). We have that $E_\varphi(H^{\frac{1}{2}}(\mathbb{R}^{d-1}, \mathbb{R})) = \mathfrak{H}_\varphi$ and $E_\pi(H^{-\frac{1}{2}}(\mathbb{R}^{d-1}, \mathbb{R})) = \mathfrak{H}_\pi$. For each $\boldsymbol{h}_\varphi \in \mathfrak{H}_\varphi$ and $\boldsymbol{h}_\pi \in \mathfrak{H}_\pi$ and using the map (3.10), we define the Weyl unitaries

$$W_\varphi(\boldsymbol{h}_\varphi) \coloneqq W(\boldsymbol{h}_\varphi) \quad \text{and} \quad W_\pi(\boldsymbol{h}_\pi) \coloneqq W(\boldsymbol{h}_\pi), \tag{3.23}$$

which satisfy the CCR in the Weyl form [27]

$$\begin{aligned} &W_\varphi(\boldsymbol{h}_\varphi)W_\pi(\boldsymbol{h}_\pi)W_\varphi(\boldsymbol{h}'_\varphi)W_\pi(\boldsymbol{h}'_\pi) \\ &\quad = W_\varphi(\boldsymbol{h}_\varphi + \boldsymbol{h}'_\varphi)W_\pi(\boldsymbol{h}_\pi + \boldsymbol{h}'_\pi)\mathrm{e}^{2i\mathrm{Im}\langle \boldsymbol{h}'_\varphi, \boldsymbol{h}_\pi\rangle_{\mathfrak{H}}} \end{aligned} \tag{3.24}$$

$$W_\varphi(\boldsymbol{h}_\varphi)^* = W_\varphi(-\boldsymbol{h}_\varphi) \tag{3.25}$$

$$W_\pi(\boldsymbol{h}_\pi)^* = W_\pi(-\boldsymbol{h}_\pi), \tag{3.26}$$

The field operator at a fixed time $\varphi(f_\varphi)$ and its canonical conjugate momentum field $\pi(f_\pi)$ are defined through the formulas

$$\begin{aligned} &W_\varphi(E_\varphi(f_\varphi)) = \mathrm{e}^{i\varphi(f_\varphi)} \eqqcolon W_\varphi(f_\varphi) \quad \text{and} \\ &W_\pi(E_\pi(f_\pi)) \eqqcolon \mathrm{e}^{i\pi(f_\pi)} \eqqcolon W_\pi(f_\pi). \end{aligned} \tag{3.27}$$

Here again, the local algebras at a fixed time are also defined through the first and second quantization map.

First quantization map. We say that $\mathcal{C} \subset \mathbb{R}^{d-1}$ is a spatially complete region if it is open, regular[6] and with regular boundary.[7] Here we work with this kind of region. Given any such region $\mathcal{C} \subset \mathbb{R}^{d-1}$, we define its (open) space complement as $\mathcal{C}' \coloneqq \mathbb{R}^{d-1} - \bar{\mathcal{C}}$.

Then the first quantization map is defined as

$$\mathcal{C} \subset \mathbb{R}^{d-1} \to K_\varphi(\mathcal{C}) \coloneqq \overline{\{E_\varphi(f) : f \in \mathcal{S}(\mathbb{R}^{d-1}, \mathbb{R}) \quad \text{and} \quad \mathrm{suppp}(f) \subset \mathcal{C}\}} \subset \mathfrak{H}_\varphi, \tag{3.28}$$

$$\mathcal{C} \subset \mathbb{R}^{d-1} \to K_\pi(\mathcal{C}) \coloneqq \overline{\{E_\pi(f) : f \in \mathcal{S}(\mathbb{R}^{d-1}, \mathbb{R}) \quad \text{and} \quad \mathrm{supp}(f) \subset \mathcal{C}\}} \subset \mathfrak{H}_\pi. \tag{3.29}$$

It can be shown that

[6]An open set $U \subset \mathbb{R}^n$ is regular if $U \equiv Int(\bar{U})$.

[7]The boundary $\partial\mathcal{C} \subset \mathbb{R}^{d-1}$ is a smooth submanifold of dimension $d-2$, or several manifolds joined together along smooth manifolds of dimension $d-3$ [28].

$$K_\varphi(\mathcal{C}) = \{E_\varphi(f) : f \in H^{-\frac{1}{2}}(\mathbb{R}^{d-1}, \mathbb{R}) \quad \text{and} \quad \mathrm{supp}(f) \subset \bar{\mathcal{C}}\,\text{a.e.}\}, \tag{3.30}$$

$$K_\pi(\mathcal{C}) = \{E_\pi(f) : f \in H^{\frac{1}{2}}(\mathbb{R}^{d-1}, \mathbb{R}) \quad \text{and} \quad \mathrm{supp}(f) \subset \bar{\mathcal{C}}\,\text{a.e.}\}. \tag{3.31}$$

Second quantization map. For each pair $K_\varphi \subset \mathfrak{H}_\varphi$ and $K_\pi \subset \mathfrak{H}_\pi$ of $\mathbb{R}$-linear closed subspaces, we define the von Neumann algebra

$$(K_\varphi, K_\pi) \to \mathcal{R}_0(K_\varphi, K_\pi) \coloneqq \{W_\varphi(k_\varphi) W_\pi(k_\pi) : k_\varphi \in K_\varphi, k_\pi \in K_\pi\}'' \subset \mathcal{B}(\mathcal{H}). \tag{3.32}$$

Net of local algebras at a fixed time. The net of local algebras $\mathcal{C} \subset \mathbb{R}^{d-1} \to \mathcal{R}_0(\mathcal{C}) \subset \mathcal{B}(\mathcal{H})$ of the free Hermitian scalar field at a fixed time is then defined as the composition of the first and second quantization maps, i.e.,

$$\mathcal{R}_0(\mathcal{C}) \coloneqq \mathcal{R}_0(K_\varphi(\mathcal{C}), K_\pi(\mathcal{C})). \tag{3.33}$$

The above net satisfies the following expected properties [28]:

$$\mathcal{R}_0(\mathcal{C}_1) \subset \mathcal{R}_0(\mathcal{C}_2) \quad \text{if } \mathcal{C}_1 \subset \mathcal{C}_2, \tag{3.34}$$

$$\mathcal{R}_0(\mathcal{C}_1 \cup \mathcal{C}_2) = \mathcal{R}_0(\mathcal{C}_1) \vee \mathcal{R}_0(\mathcal{C}_2), \tag{3.35}$$

$$\mathcal{R}_0(\mathcal{C}') = \mathcal{R}_0(\mathcal{C})', \tag{3.36}$$

$$\mathcal{R}_0(\mathbb{R}^{d-1}) = \mathcal{B}(\mathcal{H}), \tag{3.37}$$

where the $\mathcal{R}_1 \vee \mathcal{R}_2$ is the von Neumann algebra generated by $\mathcal{R}_1 \cup \mathcal{R}_2$.

D. Relation between the two approaches

In this subsection, we explain the relation existing between the two approaches of Secs. III B and III C.

The relation between nets. Given any spatially complete region $\mathcal{C} \subset \mathbb{R}^{d-1}$, we define its domain of dependence $\mathcal{O}_\mathcal{C} \subset \mathbb{R}^d$ as

$$\mathcal{O}_\mathcal{C} \coloneqq \{x \in \mathbb{R}^d : (x - (0, \bar{y}))^2 < 0 \ \text{ for all } \bar{y} \in \mathcal{C}'\}. \tag{3.38}$$

Then the following relation holds [29],

$$K(\mathcal{O}_\mathcal{C}) = K_\varphi(\mathcal{C}) \oplus_\mathbb{R} K_\pi(\mathcal{C}), \tag{3.39}$$

and hence we have the equality between the von Neumann algebras

$$\mathcal{R}_0(\mathcal{C}) = \mathcal{R}(\mathcal{O}_\mathcal{C}). \tag{3.40}$$

The relations developed along the above subsections can be summarized in the following schematic diagram:

$$\begin{array}{ccccc} \mathcal{O}_\mathcal{C} \subset \mathbb{R}^d & \xrightarrow{E} & K \subset \mathfrak{H} & \xrightarrow{W} & \mathcal{R} \subset \mathcal{B}(\mathcal{H}) \\ \uparrow & & \uparrow \oplus_\mathbb{R} & & \| \\ \mathcal{C} \subset \mathbb{R}^{d-1} & \xrightarrow{E_{\varphi,\pi}} & (K_\varphi, K_\pi) \subset \mathfrak{H}_\varphi \oplus_\mathbb{R} \mathfrak{H}_\pi & \xrightarrow{W_{\varphi,\pi}} & \mathcal{R}_0 \subset \mathcal{B}(\mathcal{H}). \end{array} \tag{3.41}$$

The relation between test functions. Given $f \in \mathcal{S}(\mathbb{R}^d, \mathbb{R})$ we can define

$$F(x) \coloneqq \int_{\mathbb{R}^d} \Delta(x - y) f(y) d^d y, \tag{3.42}$$

where $\Delta(x) \coloneqq -i(2\pi)^{-(d-1)} \int_{\mathbb{R}^d} e^{-ip\cdot x} \delta(p^2 - m^2) sgn(p^0) \times d^d p$. Indeed $(\Box + m^2)F = 0$ and we can take its initial Cauchy data at $x^0 = 0$ through

$$f_\varphi(\bar{x}) = -\frac{\partial F}{\partial x^0}(0, \bar{x}) \quad \text{and} \quad f_\pi(\bar{x}) = F(0, \bar{x}). \tag{3.43}$$

Finally, it can be shown $f_\varphi, f_\pi \in \mathcal{S}(\mathbb{R}^{d-1}, \mathbb{R})$ and

$$E(f) = E_\varphi(f_\varphi) + E_\pi(f_\pi). \tag{3.44}$$

Moreover, since $F(x) = 0$ if $x \in \mathrm{supp}(f)'$, then we have $\mathrm{supp}(f) \subset \mathcal{O}_\mathcal{C} \Rightarrow \mathrm{supp}(f_\varphi), \mathrm{supp}(f_\pi) \subset \mathcal{C}$.

The relation between Weyl unitaries. For the particular case of Weyl unitaries, it follows that

$$W(f) = e^{i\mathrm{Im}\langle f_\varphi, f_\pi\rangle_\mathfrak{H}} W_\varphi(f_\varphi) W_\pi(f_\pi), \tag{3.45}$$

where

$$\mathrm{Im}\langle f_\varphi, f_\pi\rangle_\mathfrak{H} = \frac{1}{2}\int_{\mathbb{R}^{d-1}} f_\varphi(\bar{x}) f_\pi(\bar{x}) d^{d-1}x. \tag{3.46}$$

IV. MODULAR THEORY

In this section, we discuss the key points of the modular theory in the framework of vN algebras. The main purpose of this section is to introduce the Araki formula for the relative entropy. For more details about the content of this section, see e.g., [15,30–33].

A. Modular Hamiltonian and modular flow

Lemma 4.1: Let $\mathcal{R} \subset \mathcal{B}(\mathcal{H})$ be a vN algebra and $\Omega \in \mathcal{H}$ be a cyclic and separating vector. Then there exists a unique closed antilinear (generally unbounded) operator S_Ω such that

$$S_\Omega A\Omega = A^*\Omega, \quad \forall \ A \in \mathcal{R}. \tag{4.1}$$

The operator S_Ω is called the *modular involution* associated to the pair $\{\mathcal{R}, \Omega\}$.

Let $S_\Omega = J_\Omega \Delta_\Omega^{\frac{1}{2}}$ be the polar decomposition of S_Ω. Then, Δ_Ω (positive self-adjoint and generally unbounded) is called the modular operator and J_Ω (antiunitary) is called the modular conjugation. Finally, the modular Hamiltonian is defined as

$$K_\Omega \coloneqq -\log(\Delta_\Omega), \tag{4.2}$$

and the one-parameter (strongly continuous) group of unitaries Δ_Ω^{it} is called the modular flow.

Theorem 4.2: (*Tomita-Takesaki*) Let $\mathcal{R} \subset \mathcal{B}(\mathcal{H})$ be a vN algebra, $\Omega \in \mathcal{H}$ be a cyclic and separating vector and $S_\Omega = J_\Omega \Delta_\Omega^{\frac{1}{2}}$ be the operator defined above. The one-parameter (strongly continuous) group of unitaries Δ_Ω^{it} is called modular group or modular flow. The Tomita-Takesaki theorem states that

$$J_\Omega \mathcal{R} J_\Omega = \mathcal{R}' \tag{4.3}$$

$$\Delta_\Omega^{it} \mathcal{R} \Delta_\Omega^{-it} = \mathcal{R} \quad \text{and} \quad \Delta_\Omega^{it} \mathcal{R}' \Delta_\Omega^{-it} = \mathcal{R}', \tag{4.4}$$

for all $t \in \mathbb{R}$.

Remark 4.3.—In general, the modular flow Δ_Ω^{it} does not belong to $\mathcal{R}$ or $\mathcal{R}'$.

Before we state the Bisognano-Whichmann theorem, we need to introduce some conventions. Let $\mathcal{W} \coloneqq \{x \in \mathbb{R}^d : x^1 > |x^0|\}$ be the right Rindler wedge and $\Sigma \coloneqq \{\bar{x} \in \mathbb{R}^{d-1} : x^1 \geq 0\}$. Then by (3.38) we have that $\mathcal{O}_\Sigma = \mathcal{W}$. From now on, we denote the orthogonal coordinates to the Rindler wedge as $\bar{x}_\perp \coloneqq (x^2, \ldots, x^{d-1})$ and hence any spacetime point can be expressed as $x = (x^0, x^1, \bar{x}_\perp)$. We also denote the following vN algebras simply as

$$\mathcal{R}_\mathcal{W} \coloneqq \mathcal{R}(\mathcal{W}) = \mathcal{R}_0(\Sigma), \tag{4.5}$$

$$\mathcal{R}_{\mathcal{W}'} \coloneqq \mathcal{R}(\mathcal{W}') = \mathcal{R}_0(\Sigma'). \tag{4.6}$$

From relations (3.34)–(3.37) we have

$$\mathcal{R}_\mathcal{W}' = \mathcal{R}_{\mathcal{W}'} \quad \text{and} \quad \mathcal{R}_\mathcal{W} \vee \mathcal{R}_{\mathcal{W}'} = \mathcal{B}(\mathcal{H}). \tag{4.7}$$

Reeh-Schlieder theorem 3. 1 asserts that the vacuum vector Ω is cyclic and separating for $\mathcal{R}_\mathcal{W}$.

Theorem 4.4: (*Bisognano-Wichmann [33]*) The modular operator Δ_Ω and the modular conjugation J_Ω for the pair $\{\mathcal{R}_\mathcal{W}, \Omega\}$ are

$$J_\Omega = \Theta U(R_1(\pi)) \quad \text{and} \quad \Delta_\Omega = \mathrm{e}^{-2\pi K_1}, \tag{4.8}$$

where Θ is the *CPT* operator, $U(R_1(\pi))$ is the Lorentz unitary operator representing a space rotation of angle π along the x^1 axes and K_1 is the infinitesimal generator of the one-parameter group of boost symmetries in the plane (x^0, x^1), i.e.,

$$U(\Lambda_1^s, 0) = \mathrm{e}^{iK_1 s}, \quad \text{with}$$
$$\Lambda_1^s \coloneqq \begin{pmatrix} \cosh(s) & \sinh(s) & \mathbf{0} \\ \sinh(s) & \cosh(s) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}. \tag{4.9}$$

Remark 4.5.—Although we are working with the net of local algebras for the real scalar field, the above result holds for any relativistic QFT which satisfies the Wightman axioms.

B. Relative modular Hamiltonian and relative modular flow

Lemma 4.6: Let $\mathcal{R} \subset \mathcal{B}(\mathcal{H})$ be a vN algebra and two cyclic and separating vectors $\Omega, \Phi \in \mathcal{H}$. Then there exists a unique (generally unbounded) closed antilinear operator such that

$$S_{\Phi,\Omega} A\Omega = A^*\Phi, \quad \forall \ A \in \mathcal{R}. \tag{4.10}$$

The operator S_Ω is called the *relative modular involution* associated to the pair $\{\mathcal{R}, \Omega, \Phi\}$.

Let $S_{\Phi,\Omega} = J_{\Phi,\Omega} \Delta_{\Phi,\Omega}^{\frac{1}{2}}$ be the polar decomposition of $S_{\Phi,\Omega}$. Then, $\Delta_{\Phi,\Omega}$ is called the relative modular operator and $J_{\Phi,\Omega}$ (antiunitary) is called the relative modular conjugation. Then the relative modular Hamiltonian is defined as

$$K_{\Phi,\Omega} \coloneqq -\log(\Delta_{\Phi,\Omega}). \tag{4.11}$$

The relative modular flow $\Delta_{\Phi,\Omega}^{it}$ acts as the modular flow Δ_Φ^{it} for the algebra $\mathcal{R}$ and as Δ_Ω^{it} for the algebra $\mathcal{R}'$, i.e.,

$$\Delta_{\Phi,\Omega}^{it} A \Delta_{\Phi,\Omega}^{-it} = \Delta_\Phi^{it} A \Delta_\Phi^{-it} \quad A \in \mathcal{R}, \tag{4.12}$$

$$\Delta_{\Phi,\Omega}^{it} A' \Delta_{\Phi,\Omega}^{-it} = \Delta_\Omega^{it} A' \Delta_\Omega^{-it} \quad A' \in \mathcal{R}'. \tag{4.13}$$

The following theorem summarize the analytics properties of the relative modular flow.

Theorem 4.7: (*Kubo-Martin-Schwinger (KMS) condition [34]*) Under the hypothesis of the previous lemma, given any $A, B \in \mathcal{R}$, there exits a unique continuous function $G_{A,B} : \mathbb{R} + i[-1, 0] \to \mathbb{C}$, analytic on $\mathbb{R} + i(-1, 0)$ such that

$$G_{A,B}(t) = \langle \Omega, \Delta^{it}_{\Phi,\Omega} A \Delta^{-it}_{\Omega} B\Omega \rangle_{\mathcal{H}}, \tag{4.14}$$

$$G_{A,B}(t - i) = \langle \Phi, B\Delta^{it}_{\Phi,\Omega} A \Delta^{-it}_{\Omega} \Phi \rangle_{\mathcal{H}}, \tag{4.15}$$

for all $t \in \mathbb{R}$. Moreover, the function above is uniquely determined by one of its boundary values.

As it happens for the modular flow, $\Delta^{it}_{\Phi,\Omega} \notin \mathcal{R} \cup \mathcal{R}'$ in general. However, we can define the one-parameter family of unitaries[8]

$$u_{\Phi,\Omega}(t) = \Delta^{it}_{\Phi,\Omega} \Delta^{-it}_{\Omega}, \tag{4.16}$$

who belong to $u_{\Phi,\Omega}(t) \in \mathcal{R}$ for all $t \in \mathbb{R}$. This family of unitaries is best known as Connes Radon-Nikodym cocycle.

C. Araki formula for relative entropy

The definition of the relative entropy for a general von Neumann algebra is due to Araki [15].

Definition 4.8: Let be $\mathcal{R} \subset \mathcal{B}(\mathcal{H})$ be a vN algebra in standard form. For any given two ω, ϕ two faithful normal states, there exists cyclic and separating vector representatives $\Omega, \Phi \in \mathcal{H}$.[9] Then the relative entropy $S_R(\phi|\omega)$ is defined using the relative modular Hamiltonian $K_{\Omega,\Phi}$ as[10]

$$S_R(\phi|\omega) \coloneqq \langle \Phi, K_{\Omega,\Phi}\Phi \rangle_{\mathcal{H}}. \tag{4.17}$$

It can be shown that the above formula is independent of the choice of the vector representatives for the given states, and it also satisfies the well-known properties of strict positivity, monotonicity, convexity and lower semicontinuity. All these are well discussed in Araki's original work [15]. When the relative entropy is finite (in particular, when Ω belongs to the domain of $K_{\Omega,\Phi}$), the following useful expression holds:

$$S_R(\phi|\omega) = i \lim_{t\to 0} \frac{\langle \Phi, \Delta^{it}_{\Omega,\Phi}\Phi \rangle_{\mathcal{H}} - 1}{t}. \tag{4.18}$$

V. RELATIVE ENTROPY FOR COHERENT STATES

In this section, we compute the relative entropy between a coherent state and the vacuum for the Rindler wedge. Before doing that, we study some relations concerning the relative entropy which are valid for any kind of regions. These are explained in the following subsection.

A. Generalities

Coherent states come from acting with a Weyl operator to the vacuum vector. Weyl unitaries have the very interesting property that implements, by adjoint action, automorphism for any local algebra $\mathcal{R}(\mathcal{O})$. Indeed, for any $\boldsymbol{h} \in \mathfrak{H}$ and any Weyl operator $W(f) \in \mathcal{R}(\mathcal{O})$ ($\mathrm{supp}(f) \subset \mathcal{O}$) we have that

$$W(\boldsymbol{h})^* W(f) W(\boldsymbol{h}) = e^{2i\mathrm{Im}\langle f,\boldsymbol{h}\rangle_{\mathfrak{H}}} W(f) \in \mathcal{R}(\mathcal{O}), \tag{5.1}$$

which implies that $W(\boldsymbol{h})^* \mathcal{R}(\mathcal{O}) W(\boldsymbol{h}) = \mathcal{R}(\mathcal{O})$. This property has an interesting implication for the relative entropy itself. Indeed, it implies that the relative entropy between a coherent state and the vacuum is symmetric. In order to justify this property, we prove the following lemmas.

Lemma 5.1: Let $\mathcal{R} \subset \mathcal{B}(\mathcal{H})$ be a vN algebra and $\Omega, \Phi \in \mathcal{H}$ a cyclic and separating vectors and $U \in \mathcal{B}(\mathcal{H})$ unitary such $U^* \mathcal{R} U = \mathcal{R}$. Then,

(a) $U\Omega$ and $U\Phi$ are cyclic and separating.
(b) $S_{U\Omega} = US_\Omega U^* \Rightarrow \Delta_{U\Omega} = U\Delta_\Omega U^*$.
(c) $S_{U\Omega,U\Phi} = US_{\Omega,\Phi}U^* \Rightarrow \Delta_{U\Omega,U\Phi} = U\Delta_{\Omega,\Phi}U^*$.

Proof.—(1) $\overline{\mathcal{R}U\Omega} = \overline{U\mathcal{R}\Omega} = U\overline{\mathcal{R}\Omega} = \mathcal{H}$ implies that $U\Omega$ is cyclic. $AU\Omega = 0 \Leftrightarrow U^*AU\Omega = 0 \Leftrightarrow U^*AU = 0 \Leftrightarrow A = 0$ implies that $U\Omega$ is separating. Idem for $U\Phi$. (2) For any $A \in \mathcal{R}$, we have $(US_\Omega U^*)AU\Omega = US_\Omega(U^*AU)\Omega = U(U^*AU)^*\Omega = A^*U\Omega$. Then, applying the polar decomposition we have $\Delta_{U\Omega} = U\Delta_\Omega U^*$. (3) For any $A \in \mathcal{R}$, we have $(US_{\Omega,\Phi}U^*)AU\Phi = US_{\Omega,\Phi}(U^*AU)\Phi = U(U^*AU)^*\Omega = A^*U\Omega$. Then $\Delta_{U\Omega,U\Phi} = U\Delta_{\Omega,\Phi}U^*$ follows from the polar decomposition. ∎

Given a state ω of a vN algebra $\mathcal{R} \subset \mathcal{B}(\mathcal{H})$ and a unitary $U \in \mathcal{B}(\mathcal{H})$, we denote by ω_U the state defined through $\omega_U(\cdot) \coloneqq \omega(U^* \cdot U)$.

Lemma 5.2: Given $\mathcal{R} \subset \mathcal{B}(\mathcal{H})$ a vN algebra in standard form, ω a faithful normal state and $U \in \mathcal{B}(\mathcal{H})$ unitary such that $U^*\mathcal{R}U = \mathcal{R}$, then

$$S_R(\omega_U|\omega) = S_R(\omega|\omega_{U^*}). \tag{5.2}$$

Proof.—Let Ω be the cyclic and separating vector representative of ω. Then $U\Omega$, $U^*\Omega$ are the vector representatives of ω_U,ω_{U^*} and they are cyclic and separating because of 1. in lemma 5. 1. Using 3. of the same lemma we have $S_{\Omega,U\Omega} = S_{UU^*\Omega,U\Omega} = US_{U^*\Omega,\Omega}U^*$. Then $S_R(\omega_U|\omega) = \langle U\Omega, K_{\Omega,U\Omega}U\Omega\rangle_{\mathcal{H}} = \langle U\Omega, UK_{U^*\Omega,\Omega}U^*U\Omega\rangle_{\mathcal{H}} = \langle \Omega, K_{U^*\Omega,\Omega}\Omega\rangle_{\mathcal{H}} = S_R(\omega|\omega_{U^*})$. ∎

Now, we come back to coherent states. From now on $\omega(\cdot) = \langle \Omega, \cdot\Omega\rangle_{\mathcal{H}}$ denotes the vacuum state. And given any $f \in \mathcal{S}(\mathbb{R}^d, \mathbb{R})$ we define the coherent state $\omega_f(\cdot) = \langle \Omega, W(f)^* \cdot W(f)\Omega\rangle_{\mathcal{H}}$. The Reeh-Schlieder theorem asserts that the vacuum vector Ω is cyclic and separating

[8]This one-parameter family of operators is not a one-parameter group.

[9]In particular, choose them in the natural cone of the standard vector of $\mathcal{R}$.

[10]Contrary to the notation employed in Secs. I and II, on the lhs expression (4.17) we emphasize that the relative entropy depends on the states rather than the vector representatives used to define it. We use this new notation in the rest of the paper.

for any local algebra $\mathcal{R}(\mathcal{O})$, and lemma 5.1 ensures the same for the coherent vector $W(f)\Omega$. Then, lemma 5.2 implies

$$S_R(\omega_f|\omega) = S_R(\omega|\omega_{-f}), \tag{5.3}$$

for any coherent state ω_f and any local algebra $\mathcal{R}(\mathcal{O})$. Moreover, the net algebra of the free scalar field has a global $\mathbb{Z}_2$-symmetry implemented by an operator $z = z^{-1} = z^*$ such that[11]

$$zW(f)z = W(-f) = W(f)^*, \quad \text{and} \quad z\Omega = \Omega. \tag{5.4}$$

This motivates the following lemma.

Lemma 5.3: For any local algebra $\mathcal{R}(\mathcal{O})$, the relative entropy between a coherent state ω_f and the vacuum state ω satisfies

$$S_R(\omega_f|\omega) = S_R(\omega_{-f}|\omega). \tag{5.5}$$

Proof.—Let Ω, $W(f)\Omega$ and $W(f)^*\Omega$ be the vector representatives of the states ω, ω_f and ω_{-f}. If $S_{\Omega,f}$ is the relative modular involution associated to $\{\mathcal{R}(\mathcal{O}), W(f)\Omega, \Omega\}$ and employing the $\mathbb{Z}_2$-symmetry (5.4), we have that

$$\begin{aligned}(zS_{\Omega,f}z)W(g)W(f)^*\Omega &= zS_{\Omega,f}W(g)^*W(f)\Omega \\ &= zW(g)\Omega = W(g)^*\Omega, \end{aligned} \tag{5.6}$$

for all $W(g) \in \mathcal{R}(\mathcal{O})$. Then $S_{\Omega,-f} = zS_{\Omega,f}z$ and hence $K_{\Omega,-f} = zK_{\Omega,f}z$. Finally, $S_R(\omega_f|\omega) = \langle\Omega, K_{\Omega,f}\Omega\rangle_{\mathcal{H}} = \langle\Omega, K_{\Omega,-f}\Omega\rangle_{\mathcal{H}} = S_R(\omega_{-f}|\omega)$. ■

Remark 5.4.—The above lemma should apply to any scalar theory with $\mathbb{Z}_2$-symmetry as above, satisfying the Wightman axioms.

Finally, combining (5.3) and (5.5) we have the following theorem concerning the symmetry for the relative entropy between coherent states.

Theorem 5.5: For any local algebra $\mathcal{R}(\mathcal{O})$, the relative entropy between a coherent state ω_f and the vacuum state ω is symmetric, i.e.,

$$S_R(\omega_f|\omega) = S_R(\omega|\omega_f). \tag{5.7}$$

To end, we have the following theorem concerning the relative entropy between two coherent states.[12]

Theorem 5.6: For any local algebra $\mathcal{R}(\mathcal{O})$, the relative entropy between two coherent states ω_f and ω_g satisfies

$$S_R(\omega_f|\omega_g) = S_R(\omega_{f-g}|\omega). \tag{5.8}$$

Proof.—Let Ω, $W(f)\Omega$, $W(g)\Omega$ and the vector representatives of the states be ω, ω_f and ω_g. If $S_{g,f}$ is the relative modular involution associated to $\{\mathcal{R}(\mathcal{O}), W(f)\Omega, W(g)\Omega\}$, then because of 3. in lemma 5.1 we have that $S_{U\Omega,U\Psi} = W(g)^*S_{g,f}W(g)$ is the relative modular involution associated to $\{\mathcal{R}(\mathcal{O}), W(g)^*W(f)\Omega, \Omega\}$. Since $W(g)^*W(f)\Omega$ is a vector representative of ω_{f-g}, we have $S_R(\omega_f|\omega_g) = \langle W(f)\Omega, S_{g,f}W(f)\Omega\rangle_{\mathcal{H}} = S_R(\omega_{f-g}|\omega)$. ■

B. Relative entropy for the Rindler wedge

Let $\mathcal{R}_{\mathcal{W}}$ be the right Rindler wedge algebra, ω be the vacuum state and ω_f be a coherent state with $f \in \mathcal{S}(\mathbb{R}^d, \mathbb{R})$. Let us call Ω and $\Phi \coloneqq W(f)\Omega$ its vector representatives. The aim of this subsection is to compute the relative entropy $S_R(\omega_f|\omega)$, and for that we need to calculate the relative modular Hamiltonian $K_{\Omega,\Phi}$ (or $K_{\Phi,\Omega}$ according to theorem 5.5). As we explained in the last subsection, the vectors Ω and $\Phi \coloneqq W(f)\Omega$ are cyclic and separating. We distinguish between two cases,

$$\text{easy case}: \; f = f_L + f_R, \tag{5.9}$$

$$\text{hard case}: \; f \neq f_L + f_R, \tag{5.10}$$

where $\mathrm{supp}(f_L) \in \mathcal{W}'$ and $\mathrm{supp}(f_R) \in \mathcal{W}$.[13] In the following subsections, we deal with each case (5.9) and (5.10) separately.

1. Easy case: $f = f_L + f_R$

In this case, we have that the coherent vector can be written as $W(f) = W(f_L)W(f_R)$ with $W(f_L) \in \mathcal{R}_{\mathcal{W}'}$ and $W(f_R) \in \mathcal{R}_{\mathcal{W}}$. This case can be solved in general using the following lemma.

Lemma 5.7: Given $\mathcal{R} \subset \mathcal{B}(\mathcal{H})$ a vN algebra, Ω a cyclic and separating and $U \in \mathcal{R}$ and $U' \in \mathcal{R}'$ unitaries. Then $\Phi = U'U\Omega$ is cyclic and separating and

$$S_{\Omega,\Phi} = US_\Omega U'^*, \tag{5.11}$$

and by polar decomposition we have $J_{\Omega,\Phi} = UJ_\Omega U'^*$, $\Delta_{\Omega,\Phi} = U'\Delta_\Omega U'^*$ and $K_{\Omega,\Phi} = U'K_\Omega U'^*$.

Proof.—$\overline{\mathcal{R}\Phi\mathcal{H}} = \overline{\mathcal{R}U'U\Omega} = U'\overline{\mathcal{R}U\Omega} = U'\overline{\mathcal{R}_{\mathcal{W}}\Omega} = \overline{\mathcal{R}\Omega} = \mathcal{H}$ implies Φ is cyclic. Since the same argument holds for $\mathcal{R}'$, Φ is separating for $\mathcal{R}$. For any $A \in \mathcal{R}$, we have that $(US_\Omega U'^*)A\Phi = US_\Omega U'^*AU'U\Phi = U_R S_\Omega(AU)\Omega = U(AU)^*\Omega = A^*\Omega \Rightarrow S_{\Omega,\Phi} = US_\Omega U'^*$. ■

Corollary 5.8: In the context of the above lemma, if Ω and Φ are vector representatives of the states ω and ϕ, then $S_R(\phi|\omega) = \langle\Phi, U'K_\Omega U'^*\Phi\rangle_{\mathcal{H}} = \langle\Omega, U^*K_\Omega U\Omega\rangle_{\mathcal{H}}$.

The above corollary shows explicitly that the relative entropy does not depend on the unitary U'. This is expected

[11] In the Lagrangian approach to QFT, this is the usual symmetry $\phi(x) \to -\phi(x)$.

[12] This result has been found in the past using other methods. For example, see [35] for a derivation using the replica trick for $2d$ free CFTs.

[13] In particular, the easy case includes the cases when $W(f) \in \mathcal{R}_{\mathcal{W}}$ or $W(f) \in \mathcal{R}_{\mathcal{W}'}$.

because the relative entropy is a measure of indistinguishability of the states in $\mathcal{R}$, and indeed has to be invariant under changes of the states outside $\mathcal{R}$.

Now we apply the corollary 5.8 to the case of a coherent state, i.e., $U = W(f_R)$ with $\mathrm{supp}(f_R) \subset \mathcal{W}$. Remembering that the second quantized Poincaré unitary operator $U(\Lambda_1^s, 0) = e^{iK_1 s}$, acting on the Fock space $\mathcal{H}$, is constructed from the Poincaré unitary operator $u(\Lambda_1^s, 0) = e^{ik_1 s}$, acting on the one-particle Hilbert space $\mathfrak{H}$, then we have that

$$\begin{aligned} S_R(\phi|\omega) &= \langle \Omega, U^* K_\Omega U \Omega \rangle_{\mathcal{H}} = 2\pi \langle \Omega, W(f_R)^* K_1 W(f_R) \Omega \rangle_{\mathcal{H}} \\ &= 2\pi \langle f_R, k_1 f_R \rangle_{\mathfrak{H}}, \end{aligned} \tag{5.12}$$

where the last equality is fully calculated in Appendix A 2. Thus, the relative entropy between the coherent state and the vacuum, can be expressed, in the one-particle Hilbert space $\mathfrak{H}$, in terms of the expectation value of the boost operator k_1 in the vector $E(f) \in \mathfrak{H}$ which generates the coherent state. At the end, following from (5.12) we get the following theorem.

Theorem 5.9: Let $f_L, f_R \in \mathcal{S}(\mathbb{R}^d, \mathbb{R})$ with $\mathrm{supp}(f_L) \in \mathcal{W}$ and $\mathrm{supp}(f_R) \in \mathcal{W}'$, and $f = f_L + f_R$. Then the relative entropy between the coherent state ω_f and the vacuum ω, for the right Rindler wedge algebra $\mathcal{R}_\mathcal{W}$, is

$$\begin{aligned} & S_R(\omega_f|\omega) \\ & = 2\pi \int_{x^1>0} d^{d-1}x x^1 \frac{1}{2} \left(\left(\frac{\partial F}{\partial x^0} \right)^2 + |\nabla F|^2 + m^2 F^2 \right) \Bigg|_{x^0=0}, \end{aligned} \tag{5.13}$$

where $F(x) = \int_{\mathbb{R}^d} \Delta(x-y) f(y) d^d y = \int_{\mathbb{R}^d} \Delta(x-y)[f_L(y) + f_R(y)] d^d y$. In addition, formula (5.13) does not depend in the function f_L (with support in $\mathcal{W}'$) chosen.

Proof.—A straightforward calculation explained in Appendix A 3 allows us to rewrite the expression (5.12) as Eq. (A17). However, there are already two differences between (A17) and (5.13) (beyond the obvious 2π in front of the expression). The first one is that in (A17) the integral is along the whole space $\mathbb{R}^{d-1}$, and the second one is that the function F in (A17) is computed using only f_R. To finally pass from (A17) to (5.13) we have to make the following two changes. First notice that because $\mathrm{supp}(f_R) \subset \mathcal{W} \Rightarrow \mathrm{supp}(F|_{x^0=0}) \subset \Sigma$, this allows us to replace the integration region in (A17) by Σ. Similarly, because $\mathrm{supp}(f_L) \subset \mathcal{W}' \Rightarrow$ the function $F_L(x) := \int_{\mathbb{R}^d} \Delta(x-y) f_L(y) d^d y$ vanishes along Σ and hence (5.13) holds. This also implies that (5.13) does not depend on f_L. ∎

As a remark, the outcome of the above theorem coincides with (1.18) for the canonical stress tensor (1.10).

2. Hard case: $f \neq f_L + f_R$

In this section, we assume that the function $f \in \mathcal{S}(\mathbb{R}^d, \mathbb{R})$ has $\mathrm{supp}(f) \not\subset \mathcal{W}, \mathcal{W}'$. Moreover, we assume that $\mathrm{supp}(f)$ is compact in order to avoid some possible complications coming from integrals along regions of infinite size. At the end, we are interested in the behavior of the relative entropy around the boundary of the wedge region $\partial\Sigma = \{\bar{x} \in \mathbb{R}^{d-1} : x^1 = 0\}$, which can be captured with a compactly supported coherent state.

Before we continue, we remark that, in this case, the relative entropy must be finite. The proof is as follows. Since $\mathrm{supp}(f)$ is compact, there exists a "bigger" right wedge $\tilde{\mathcal{W}}_R \supset \mathcal{W}$ such that $W(f) \in \tilde{\mathcal{W}}_R$. Then the relative entropy between this coherent and the vacuum in the algebra $\mathcal{R}(\tilde{\mathcal{W}}_R)$ is as the one computed in the previous section, which is finite because the generating function f is smooth. Then by monotonicity, the relative entropy for the original wedge $\mathcal{W}$ must be finite. In particular, we are allowed to use expression (4.18).

The first question which arises is whether we could split the unitary into two unitaries, one belonging to the right wedge $\mathcal{W}$ and the other to the left wedge $\mathcal{W}'$. In other words, if there exists unitaries $U_R \in \mathcal{R}_\mathcal{W}$ and $U_L \in \mathcal{R}_{\mathcal{W}'}$ unitaries such that $W(f) = U_L U_R$. Unfortunately the answer is no, almost for the most general interesting case. This fact arises when we try to explicitly split $W(f)$. To begin, it seems natural to split the function f simply as

$$f_R(x) := \Theta_\mathcal{W}(x) f(x), \tag{5.14}$$

$$f_L(x) := \Theta_{\mathcal{W}'}(x) f(x), \tag{5.15}$$

where $\Theta_\mathcal{W}$ is the characteristic function of the right Rindler wedge (equivalently for $\Theta_{\mathcal{W}'}$). However, it leads to a wrong result, since $f_R + f_L \neq f$. Moreover, if for example we start with a function f supported in the upper light cone $V^+ := \{x \in \mathbb{R}^d : x^0 > |\bar{x}|\}$, then Eq. (5.14) implies that $f_R \equiv 0$ and hence we obtain $S_R(\phi|\omega) = 0$, which is obviously the wrong result. To make a consistent splitting, we must use the relations explained in Sec. III D. Given the spacetime function $f \in \mathcal{S}(\mathbb{R}^d, \mathbb{R})$ we can construct $f_\varphi, f_\pi \in \mathcal{S}(\mathbb{R}^{d-1}, \mathbb{R})$ satisfying the relation (3.45). The correct result is to split these functions f_φ, f_π, which are the initial data at $x^0 = 0$ of the Klein-Gordon solution generated by f. The assumption $\mathrm{supp}(f) \not\subset \mathcal{W}, \mathcal{W}'$ implies that an open neighborhood of the origin $x = 0$ is included in the supports of f_φ and f_π. Now, we write

$$f_\varphi = f_{\varphi,L} + f_{\varphi,R} \quad \text{and} \quad f_\pi = f_{\pi,L} + f_{\pi,R}, \tag{5.16}$$

with $\mathrm{supp}(f_{\nu,L}) \in \Sigma'$ and $\mathrm{supp}(f_{\nu,R}) \in \Sigma$ $(\nu = \varphi, \pi)$. The right way to do this is taking

$$f_{\nu,L}(\bar{x}) := f_\nu(\bar{x}) \cdot \Theta(-x^1) \quad \text{and} \quad f_{\nu,R}(\bar{x}) := f_\nu(\bar{x}) \cdot \Theta(x^1), \tag{5.17}$$

where Θ is the usual step Heaviside function. The problem is that $f_{\nu,L}$ and $f_{\nu,R}$ are no longer smooth, and nothing guarantees that $E_\nu(f_{\nu,R}) \in \mathfrak{H}_\nu$ (the same problem occurs for $f_{\nu,L}$). More precisely, since $f_{\nu,R} \in L^2(\mathbb{R}^{d-1},\mathbb{R}) = H^0(\mathbb{R}^{d-1},\mathbb{R})$, and because of the inclusions (see Appendix A 1)

$$H^{\frac{1}{2}}(\mathbb{R}^{d-1},\mathbb{R}) \subset H^0(\mathbb{R}^{d-1},\mathbb{R}) \subset H^{-\frac{1}{2}}(\mathbb{R}^{d-1},\mathbb{R}), \quad (5.18)$$

we have that $f_{\varphi,R} \in \mathfrak{H}_\varphi$ but $f_{\pi,R} \notin \mathfrak{H}_\pi$. In other words, $f_{\pi,R}$ is not an appropriate smear function for the canonical conjugate field $\pi(\bar{x})$. This problem does not arise because the test function is no longer smooth; it is just because $f_{\pi,R}$ is no longer continuous. On the other hand, if $f_{\pi,R}$ is continuous, the problem can be solved due to the following lemma.

Theorem 5.10: Let $f \in L^2(\mathbb{R}^n) \cap C^0(\mathbb{R}^n) \cap C^1_t(\mathbb{R}^n)$ and $\partial_j f \in L^2(\mathbb{R}^n)$ for $j = 1, \ldots, n$.[14] Then $f \in H^1(\mathbb{R}^n)$.

Proof.—See Appendix A 1. ∎

Then, having this in mind, the strategy we adopt below is to make a splitting for some other smear function which, by construction, we know is continuous.

3. A lemma for the relative modular flow

In this subsection, we prove a lemma that gives a general expression for the relative modular flow, under the assumption that some nonlocal operator can be written as a product of two new operators, one belonging to $\mathcal{R}$ and another to $\mathcal{R}'$. In the following subsection, we prove that this assumption is already valid for the free Hermitian scalar field. For simplicity and due to the symmetry relation (5.7), in the following we work with the modular operator $\Delta_{\Phi,\Omega}$ instead of $\Delta_{\Omega,\Phi}$.

As a motivation, we remember that, contrary to the modular flow Δ_Ω^{it} and the relative modular flow $\Delta_{\Phi,\Omega}^{it}$, the Connes Radon-Nikodym cocycle $u_{\Phi,\Omega}(t) = \Delta_{\Phi,\Omega}^{it}\Delta_\Omega^{it}$ belongs to the algebra $\mathcal{R}$. This makes us think that the computation of $u_{\Phi,\Omega}(t)$ may involve the splitting of some test function, which at the end, will lead to a well-defined operator. To gain some intuition, using lemmas 5. 1 and 5. 7, we know that

$$u_{\Phi,\Omega}(t) = U^*\Delta_\Omega^{it}U\Delta_\Omega^{-it} \quad \text{when } \Phi = U'U\Omega \quad \text{with} \\ U \in \mathcal{R}, U' \in \mathcal{R}'. \quad (5.19)$$

This expression motivates the following lemma.

Lemma 5.11: Let $\mathcal{R} \subset \mathcal{B}(\mathcal{H})$ be a vN factor,[15] Ω a cyclic and separating vector, $U \in \mathcal{B}(\mathcal{H})$ a unitary such that $U^*\mathcal{R}U = \mathcal{R}$ and $\Phi = U\Omega$. If there exist families of unitaries[16] $V(t) \in \mathcal{R}$, $V'(t) \in \mathcal{R}'$ such that

$$\begin{cases} U^*\Delta_\Omega^{it}U\Delta_\Omega^{-it} = V(t)V'(t), & \forall\ t \in \mathbb{R}, \\ V(0) = V'(0) = \mathbf{1}. \end{cases} \quad (5.20)$$

Then there exists a real function $\alpha\colon \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = 0$ such that

$$\Delta_{\Phi,\Omega}^{it} = \mathrm{e}^{-i\alpha(t)}V(t)\Delta_\Omega^{it}. \quad (5.21)$$

Proof.—We first see that $V(t)\Delta_\Omega^{it}$ has the same action as $\Delta_{\Phi,\Omega}^{it}$ over every $A \in \mathcal{R}$ and $A' \in \mathcal{R}'$. Indeed

$$\begin{aligned}\mathcal{R} \ni &V(t)\Delta^{it}A\Delta^{-it}V(t)^* \\ &= V(t)V'(t)\Delta_\Omega^{it}A\Delta_\Omega^{-it}V(t)^*V'(t)^* \\ &= U\Delta_\Omega^{it}U^*\Delta_\Omega^{-it}\Delta_\Omega^{it}A\Delta_\Omega^{-it}\Delta_\Omega^{it}U\Delta_\Omega^{-it}U^* \\ &= U\Delta_\Omega^{it}U^*AU\Delta_\Omega^{-it}U^* = \Delta_\Phi^{it}A\Delta_\Phi^{-it} = \Delta_{\Phi,\Omega}^{it}A\Delta_{\Phi,\Omega}^{-it},\end{aligned} \quad (5.22)$$

where we have used 2. in lemma 5.1. Similarly,

$$\begin{aligned}V(t)\underbrace{\Delta_\Omega^{it}A'\Delta_\Omega^{-it}}_{\in\mathcal{R}_{\mathcal{W}'}}V(t)^* &= V(t)V(t)^*\Delta_\Omega^{it}A'\Delta_\Omega^{-it} \\ &= \Delta_\Omega^{it}A'\Delta_\Omega^{-it} = \Delta_{\Phi,\Omega}^{it}A'\Delta_{\Phi,\Omega}^{-it}.\end{aligned} \quad (5.23)$$

Then for all $B \in \mathcal{R} \cup \mathcal{R}'$ we have

$$\begin{aligned}&(V(t)\Delta_\Omega^{it})B(V(t)\Delta_\Omega^{it})^* \\ &= \Delta_{\Phi,\Omega}^{it}B\Delta_{\Phi,\Omega}^{-it} \Rightarrow [B,(V(t)\Delta_\Omega^{it})^*\Delta_{\Phi,\Omega}^{it}] = 0,\end{aligned} \quad (5.24)$$

and hence $(V(t)\Delta_\Omega^{it})^*\Delta_{\Phi,\Omega}^{it}$ belongs to the center $(\mathcal{R} \cup \mathcal{R}')' = \mathcal{R} \cap \mathcal{R}' = \{\lambda \cdot \mathbf{1}\}$, which is trivial since $\mathcal{R}$ is a factor. This means that there exists a function $\lambda\colon \mathbb{R} \to \mathbb{C}$ such that

$$\Delta_{\Phi,\Omega}^{it} = \lambda(t)V(t)\Delta_\Omega^{it}. \quad (5.25)$$

Moreover, evaluating the above expression at $t = 0$ we get that $\lambda(0) = 1$. Finally, since all operators in (5.25) are unitaries, we have that $\lambda(t) = \mathrm{e}^{-i\alpha(t)}$ for some function $\alpha\colon \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = 0$, and then (5.21) holds. ∎

Under the hypothesis of the above lemma, we obtain the relative modular Hamiltonian deriving (5.21) at $t = 0$,

$$\begin{aligned}K_{\Phi,\Omega} &= i\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\Delta_{\Phi,\Omega}^{it} = i\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\mathrm{e}^{-i\alpha(t)}V(t)\Delta_\Omega^{it} \\ &= \alpha'(0)\mathbf{1} + i\dot{V}(0) + K_\Omega,\end{aligned} \quad (5.26)$$

[14] $C^1_t(\mathbb{R}^n)$ is the set of piecewise differentiable functions. See Appendix A 1 for a proper definition.

[15] A vN algebra $\mathcal{R} \subset \mathcal{B}(\mathcal{H})$ is said to be a *factor* if its center is trivial, i.e., $\mathcal{R} \cap \mathcal{R}' = \{\lambda \cdot \mathbf{1}\}$.

[16] They are not necessarily one-parameter groups for $t \in \mathbb{R}$.

where the derivative in (5.26) has to be understood as a limit in the strong operator topology of $\mathcal{H}$. This formula gives a well-defined expression for the relative modular Hamiltonian up to a constant. One way to determine such a constant is using that $\Delta^{it}_{\Phi,\Omega}$ is a one-parameter group of unitaries and must fulfil the concatenation equation

$$\Delta^{it_1}_{\Phi,\Omega}\Delta^{it_2}_{\Phi,\Omega} = \Delta^{i(t_1+t_2)}_{\Phi,\Omega}, \quad \forall\ t_1, t_2 \in \mathbb{R}. \tag{5.27}$$

We discuss the computation to determine $\alpha'(0)$ in Sec. V B 5.

4. Relative modular flow for coherent states

In this subsection, we apply lemma 5.11 for the theory of a real scalar field. More concretely, we show that the splitting of such lemma can be done for a general coherent state. Indeed we have the following theorem.

Theorem 5.12: For the algebra of the Rindler wedge $\mathcal{R}_{\mathcal{W}}$, a Weyl unitary $U = W(f)$ with $f \in \mathcal{S}(\mathbb{R}^d, \mathbb{R})$, the vacuum vector Ω and $\Phi = U\Omega$, the hypothesis in lemma (5.11) holds. In particular we have that

$$\begin{aligned}\Delta^{it}_{\Phi,\Omega} &= e^{i\alpha(s)} W_\varphi(g^s_{\varphi,R}) W_\pi(g^s_{\pi,R}) \Delta^{it}_{\Omega} \\ &= e^{i\alpha(s)} e^{i\varphi(g^s_{\varphi,R})} e^{i\pi(g^s_{\pi,R})} e^{isK_1},\end{aligned} \tag{5.28}$$

where we have denoted $s \coloneqq -2\pi t$ and

$$g^s_{\varphi,R}(\bar{x}) = -\frac{\partial G^s}{\partial x^0}(0,\bar{x})\Theta(x^1), \tag{5.29}$$

$$g^s_{\pi,R}(\bar{x}) = G^s(0,\bar{x})\Theta(x^1), \tag{5.30}$$

$$G^s(x) = \int_{\mathbb{R}^d} \Delta(x-y)[f(\Lambda_1^{-s}y) - f(y)]d^d y. \tag{5.31}$$

Proof.—From relations (3.34)–(3.37) we have that $\mathcal{R}_{\mathcal{W}}$ is a vN factor. From the Reeh-Schlieder theorem, we have that the vacuum vector Ω is cyclic and separating. And we have already discussed that any Weyl unitary satisfies $W(\boldsymbol{h})^*\mathcal{R}_{\mathcal{W}}W(\boldsymbol{h}) = \mathcal{R}_{\mathcal{W}}$. From now on, we set $s = -2\pi t$ and we replace $U = W(f)$ in (5.20)

$$\begin{aligned}&W(f)^*\Delta^{it}_{\Omega}W(-f)\Delta^{-it}_{\Omega} \\ &\quad = W(-f)e^{isK_1}W(f)e^{-isK_1} = W(-f)W(f_{(\Lambda_1^s,0)}) \\ &\quad = e^{i\mathrm{Im}\langle f, f_{(\Lambda_1^s,0)}\rangle_{\mathfrak{H}}} W(f_{(\Lambda_1^s,0)} - f) = e^{i\mathrm{Im}\langle f, f^s\rangle_{\mathfrak{H}}} W(f^s - f),\end{aligned} \tag{5.32}$$

where we have defined $f^s \coloneqq f_{(\Lambda_1^s,0)}$. Applying the decomposition (3.44) to $g^s \coloneqq f^s - f$ we have

$$\begin{aligned}W(f)^*\Delta^{it}_{\Omega}W(f)\Delta^{-it}_{\Omega} &= e^{i\mathrm{Im}\langle f, f^s\rangle_{\mathfrak{H}}} W(g^s) \\ &= e^{i\mathrm{Im}\langle f, f^s\rangle_{\mathfrak{H}}} e^{i\mathrm{Im}\langle g^s_\varphi, g^s_\pi\rangle_{\mathfrak{H}}} W_\varphi(g^s_\varphi)W_\pi(g^s_\pi),\end{aligned} \tag{5.33}$$

with

$$\begin{aligned}g^s_\varphi(\bar{x}) &= -\frac{\partial G^s}{\partial x^0}(0,\bar{x}) = -\cosh(s)\frac{\partial F}{\partial x^0}(\bar{x}^s) \\ &\quad + \sinh(s)\frac{\partial F}{\partial x^1}(\bar{x}^s) + \frac{\partial F}{\partial x^0}(0,\bar{x}),\end{aligned} \tag{5.34}$$

$$g^s_\pi(\bar{x}) = G^s(0,\bar{x}) = F(\bar{x}^s) - F(0,\bar{x}), \tag{5.35}$$

where $\bar{x}^s \coloneqq (\Lambda_1^{-s}x)_{x^0=0} = (-x^1\sinh(s), x^1\cosh(s), \bar{x}_\perp)$ and

$$G^s(x) = \int_{\mathbb{R}^d} \Delta(x-y)[f^s(y) - f(y)]d^d y. \tag{5.36}$$

Now, we explicitly split the unitaries $W_\varphi(g^s_\varphi)$ and $W_\pi(g^s_\pi)$ in Eq. (5.33) defining

$$g^s_{\varphi,R}(\bar{x}) \coloneqq g^s_\varphi(\bar{x})\Theta(x^1) \quad \text{and} \quad g^s_{\varphi,L}(\bar{x}) \coloneqq g^s_\varphi(\bar{x})\Theta(-x^1), \tag{5.37}$$

$$g^s_{\pi,R}(\bar{x}) \coloneqq g^s_\pi(\bar{x})\Theta(x^1) \quad \text{and} \quad g^s_{\pi,L}(\bar{x}) \coloneqq g^s_\pi(\bar{x})\Theta(-x^1), \tag{5.38}$$

which clearly implies that $g^s_{\varphi,L} + g^s_{\varphi,R} = g^s_\varphi$ and $g^s_{\pi,L} + g^s_{\pi,R} = g^s_\pi$. Moreover

$$\left.\begin{aligned}g^s_{\varphi,R}, g^s_{\varphi,L} \in L^2(\mathbb{R}^{d-1},\mathbb{R}) \subset H^{-\frac{1}{2}}(\mathbb{R}^{d-1},\mathbb{R}) \\ \mathrm{supp}(g^s_{\varphi,R}) \subset \Sigma \quad \text{and} \quad \mathrm{supp}(g^s_{\pi,L}) \subset \Sigma'\end{aligned}\right\} \Rightarrow g^s_{\varphi,R} \in K_\varphi(\Sigma) \quad \text{and} \quad g^s_{\varphi,L} \in K_\varphi(\Sigma'). \tag{5.39}$$

Furthermore, we have that $g^s_{\pi,R}, g^s_{\pi,L}$ are real-valued functions and they satisfy the hypothesis in lemma (5.10). Then

$$\left.\begin{aligned}g^s_{\pi,R}, g^s_{\pi,L} \in H^1(\mathbb{R}^{d-1},\mathbb{R}) \subset H^{\frac{1}{2}}(\mathbb{R}^{d-1},\mathbb{R}) \\ \mathrm{supp}(g^s_{\pi,R}) \subset \Sigma \quad \text{and} \quad \mathrm{supp}(g^s_{\pi,L}) \subset \Sigma'\end{aligned}\right\} \Rightarrow g^s_{\pi,R} \in K_\pi(\Sigma) \quad \text{and} \quad g^s_{\pi,L} \in K_\pi(\Sigma'), \tag{5.40}$$

which means that the splits (5.37) and (5.38) work. Coming back to (5.33), we have that

$$\begin{aligned} W(f)\Delta_\Omega^{it}W(f)^*\Delta_\Omega^{-it} &= \mathrm{e}^{i\mathrm{Im}\langle f,f^s\rangle_{\mathfrak{H}}}\mathrm{e}^{i\mathrm{Im}\langle g_\varphi^s,g_\pi^s\rangle_{\mathfrak{H}}}W_\varphi(g_{\varphi,L}^s+g_{\varphi,R}^s)W_\pi(g_{\pi,L}^s+g_{\pi,R}^s) \\ &= \mathrm{e}^{i\mathrm{Im}\langle f,f^s\rangle_{\mathfrak{H}}}\mathrm{e}^{i\mathrm{Im}\langle g_\varphi^s,g_\pi^s\rangle_{\mathfrak{H}}}W_\varphi(g_{\varphi,L}^s)W_\varphi(g_{\varphi,R}^s)W_\pi(g_{\pi,L}^s)W_\pi(g_{\pi,R}^s) \\ &= \mathrm{e}^{i\mathrm{Im}\langle f,f^s\rangle_{\mathfrak{H}}}\mathrm{e}^{i\mathrm{Im}\langle g_\varphi^s,g_\pi^s\rangle_{\mathfrak{H}}}\underbrace{\mathrm{e}^{-2i\mathrm{Im}\langle g_{\varphi,R}^s,g_{\pi,L}^s\rangle_{\mathfrak{H}}}}_{=0}\underbrace{W_\varphi(g_{\varphi,R}^s)W_\pi(g_{\pi,R}^s)}_{\in\mathcal{R}_\mathcal{W}}\underbrace{W_\varphi(g_{\varphi,L}^s)W_\pi(g_{\pi,L}^s)}_{\in\mathcal{R}_{\mathcal{W}'}}. \end{aligned} \tag{5.41}$$

Finally, replacing $V(t) = W_\varphi(g_{\varphi,R}^s)W_\pi(g_{\pi,R}^s)$ into (5.21) we arrive at (5.28). ■

Remark 5.13.—Using the fact that the relative modular flow $\Delta_{\Phi,\Omega}^{it}$ is strongly continuous and that the relative entropy $S_R(\omega_f|\omega)$ is finite (see the discussion at the beginning of Sec. V B 2) and hence the expression (4.18) holds, it is not difficult to show that the function $t\mapsto\langle\Omega,\Delta_{\Phi,\Omega}^{it}\Omega\rangle_\mathcal{H}$ is continuous differentiable. Furthermore, taking the vacuum expectation value on the rhs of (5.28), it can be proven that the function $\alpha(s)\in C^1(\mathbb{R})$.

Finally, from (5.26) we get the following expression for the relative modular Hamiltonian,

$$K_{\Phi,\Omega} = 2\pi(\alpha'(0)\mathbf{1}+\varphi(h_{\varphi,R})+\pi(h_{\pi,R})+K_1), \tag{5.42}$$

where

$$h_{\varphi,R}(\bar{x}) \coloneqq \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} g_{\varphi,R}^s(\bar{x}) = \left(x^1\frac{\partial^2 F}{(\partial x^0)^2}(0,\bar{x})+\frac{\partial F}{\partial x^1}(0,\bar{x})\right)\cdot\Theta(x^1), \tag{5.43}$$

$$h_{\pi,R}(\bar{x}) \coloneqq \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} g_{\pi,R}^s(\bar{x}) = \left(-x^1\frac{\partial F}{\partial x^0}(0,\bar{x})\right)\cdot\Theta(x^1). \tag{5.44}$$

With similar arguments used above, we have that $h_{\varphi,R}\in K_\varphi(\Sigma)$ and $h_{\pi,R}\in K_\pi(\Sigma)$.[17]

Before we proceed to obtain the constant $\alpha'(0)$ we emphasize its importance,

$$S_R(\omega_f|\omega) = \langle\Omega,K_{\Phi,\Omega}\Omega\rangle_\mathcal{H} = 2\pi\alpha'(0). \tag{5.45}$$

Thus, the constant $\alpha'(0)$ gives the desired result for the relative entropy. Regardless of the problem of computing the value of $\alpha'(0)$, expressions (5.42)–(5.44) give us an explicit exact expression for the relative modular Hamiltonian $K_{\Phi,\Omega}$ up to a constant. It is interesting to notice that the difference $K_{\Phi,\Omega}-K_\Omega$ is just a linear term on the fields operators plus a constant term. We expect that this structure holds not just for the Rindler wedge, but for any kind of region as long as $\Phi = W(f)\Omega$ is a coherent vector.

[17]An explicit computation of the strong derivative in Eq. (5.42) shows that the vacuum vector Ω, any coherent vector and any vector of finite number of particles belong to the domain of $K_{\Psi,\Omega}$.

5. Determination of $\alpha'(0)$ and the relative entropy

As we have already explained in Eq. (5.45), we need to determine the constant $\alpha'(0)$. Most of the calculation is straightforward and we present the detailed computations in Appendix A 4. As in theorem 5.12, throughout this section we set $s \coloneqq -2\pi t$.

We start taking the vacuum expectation value on both sides in expression (5.27),

$$\langle\Omega,\Delta_{\Psi,\Omega}^{it_1}\Delta_{\Psi,\Omega}^{it_2}\Omega\rangle_\mathcal{H} = \langle\Omega,\Delta_{\Psi,\Omega}^{i(t_1+t_2)}\Omega\rangle_\mathcal{H}, \tag{5.46}$$

and we replace the expression (5.28) obtained for the relative modular flow (see Eqs. (A18) and (A19). Applying $\frac{\mathrm{d}}{\mathrm{d}s_1}\big|_{s_1=0} = -\frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}t_1}\big|_{t_1=0}$ on both sides of (5.46) (Eqs. (A20) and (A21),[18] and matching real and imaginary parts separately we get[19]

$$\begin{aligned} &\alpha'(s_2)-\frac{\mathrm{d}}{\mathrm{d}s_2}\mathrm{Im}\langle g_{\varphi,R}^{s_2},g_{\pi,R}^{s_2}\rangle_{\mathfrak{H}} \\ &\quad= \alpha'(0)-\frac{\mathrm{d}}{\mathrm{d}s_1}\bigg|_{s_1=0}\mathrm{Im}\langle g_R^{s_1},g_R^{s_2}\rangle_{\mathfrak{H}}, \end{aligned} \tag{5.47}$$

$$\frac{\mathrm{d}}{\mathrm{d}s_1}\bigg|_{s_1=0}\|g_R^{s_1+s_2}\|_{\mathfrak{H}}^2 = \frac{\mathrm{d}}{\mathrm{d}s_1}\bigg|_{s_1=0}\|g_R^{s_1}+u(\Lambda_1^{s_1},0)g_R^{s_2}\|_{\mathfrak{H}}^2, \tag{5.48}$$

where $g_R^s = E_\varphi(g_{\varphi,R}^s)+E_\pi(g_{\pi,R}^s)$. The second equation is useless to determine $\alpha'(0)$; then we concentrate in the first one which is a differential equation for $\alpha'(s)$, with the particularity that $\alpha'(0)$ appears on it. To solve it, let us analyze the second term on the right-hand side of Eq. (5.47). In Appendix A 4 we compute

[18]Analytic properties of the relative modular flow ensures that both sides of (5.46) are continuous differentiable functions on t_1 and t_2.

[19]The $\frac{\mathrm{d}}{\mathrm{d}s_2}$ in (5.47) appears because in some terms the dependance on s_1 of the expression is through s_1+s_2.

$$
\begin{aligned}
&2\mathrm{Im}\langle g_R^{s_1}, g_R^{s_2}\rangle_{\mathfrak{H}} \\
&\quad = 2\mathrm{Im}\langle g_{\varphi,R}^{s_1} + g_{\pi,R}^{s_1}, g_{\varphi,R}^{s_2} + g_{\pi,R}^{s_2}\rangle_{\mathfrak{H}} \\
&\quad = \underbrace{\int_{x^1>0} f_\varphi(\bar{x}) f_\pi^{s_1}(\bar{x}) d^{d-1}x - \int_{x^1>0} f_\varphi^{s_1}(\bar{x}) f_\pi(\bar{x}) d^{d-1}x}_{:=P(s_1)} \\
&\qquad + \underbrace{\int_{x^1>0} f_\varphi^{s_1}(\bar{x}) f_\pi^{s_2}(\bar{x}) d^{d-1}x}_{:=Q(s_1,s_2)} - \underbrace{\int_{x^1>0} f_\varphi^{s_2}(\bar{x}) f_\pi^{s_1}(\bar{x}) d^{d-1}x}_{:=R(s_1,s_2)} \\
&\qquad + \gamma(s_2).
\end{aligned} \tag{5.49}
$$

The function γ includes all the s_1-independent terms, which they do not contribute to (5.47). In the same Appendix we analyze P, Q, R carefully and we get

$$
\frac{\mathrm{d}P}{\mathrm{d}s_1}\bigg|_{s_1=0} = \int_{x^1\geq 0} d^{d-1}x x^1 \left(\left(\frac{\partial F}{\partial x^0}\right)^2 + (\nabla F)^2 + m^2 F^2\right)\bigg|_{x^0=0} =: \boldsymbol{S}, \tag{5.50}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}s_1}(Q-R)\bigg|_{s_1=0} = -\frac{\mathrm{d}}{\mathrm{d}s_2}(Q-R)\bigg|_{s_1=0}. \tag{5.51}
$$

Coming back to (5.47), we have that

$$
\begin{aligned}
&\alpha'(s_2) - \frac{\mathrm{d}}{\mathrm{d}s_2}\mathrm{Im}\langle g_{\varphi,R}^{s_2}, g_{\pi,R}^{s_2}\rangle_{\mathfrak{H}} \\
&\quad = \alpha'(0) - \frac{\mathrm{d}}{\mathrm{d}s_1}\bigg|_{s_1=0} \mathrm{Im}\langle g_R^{s_1}, g_R^{s_2}\rangle_{\mathfrak{H}} \\
&\quad = \alpha'(0) - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s_1}\bigg|_{s_1=0} (P(s_1) + Q(s_1,s_2) - R(s_1,s_2)) \\
&\quad = \alpha'(0) - \frac{1}{2}\boldsymbol{S} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s_2}(Q(0,s_2) - R(0,s_2)).
\end{aligned} \tag{5.52}
$$

Then, integrating this last equation with respect to s_2 we have

$$
\begin{aligned}
\alpha(s_2) - \mathrm{Im}\langle g_{\varphi,R}^{s_2}, g_{\pi,R}^{s_2}\rangle_{\mathfrak{H}} &= \alpha'(0)s_2 - \frac{1}{2}\boldsymbol{S}s_2 \\
&\quad + \frac{1}{2}(Q(0,s_2) - R(0,s_2)),
\end{aligned} \tag{5.53}
$$

where we have used $g_{\varphi,R}^{s_2=0} = g_{\pi,R}^{s_2=0} = 0 \Rightarrow \mathrm{Im}\langle g_{\varphi,R}^{s_2=0}, g_{\pi,R}^{s_2=0}\rangle_{\mathfrak{H}} = 0$, and $Q(0,0) - R(0,0) = 0$ which follows from the definitions of Q and R. To determine $\alpha'(0)$, we use the Kubo-Martin-Schwinger condition stated in theorem 4. 7. Using $A = B = \mathbf{1}$ in Eq. (4.14) and simply calling $G(z)$ to the underlying function, we have that

$$
G(t) = \langle \Omega, \Delta_{\Psi,\Omega}^{it}\Omega\rangle_{\mathcal{H}} \underset{t\to -i}{\to} G(-i) = \langle \Phi, \Phi\rangle_{\mathcal{H}} = 1. \tag{5.54}
$$

In terms of the real variable $s = -2\pi t$, the function $G(s)$ is in analytic on $\mathbb{R} + i(0, 2\pi)$, and relation (5.54) must hold for $s \to 2\pi i$. Using (5.28), we have that

$$
\begin{aligned}
G(s) &= \mathrm{e}^{i\alpha(s)}\langle \Omega, \mathrm{e}^{i\varphi(g_{\varphi,R}^s)}\mathrm{e}^{i\pi(g_{\pi,R}^s)}\Omega\rangle_{\mathcal{H}} \\
&= \mathrm{e}^{i\alpha(s) - i\mathrm{Im}\langle g_{\varphi,R}^s, g_{\pi,R}^s\rangle_{\mathfrak{H}} - \frac{1}{2}\|g_R^s\|_{\mathfrak{H}}^2},
\end{aligned} \tag{5.55}
$$

and hence

$$
i\alpha(s) - i\mathrm{Im}\langle g_{\varphi,R}^s, g_{\pi,R}^s\rangle_{\mathfrak{H}} - \frac{1}{2}\|g_R^s\|_{\mathfrak{H}}^2 \underset{s\to 2\pi i}{\to} i2n\pi, \quad n \in \mathbb{Z}. \tag{5.56}
$$

Taking this into account, we come back to (5.53) and write

$$
\begin{aligned}
&i\alpha(s) - i\mathrm{Im}\langle g_{\varphi,R}^s, g_{\pi,R}^s\rangle_{\mathfrak{H}} - \frac{1}{2}\|g_R^s\|_{\mathfrak{H}}^2 \\
&\quad = i\alpha'(0)s - \frac{i}{2}\boldsymbol{S}s + \frac{i}{2}(Q(0,s) - R(0,s)) - \frac{1}{2}\|g_R^s\|_{\mathfrak{H}}^2.
\end{aligned} \tag{5.57}
$$

Before we take limit $s \to 2\pi i$, we may notice that $\bar{x}^s = (-x^1\sinh(s), x^1\cosh(s), \bar{x}_\perp) \underset{s\to 2\pi i}{\to} (0, \bar{x})$, which informally suggests that

$$
g_R^s \underset{s\to 2\pi i}{\longrightarrow} 0 \Rightarrow \|g_R^s\|_{\mathfrak{H}}^2 \underset{s\to 2\pi i}{\longrightarrow} 0, \tag{5.58}
$$

$$
f_\nu^s \underset{s\to 2\pi i}{\longrightarrow} f_\nu \Rightarrow Q(0,s) - R(0,s) \underset{s\to 2\pi i}{\longrightarrow} 0, \quad \text{where } \nu = \varphi, \pi. \tag{5.59}
$$

We prove in Appendix A 5 that the function

$$
N(s) := \frac{i}{2}(Q(0,s) - R(0,s)) - \frac{1}{2}\|g_R^s\|_{\mathfrak{H}}^2, \tag{5.60}
$$

of the variable $s \in \mathbb{R}$, can be analytically continued on the strip $\mathbb{R} + i(0, 2\pi)$ and that $\lim_{s\to 2\pi i} N(s) = 0$. Then, taking the limit $s \to 2\pi i$ on (5.57) we get

$$
i2n\pi = -\alpha'(0)2\pi + \frac{1}{2}\boldsymbol{S}2\pi. \tag{5.61}
$$

Since $\alpha'(0), \boldsymbol{S} \in \mathbb{R}$ then it must be $n = 0$, an hence we finally get $\alpha'(0) = \frac{1}{2}\boldsymbol{S}$.

All these together can be summarized in the following theorem which generalizes theorem 5.9.

Theorem 5.14: Let $f \in \mathcal{S}(\mathbb{R}^d, \mathbb{R})$ with $\mathrm{supp}(f)$ compact. Then the relative entropy between the coherent state

ω_f and the vacuum ω, for the right Rindler wedge algebra $\mathcal{R}_{\mathcal{W}}$, is

$$S_R(\omega_f|\omega) = 2\pi \int_{x^1>0} d^{d-1}x x^1 \times \frac{1}{2}\left(\left(\frac{\partial F}{\partial x^0}\right)^2 + |\nabla F|^2 + m^2 F^2\right)\Bigg|_{x^0=0}, \tag{5.62}$$

where $F(x) = \int_{\mathbb{R}^d} \Delta(x-y) f(y) d^d y$. In addition, formula (5.62) only depends in the behavior of f in $\mathbb{R}^d - \mathcal{W}'$.

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APPENDIX

1. Sobolev spaces

For the definition and properties of Sobolev spaces, we follow [36]. Here we adapt the notation to our convenience.

Consider the test function space $\mathcal{D}(\mathbb{R}^n) \coloneqq C_c^\infty(\mathbb{R}^n) \not\subseteq \mathcal{S}(\mathbb{R}^n)$ of smooth and compactly supported functions, with its usual topology. The n-dimensional complex Sobolev space of order $\alpha \in \mathbb{R}$ is defined as

$$H^\alpha(\mathbb{R}^n) \coloneqq \{f \in \mathcal{D}'(\mathbb{R}^n) : \hat{f}(\bar{p})\omega_{\bar{p}}^\alpha \in L^2(\mathbb{R}^n)\}, \tag{A1}$$

where $\omega_{\bar{p}} = \sqrt{\bar{p}^2+1}$ and $\hat{f}(\bar{p}) \coloneqq (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(\bar{x}) e^{-i\bar{p}\cdot\bar{x}} d^n x$ is the usual Fourier transform. From the definition it follows that $H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ and $H^\alpha(\mathbb{R}^n) \subset H^{\alpha'}(\mathbb{R}^n)$ if $\alpha > \alpha'$.

The Sobolev space $H^\alpha(\mathbb{R}^n)$ is a Hilbert space under the inner product

$$\langle f, g\rangle_{H^\alpha} \coloneqq \langle \hat{f}\omega_{\bar{p}}^\alpha, \hat{g}\omega_{\bar{p}}^\alpha\rangle_{L^2} = \int_{\mathbb{R}^n} d^n p \hat{f}(\bar{p})^* \hat{g}(\bar{p}) \omega_{\bar{p}}^{2\alpha}. \tag{A2}$$

Furthermore, for $f \in H^\alpha(\mathbb{R}^n)$ we have that $\|f\|_{H^{\alpha'}} \leq \|f\|_{H^\alpha}$ if $\alpha > \alpha'$, and hence the natural injections $H^\alpha(\mathbb{R}^n) \hookrightarrow H^{\alpha'}(\mathbb{R}^n)$ for $\alpha > \alpha'$ are continuous. We also have that the set $C^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ is dense in $H^\alpha(\mathbb{R}^n)$.

When $\alpha = k \in \mathbb{N}_0$, there is also another useful equivalent characterization of the Sobolev spaces in term of weak derivatives,[20]

$$H^k(\mathbb{R}^n) = \{f \in \mathcal{D}'(\mathbb{R}^n) : D^\mu f \in L^2(\mathbb{R}^n), \quad \text{for all } |\mu| \leq k\}. \tag{A3}$$

It is useful to introduce a new norm in $H^k(\mathbb{R}^n)$ as

$$\|f\|'_{H^k} \coloneqq \left(\sum_{|\mu|\leq k} \int_{\mathbb{R}^n} d^n x |D^\mu f(x)|^2\right)^{\frac{1}{2}}, \tag{A4}$$

which is equivalent to the former norm $\|\cdot\|_{H^k}$.

The real Sobolev spaces $H^\alpha(\mathbb{R}^n, \mathbb{R})$ are defined in a similar manner as above, but restricting to real-valued functions.

In general, it is easier to calculate the usual pointwise derivatives rather than the weak derivatives. Then, the following lemma states sufficient conditions for both notions of derivatives coincide. Before we formulate it, we need to introduce the notions of the C^k-piecewise function.

Definition A.1: Let $U \subset \mathbb{R}^n$ open, $f \in L^1_{loc}(U)$ and $k \in \mathbb{N}_0$. We say that f is a C^k-piecewise function if there exists a finite family of pairwise disjoint open sets $\{\Omega_j\}_{j=1,\ldots,J} \subset U$ such that

(a) $\bigcup_{j=1}^J \bar{\Omega}_j = \bar{U}$.
(b) $f \in C^k(\Omega_j)$ for all $j = 1, \ldots, J$.
(c) For all $j = 1, \ldots, J$, $\forall x_0 \in \partial\Omega_j$ and for all multi-index $|\alpha| \leq k$, the $\lim_{x\to x_0} D^\alpha f(x)|_{\Omega_j}$ exist and are finite (where D^α is the usual multiorder pointwise derivative).

We denote by $C_t^k(U)$ the set of C^k-piecewise functions on U.

Now, we formulate the lemma that ensures that weak derivatives and pointwise derivatives coincide.

Lemma A.2: Let $U \subset \mathbb{R}^n$ be open and $f \in C^0(U) \cap C_t^1(U)$. Then the (first order) weak derivatives of f coincides with the usual pointwise derivatives.

Proof.—Since $f \in C^0(U) \cap C_t^1(U)$ we have that f is locally Lipschitz continuous on U (see corollary 4.1.1 on [37]). Then we have that f is locally absolute continuous on U, and of course $f \in L^1_{loc}(U)$. Then f is weakly differentiable and the (first order) weak and pointwise derivatives of f coincide a.e. ∎

Now, using the above lemma and the alternative definition [Eq. (A3)] for the Sobolev space $H^1(\mathbb{R}^n)$, the proof in lemma 5. 10 is trivial.

[20]The weak derivative of an element of $\mathcal{D}'(\mathbb{R}^n)$ is its usual derivative in the distributional sense.

2. Calculation of $\langle \Omega, W(f_R)^* K_1 W(f_R)\Omega\rangle$

Take $f_R \in \mathcal{S}(\mathbb{R}^d, \mathbb{R})$ and for simplicity call $f \coloneqq f_R$. Then

$$\langle \Omega, W(f)^* K_1 W(f)\Omega\rangle_{\mathcal{H}} = \left\langle \mathrm{e}^{-\frac{\|f\|^2_{\mathfrak{H}}}{2}} \sum_{n=0}^{\infty} \frac{i f^{\otimes n}}{\sqrt{n!}}, K_1 \mathrm{e}^{-\frac{\|f\|^2_{\mathfrak{H}}}{2}} \sum_{n=0}^{\infty} \frac{i f^{\otimes n}}{\sqrt{n!}} \right\rangle_{\mathcal{H}} \tag{A5}$$

$$\begin{aligned} &= \mathrm{e}^{-\|f\|^2_{\mathfrak{H}}} \left\langle \sum_{n=0}^{\infty} \frac{i f^{\otimes n}}{\sqrt{n!}}, K_1 \sum_{n=0}^{\infty} \frac{i f^{\otimes n}}{\sqrt{n!}} \right\rangle_{\mathcal{H}} = \mathrm{e}^{-\|f\|^2_{\mathfrak{H}}} \sum_{n=0}^{\infty} \frac{(-i)^n (i)^n}{n!} \langle f^{\otimes n}, K_1 f^{\otimes n}\rangle_{\mathfrak{H}^{\otimes n}} \\ &= \mathrm{e}^{-\|f\|^2_{\mathfrak{H}}} \sum_{n=0}^{\infty} \frac{n}{n!} \langle f^{\otimes n}, (k_1 f) \otimes f^{\otimes n-1}\rangle_{\mathfrak{H}^{\otimes n}} = \mathrm{e}^{-\|f\|^2_{\mathfrak{H}}} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \langle f, k_1 f\rangle_{\mathfrak{H}} \langle f, f\rangle_{\mathfrak{H}}^{n-1} \\ &= \mathrm{e}^{-\|f\|^2_{\mathfrak{H}}} \langle f, k_1 f\rangle_{\mathfrak{H}} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \langle f, f\rangle_{\mathfrak{H}}^{n-1} = \mathrm{e}^{-\|f\|^2_{\mathfrak{H}}} \langle f, k_1 f\rangle_{\mathfrak{H}} \mathrm{e}^{\|f\|^2_{\mathfrak{H}}} = \langle f, k_1 f\rangle_{\mathfrak{H}}. \end{aligned} \tag{A6}$$

3. Calculation of $\langle f_R, k_1 f_R\rangle_{\mathfrak{H}}$

Take $f_R \in \mathcal{S}(\mathbb{R}^d, \mathbb{R})$ and for simplicity call $f \coloneqq f_R$. Then

$$\begin{aligned} \langle f, k_1 f\rangle_{\mathfrak{H}} &= \mathrm{Re}\langle f, k_1 f\rangle_{\mathfrak{H}} = \mathrm{Re}\left(-i \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \langle f, \mathrm{e}^{i k_1 s} f\rangle_{\mathfrak{H}} \right) = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \mathrm{Im}\langle f, u(\Lambda_1^s, 0) f\rangle_{\mathfrak{H}} \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \mathrm{Im}\langle f, f_{(\Lambda_1^s, 0)}\rangle_{\mathfrak{H}} = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \mathrm{Im}\langle f, f^s\rangle_{\mathfrak{H}}, \end{aligned} \tag{A7}$$

where we have defined $f^s = f_{(\Lambda_1^s, 0)}$. As we explained in Sec. III D, there exist functions $f_\varphi, f_\pi, f^s_\varphi, f^s_\pi \in \mathcal{S}(\mathbb{R}^{d-1}, \mathbb{R})$ such that

$$E(f) = E_\varphi(f_\varphi) + E_\pi(f_\pi) \quad \text{and} \quad E(f^s) = E_\varphi(f^s_\varphi) + E_\pi(f^s_\pi). \tag{A8}$$

Replacing these in (A7) we get

$$\begin{aligned} \langle f, k_1 f\rangle_{\mathfrak{H}} &= \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \mathrm{Im}\langle f_\varphi + f_\pi, f^s_\varphi + f^s_\pi\rangle_{\mathfrak{H}} = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} (\mathrm{Im}\langle f_\varphi, f^s_\pi\rangle_{\mathfrak{H}} + \mathrm{Im}\langle f_\pi, f^s_\varphi\rangle_{\mathfrak{H}}) \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \left(\frac{1}{2} \int_{\mathbb{R}^{d-1}} f_\varphi(\bar{x}) f^s_\pi(\bar{x}) d^{d-1}x - \frac{1}{2} \int_{\mathbb{R}^{d-1}} f^s_\varphi(\bar{x}) f_\pi(\bar{x}) d^d x \right), \end{aligned} \tag{A9}$$

where we have used the relations (3.20) in the second line and (3.46) in the last line. From the Poincaré invariance of the distribution $\Delta(x)$ we have that

$$F^s(x) = \int_{\mathbb{R}^d} \Delta(x-y) f^s(x) d^d y, \tag{A10}$$

where we have defined $F^s \coloneqq F_{(\Lambda_1^s, 0)}$. Then

$$f_\varphi(\bar{x}) \coloneqq -\frac{\partial F}{\partial x^0}(0, \bar{x}), \tag{A11}$$

$$f_\pi(\bar{x}) \coloneqq F(0, \bar{x}), \tag{A12}$$

$$f^s_\varphi(\bar{x}) \coloneqq -\cosh(s) \frac{\partial F}{\partial x^0}(\bar{x}^s) + \sinh(s) \frac{\partial F}{\partial x^1}(\bar{x}^s), \tag{A13}$$

$$f^s_\pi(\bar{x}) \coloneqq F(\bar{x}^s), \tag{A14}$$

being $\bar{x}^s \coloneqq (-x^1 \sinh(s), x^1 \cosh(s), x_\perp)$, and hence

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} f^s_\varphi(\bar{x}) \coloneqq x^1 \frac{\partial^2 F}{(\partial x^0)^2}(0, \bar{x}) + \frac{\partial F}{\partial x^1}(0, \bar{x}), \tag{A15}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} f^s_\pi(\bar{x}) \coloneqq -x^1 \frac{\partial F}{\partial x^0}(0, \bar{x}). \tag{A16}$$

Replacing such expressions in (A9), using the equation of motion for F, and doing an integration by parts, we finally get

$$\langle f, k_1 f\rangle_{\mathfrak{H}} = \int_{\mathbb{R}^{d-1}} d^{d-1}x x^1 \times \frac{1}{2} \left(\left(\frac{\partial F}{\partial x^0} \right)^2 + |\nabla F|^2 + m^2 F^2 \right)\bigg|_{x^0=0}. \tag{A17}$$

4. Explicit computations of Sec. V B 5

Defining $g_R^s = E_\varphi(g_{\varphi,R}^s) + E_\pi(g_{\pi,R}^s) \in \mathfrak{H}$,

$$\begin{aligned}
\langle \Omega, \Delta_{\Psi,\Omega}^{it_1}\Delta_{\Psi,\Omega}^{it_2}\Omega\rangle_{\mathcal{H}} &= \langle \Omega, \mathrm{e}^{i\alpha(s_1)} W_\varphi(g_{\varphi,R}^{s_1}) W_\pi(g_{\pi,R}^{s_1})\Delta_\Omega^{it_1}\mathrm{e}^{i\alpha(s_2)} W_\varphi(g_{\varphi,R}^{s_2}) W_\pi(g_{\pi,R}^{s_2})\Delta_\Omega^{it_1}\Omega\rangle_{\mathcal{H}} \\
&= \mathrm{e}^{i\alpha(s_1)+i\alpha(s_2)}\langle \Omega, W_\varphi(g_{\varphi,R}^{s_1}) W_\pi(g_{\pi,R}^{s_1})\mathrm{e}^{is_1K_1} W_\varphi(g_{\varphi,R}^{s_2}) W_\pi(g_{\pi,R}^{s_2})\Omega\rangle_{\mathcal{H}} \\
&= \mathrm{e}^{i\alpha(s_1)+i\alpha(s_2)-i\mathrm{Im}\langle g_{\varphi,R}^{s_1},g_{\pi,R}^{s_1}\rangle - i\mathrm{Im}\langle g_{\varphi,R}^{s_2},g_{\pi,R}^{s_2}\rangle_{\mathfrak{H}}}\langle \Omega, W(g_R^{s_1})\mathrm{e}^{is_1K_1} W(g_R^{s_2})\Omega\rangle_{\mathcal{H}} \\
&= \mathrm{e}^{i\alpha(s_1)+i\alpha(s_2)-i\mathrm{Im}\langle g_{\varphi,R}^{s_1},g_{\pi,R}^{s_1}\rangle - i\mathrm{Im}\langle g_{\varphi,R}^{s_2},g_{\pi,R}^{s_2}\rangle_{\mathfrak{H}}}\langle \Omega, W(g_R^{s_1})\mathrm{e}^{is_1K_1} W(g_R^{s_2})\mathrm{e}^{-is_1K_1}\Omega\rangle_{\mathcal{H}} \\
&= \mathrm{e}^{i\alpha(s_1)+i\alpha(s_2)-i\mathrm{Im}\langle g_{\varphi,R}^{s_1},g_{\pi,R}^{s_1}\rangle - i\mathrm{Im}\langle g_{\varphi,R}^{s_2},g_{\pi,R}^{s_2}\rangle_{\mathfrak{H}}}\langle \Omega, W(g_R^{s_1}) W(u(\Lambda_1^{s_1})g_R^{s_2})\Omega\rangle_{\mathcal{H}} \\
&= \mathrm{e}^{i\alpha(s_1)+i\alpha(s_2)-i\mathrm{Im}\langle g_{\varphi,R}^{s_1},g_{\pi,R}^{s_1}\rangle - i\mathrm{Im}\langle g_{\varphi,R}^{s_2},g_{\pi,R}^{s_2}\rangle_{\mathfrak{H}} - i\mathrm{Im}\langle g_R^{s_1}, u(\Lambda_1^{s_1})g_R^{s_2}\rangle_{\mathfrak{H}}}\langle \Omega, W(g_R^{s_1} + u(\Lambda_1^{s_1})g_R^{s_2})\Omega\rangle_{\mathcal{H}} \\
&= \mathrm{e}^{i\alpha(s_1)+i\alpha(s_2)-i\mathrm{Im}\langle g_{\varphi,R}^{s_1},g_{\pi,R}^{s_1}\rangle - i\mathrm{Im}\langle g_{\varphi,R}^{s_2},g_{\pi,R}^{s_2}\rangle_{\mathfrak{H}} - i\mathrm{Im}\langle g_R^{s_1}, u(\Lambda_1^{s_1})g_R^{s_2}\rangle_{\mathfrak{H}} - \frac{1}{2}\|g_R^{s_1}+u(\Lambda_1^{s_1})g_R^{s_2}\|_{\mathfrak{H}}^2},
\end{aligned} \tag{A18}$$

and

$$\begin{aligned}
\langle \Omega, \Delta_{\Psi,\Omega}^{i(t_1+t_2)}\Omega\rangle_{\mathcal{H}} &= \langle \Omega, \mathrm{e}^{i\alpha(s_1+s_2)} W_\varphi(g_{\varphi,R}^{s_1+s_2}) W_\pi(g_{\pi,R}^{s_1+s_2})\Delta_\Omega^{i(t_1+t_2)}\Omega\rangle_{\mathcal{H}} \\
&= \mathrm{e}^{i\alpha(s_1+s_2)}\langle \Omega, W_\varphi(g_{\varphi,R}^{s_1+s_2}) W_\pi(g_{\pi,R}^{s_1+s_2})\Omega\rangle_{\mathcal{H}} \\
&= \mathrm{e}^{i\alpha(s_1+s_2)-i\mathrm{Im}\langle g_{\varphi,R}^{s_1+s_2},g_{\pi,R}^{s_1+s_2}\rangle_{\mathfrak{H}}}\langle \Omega, W(g_R^{s_1+s_2})\Omega\rangle_{\mathcal{H}} \\
&= \mathrm{e}^{i\alpha(s_1+s_2)-i\mathrm{Im}\langle g_{\varphi,R}^{s_1+s_2},g_{\pi,R}^{s_1+s_2}\rangle_{\mathfrak{H}} - \frac{1}{2}\|g_R^{s_1+s_2}\|_{\mathfrak{H}}^2}.
\end{aligned} \tag{A19}$$

Taking $\frac{\mathrm{d}}{\mathrm{d}s_1}|_{s_1=0}$ on both expressions above we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s_1}\Big|_{s_1=0}\langle \Omega, \Delta_{\Psi,\Omega}^{it_1}\Delta_{\Psi,\Omega}^{it_2}\Omega\rangle_{\mathcal{H}} &= i\alpha'(0) - \underbrace{i\frac{\mathrm{d}}{\mathrm{d}s_1}\Big|_{s_1=0}\mathrm{Im}\langle g_{\varphi,R}^{s_1}, g_{\pi,R}^{s_1}\rangle_{\mathfrak{H}}}_{=0} - i\frac{\mathrm{d}}{\mathrm{d}s_1}\Big|_{s_1=0}\mathrm{Im}\langle g_R^{s_1}, u(\Lambda_1^{s_1})g_R^{s_2}\rangle_{\mathfrak{H}} \\
&\quad - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s_1}\Big|_{s_1=0}\|g_R^{s_1} + u(\Lambda_1^{s_1})g_R^{s_2}\|_{\mathfrak{H}}^2 \\
&= i\alpha'(0) - i\frac{\mathrm{d}}{\mathrm{d}s_1}\Big|_{s_1=0}\mathrm{Im}\langle g_R^{s_1}, g_R^{s_2}\rangle_{\mathfrak{H}} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s_1}\Big|_{s_1=0}\|g_R^{s_1} + u(\Lambda_1^{s_1})g_R^{s_2}\|_{\mathfrak{H}}^2,
\end{aligned} \tag{A20}$$

and

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s_1}\Big|_{s_1=0}\langle \Omega, \Delta_{\Psi,\Omega}^{i(t_1+t_2)}\Omega\rangle_{\mathcal{H}} &= i\alpha'(s_2) - i\frac{\mathrm{d}}{\mathrm{d}s_1}\Big|_{s_1=0}\mathrm{Im}\langle g_{\varphi,R}^{s_1+s_2}, g_{\pi,R}^{s_1+s_2}\rangle_{\mathfrak{H}} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s_1}\Big|_{s_1=0}\|g_R^{s_1+s_2}\|_{\mathfrak{H}}^2 \\
&= i\alpha'(s_2) - i\frac{\mathrm{d}}{\mathrm{d}s_2}\mathrm{Im}\langle g_{\varphi,R}^{s_2}, g_{\pi,R}^{s_2}\rangle_{\mathfrak{H}} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s_1}\Big|_{s_1=0}\|g_R^{s_1+s_2}\|_{\mathfrak{H}}^2.
\end{aligned} \tag{A21}$$

Matching real and imaginary parts of these two last expressions, we arrive to formulas (5.47) and (5.48).

Expressions (5.49) follow from

$$\begin{aligned}
2\mathrm{Im}\langle g_R^{s_1}, g_R^{s_2}\rangle_{\mathfrak{H}} &= 2\mathrm{Im}\langle g_{\varphi,R}^{s_1} + g_{\pi,R}^{s_1}, g_{\varphi,R}^{s_2} + g_{\pi,R}^{s_2}\rangle_{\mathfrak{H}} \\
&= \int_\Sigma d^{d-1}x g_\varphi^{s_1}(\bar{x}) g_\pi^{s_2}(\bar{x}) - \int_\Sigma d^{d-1}x g_\varphi^{s_2}(\bar{x}) g_\pi^{s_1}(\bar{x}) \\
&= \int_\Sigma d^{d-1}x (f_\varphi^{s_1}(\bar{x}) - f_\varphi(\bar{x}))(f_\pi^{s_2}(\bar{x}) - f_\pi(\bar{x})) - \int_\Sigma d^{d-1}x (f_\varphi^{s_2}(\bar{x}) - f_\varphi(\bar{x}))(f_\pi^{s_1}(\bar{x}) - f_\pi(\bar{x})) \\
&= \underbrace{\int_\Sigma d^{d-1}x f_\varphi(\bar{x}) f_\pi^{s_1}(\bar{x}) - \int_\Sigma d^{d-1}x f_\varphi^{s_1}(\bar{x}) f_\pi(\bar{x})}_{:=P(s_1)} + \underbrace{\int_\Sigma d^{d-1}x f_\varphi^{s_1}(\bar{x}) f_\pi^{s_2}(\bar{x})}_{:=Q(s_1,s_2)} - \underbrace{\int_\Sigma d^{d-1}x f_\varphi^{s_2}(\bar{x}) f_\pi^{s_1}(\bar{x})}_{:=R(s_1,s_2)} + \gamma(s_2),
\end{aligned} \tag{A22}$$

where the function γ includes all the s_1-independent terms.

The function $P(s_1)$ is essentially the same as (A9) in Appendix A 3, with the difference that now the integration is over the region $\Sigma = \{\bar{x} \in \mathbb{R}^{d-1} : x^1 \geq 0\}$ instead of the whole $\mathbb{R}^{d-1}$. Despite this, the final result is the same and hence we get[21]

$$\frac{\mathrm{d}P}{\mathrm{d}s_1}\bigg|_{s_1=0} = \int_\Sigma d^{d-1}x x^1 \left(\left(\frac{\partial F}{\partial x^0}\right)^2 + (\nabla F)^2 + m^2 F^2\right)\bigg|_{x^0=0} =: \boldsymbol{S}.$$

Now we explicitly obtain the relations (5.51). Indeed,

$$\begin{aligned}
\frac{\mathrm{d}R}{\mathrm{d}s_1}\bigg|_{s_1=0} &= \frac{\mathrm{d}}{\mathrm{d}s_1}\bigg|_{s_1=0} \int_\Sigma d^{d-1}x \left(-\cosh(s_2)\frac{\partial F}{\partial x^0}(\bar{x}^{s_2}) + \sinh(s_2)\frac{\partial F}{\partial x^1}(\bar{x}^{s_2})\right) F(\bar{x}^{s_1}) \\
&= \int_\Sigma d^{d-1}x \left(-\cosh(s_2)\frac{\partial F}{\partial x^0}(\bar{x}^{s_2}) + \sinh(s_2)\frac{\partial F}{\partial x^1}(\bar{x}^{s_2})\right)\left(-x^1\frac{\partial F}{\partial x^0}(\bar{x})\right) \\
&= \int_\Sigma d^{d-1}x \left(-\frac{\partial F}{\partial x^0}(\bar{x})\right)\left(-x^1\cosh(s_2)\frac{\partial F}{\partial x^0}(\bar{x}^{s_2}) + x^1\sinh(s_2)\frac{\partial F}{\partial x^1}(\bar{x}^{s_2})\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}s_2}\int_\Sigma d^{d-1}x \left(-\frac{\partial F}{\partial x^0}(\bar{x})\right) F(\bar{x}^{s_2}) \\
&= \frac{\mathrm{d}}{\mathrm{d}s_2}\bigg|_{s_1=0} \int_\Sigma d^{d-1}x \left(-\cosh(s_1)\frac{\partial F}{\partial x^0}(\bar{x}^{s_1}) + \sinh(s_1)\frac{\partial F}{\partial x^1}(\bar{x}^{s_1})\right) F(\bar{x}^{s_2}) \\
&= \frac{\mathrm{d}}{\mathrm{d}s_2}\bigg|_{s_1=0} \int_\Sigma d^{d-1}x f_\varphi^{s_1}(\bar{x}) f_\pi^{s_2}(\bar{x}) = \frac{\mathrm{d}Q}{\mathrm{d}s_2}\bigg|_{s_1=0}.
\end{aligned} \tag{A23}$$

Similarly we start with

$$\begin{aligned}
\frac{\mathrm{d}Q}{\mathrm{d}s_1}\bigg|_{s_1=0} &= \frac{\mathrm{d}}{\mathrm{d}s_1}\bigg|_{s_1=0} \int_\Sigma d^{d-1}x f_\varphi^{s_1}(\bar{x}) f_\pi^{s_2}(\bar{x}) \\
&= \frac{\mathrm{d}}{\mathrm{d}s_1}\bigg|_{s_1=0} \int_\Sigma d^{d-1}x \left(-\cosh(s_1)\frac{\partial F}{\partial x^0}(\bar{x}^{s_1}) + \sinh(s_1)\frac{\partial F}{\partial x^1}(\bar{x}^{s_1})\right) F(\bar{x}^{s_2}) \\
&= \int_\Sigma d^{d-1}x \left(x^1\frac{\partial^2 F}{(\partial x^0)^2}(\bar{x}) + \frac{\partial F}{\partial x^1}(\bar{x})\right) F(\bar{x}^{s_2}) \\
&= \int_\Sigma d^{d-1}x \left(x^1(\nabla^2 - m^2)F(\bar{x}) + \frac{\partial F}{\partial x^1}(\bar{x})\right) F(\bar{x}^{s_2}).
\end{aligned}$$

First we integrate the Laplacian term by parts,

$$\begin{aligned}
\frac{\mathrm{d}Q}{\mathrm{d}s_1}\bigg|_{s_1=0} = &-\int_\Sigma d^{d-1}x x^1 m^2 F(\bar{x}) F(\bar{x}^{s_2}) - \int_\Sigma d^{d-1}x x^1 \nabla_\perp F(\bar{x}) \cdot \nabla_\perp F(\bar{x}^{s_2}) \\
&-\int_\Sigma d^{d-1}x x^1 \frac{\partial F}{\partial x^1}(\bar{x}) \left(-\sinh(s_2)\frac{\partial F}{\partial x^0}(\bar{x}^{s_2}) + \cosh(s_2)\frac{\partial F}{\partial x^1}(\bar{x}^{s_2})\right).
\end{aligned}$$

After a second integration by parts we get

[21] Following the computation of (A9) in Appendix A 3, there now appears a boundary term after the integration by parts. Fortunately, this term vanishes since the integrand is 0 at the boundary of Σ.

$$\left.\frac{\mathrm{d}Q}{\mathrm{d}s_1}\right|_{s_1=0} = \int_\Sigma d^{d-1}xx^1 F(\bar{x})(\nabla_\perp^2 - m^2)F(\bar{x}^{s_2}) + \int_\Sigma d^{d-1}xF(\bar{x})\left(-\sinh(s_2)\frac{\partial F}{\partial x^0}(\bar{x}^{s_2}) + \cosh(s_2)\frac{\partial F}{\partial x^1}(\bar{x}^{s_2})\right)$$
$$+ \int_\Sigma d^{d-1}xx^1 F(\bar{x})\left(\sinh^2(s_2)\frac{\partial^2 F}{(\partial x^0)^2}(\bar{x}^{s_2}) - 2\sinh(s_2)\cosh(s_2)\frac{\partial^2 F}{\partial x^0 \partial x^1}(\bar{x}^{s_2}) + \cosh^2(s_2)\frac{\partial^2 F}{(\partial x^1)^2}(\bar{x}^{s_2})\right).$$

Now we form a Laplacian term in the first line and we use the equation of motion for F,

$$\left.\frac{\mathrm{d}Q}{\mathrm{d}s_1}\right|_{s_1=0} = \int_\Sigma d^{d-1}xx^1 F(\bar{x})\frac{\partial^2 F}{(\partial x^0)^2}(\bar{x}^{s_2}) + \int_\Sigma d^{d-1}xF(\bar{x})\left(-\sinh(s_2)\frac{\partial F}{\partial x^0}(\bar{x}^{s_2}) + \cosh(s_2)\frac{\partial F}{\partial x^1}(\bar{x}^{s_2})\right)$$
$$+ \int_\Sigma d^{d-1}xx^1 F(\bar{x})\left(\sinh^2(s_2)\frac{\partial^2 F}{(\partial x^0)^2}(\bar{x}^{s_2}) - 2\sinh(s_2)\cosh(s_2)\frac{\partial^2 F}{\partial x^0 \partial x^1}(\bar{x}^{s_2}) + \sinh^2(s_2)\frac{\partial^2 F}{(\partial x^1)^2}(\bar{x}^{s_2})\right).$$

Finally, a straightforward computation shows that

$$\begin{aligned}\left.\frac{\mathrm{d}Q}{\mathrm{d}s_1}\right|_{s_1=0} &= \int_\Sigma d^{d-1}x\frac{\mathrm{d}}{\mathrm{d}s_2}\left(-\cosh(s_2)\frac{\partial F}{\partial x^0}(\bar{x}^{s_2}) + \sinh(s_2)\frac{\partial F}{\partial x^1}(\bar{x}^{s_2})\right)F(\bar{x}) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}s_2}\right|_{s_1=0}\int_\Sigma d^{d-1}x\left(-\cosh(s_2)\frac{\partial F}{\partial x^0}(\bar{x}^{s_2}) + \sinh(s_2)\frac{\partial F}{\partial x^1}(\bar{x}^{s_2})\right)F(\bar{x}^{s_1}) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}s_2}\right|_{s_1=0}\int_\Sigma d^{d-1}xf_\varphi^{s_2}(\bar{x})f_\pi^{s_1}(\bar{x}) = \left.\frac{\mathrm{d}R}{\mathrm{d}s_2}\right|_{s_1=0}.\end{aligned} \tag{A24}$$

Using (A23) and (A24) we arrive at (5.51).

5. Analytic continuation for $N(s)$

In order to show that formulas (5.59) hold, we need to explicitly show the analytic continuation for the function

$$N(s) = \frac{i}{2}(Q(0,s) - R(0,s)) - \frac{1}{2}\|g_R^s\|^2_{\mathfrak{H}}, \tag{A25}$$

or more specifically, we need to show that there exists a continuous function $\tilde{N}: \mathbb{R} + i[0,2\pi] \to \mathbb{C}$, analytic on $\mathbb{R} + i(0,2\pi)$ such that

$$\tilde{N}(s + i0) = N(s). \tag{A26}$$

To begin with, we notice that

$$\frac{i}{2}Q(0,s) = \frac{i}{2}\int_{x^1>0} d^{d-1}xf_\varphi(\bar{x})f_\pi^s(\bar{x}) = i\mathrm{Im}\langle f_{\varphi,R}, f_{\pi,R}^s\rangle_{\mathfrak{H}}, \tag{A27}$$

$$\frac{i}{2}R(0,s) = \frac{1}{2}\int_{x^1>0} d^{d-1}xf_\varphi^s(\bar{x})f_\pi(\bar{x}) = i\mathrm{Im}\langle f_{\varphi,R}^s, f_{\pi,R}\rangle_{\mathfrak{H}}, \tag{A28}$$

where the above expressions make sense regardless of $f_{\pi,R}^s \notin \mathfrak{H}$. This is because

$$\langle f_{\varphi,R}, f_{\pi,R}^s\rangle_{\mathfrak{H}} = \int_{\mathbb{R}^{d-1}}\frac{d^{d-1}p}{2\omega_{\bar{p}}}\hat{f}_{\varphi,R}(\bar{p})^* i\omega_{\bar{p}}\hat{f}_{\pi,R}^s(\bar{p}) = \frac{i}{2}\langle\hat{f}_{\varphi,R}, \hat{f}_{\pi,R}^s\rangle_{L^2}, \tag{A29}$$

which is convergent. The problem involving scalar products of split functions $f_{\varphi,R}^s$ and $f_{\pi,R}^s$ happens only when we try to compute the scalar product of two sharply cut test functions of the momentum operator, e.g.,

$$\langle f_{\pi,R}, f_{\pi,R}^s\rangle_{\mathfrak{H}} = \int_{\mathbb{R}^{d-1}}\frac{d^{d-1}p}{2\omega_{\bar{p}}}(i\omega_{\bar{p}}\hat{f}_{\pi,R}(\bar{p}))^* i\omega_{\bar{p}}\hat{f}_{\pi,R}^s(\bar{p}) = \frac{1}{2}\langle\hat{f}_{\pi,R}, \hat{f}_{\pi,R}^s\rangle_{H^{\frac{1}{2}}}, \tag{A30}$$

which is in general divergent. Such divergency comes from the noncontinuity of the function $f_{\pi,R}(\bar{x}) = f_\pi(\bar{x})\Theta(x^1)$ at $x^1 = 0$. To overcome this difficulty we introduce a family of smooth functions (for $\epsilon > 0$)

$$f^\varepsilon_{\varphi,R}(\bar{x}) \coloneqq f_\varphi(\bar{x})\Theta_\varepsilon(x^1) \quad \text{and} \quad f^\varepsilon_{\pi,R}(\bar{x}) \coloneqq f_\pi(\bar{x})\Theta_\varepsilon(x^1), \tag{A31}$$

where $\Theta_\varepsilon \in C^\infty(\mathbb{R})$ is a regularized Heaviside function such that

$$\Theta_\varepsilon(t) = \begin{cases} 0 & \text{if } t \le \frac{\varepsilon}{2} \\ 1 & \text{if } t \ge \varepsilon \end{cases}. \tag{A32}$$

Then

$$f^\varepsilon_{\varphi,R}(\bar{x}) \underset{\epsilon\to 0^+}{\to} f_{\varphi,R}(\bar{x}) \quad \text{and} \quad f^\varepsilon_{\pi,R}(\bar{x}) \underset{\epsilon\to 0^+}{\to} f_{\pi,R}(\bar{x}), \tag{A33}$$

where the above convergence must be in a sense that we specify opportunely below. Before we get into such convergence issues, we notice that $f^\varepsilon_{\varphi,R}, f^\varepsilon_{\pi,R} \in \mathcal{S}(\mathbb{R}^{d-1}, \mathbb{R})$ and hence the scalar product (A30) is now well defined. Then we define the function

$$N^\epsilon(s) \coloneqq i\mathrm{Im}\langle f^\epsilon_{\varphi,R}, f^{s,\epsilon}_{\pi,R}\rangle_{\mathfrak{H}} - i\mathrm{Im}\langle f^{s,\epsilon}_{\varphi,R}, f^\epsilon_{\pi,R}\rangle_{\mathfrak{H}} - \frac{1}{2}\|g^{s,\epsilon}_R\|^2_{\mathfrak{H}}, \tag{A34}$$

which is just the regularized version of (A25). In the next subsection we show that $N^\epsilon(s) \to N(s)$ when $\epsilon \to 0^+$. Expression (A34) can be rewritten as

$$\begin{aligned} N^\epsilon(s) &= i\mathrm{Im}\langle f^\epsilon_{\varphi,R}, f^{s,\epsilon}_{\pi,R}\rangle_{\mathfrak{H}} - i\mathrm{Im}\langle f^{s,\epsilon}_{\varphi,R}, f^\epsilon_{\pi,R}\rangle_{\mathfrak{H}} - \frac{1}{2}\|g^{s,\epsilon}_R\|^2_{\mathfrak{H}} \\ &= i\mathrm{Im}\langle f^\epsilon_{\varphi,R}, f^{s,\epsilon}_{\pi,R}\rangle_{\mathfrak{H}} + i\mathrm{Im}\langle f^\epsilon_{\pi,R}, f^{s,\epsilon}_{\varphi,R}\rangle_{\mathfrak{H}} - \frac{1}{2}\langle f^\epsilon_R - f^{s,\epsilon}_R, f^\epsilon_R - f^{s,\epsilon}_R\rangle_{\mathfrak{H}} \\ &= \langle f^\epsilon_R, f^{s,\epsilon}_R\rangle_{\mathfrak{H}} - \frac{1}{2}\langle f^\epsilon_R, f^\epsilon_R\rangle_{\mathfrak{H}} - \frac{1}{2}\langle f^{s,\epsilon}_R, f^{s,\epsilon}_R\rangle_{\mathfrak{H}} = \langle f^{-\frac{s}{2},\epsilon}_R, f^{\frac{s}{2},\epsilon}_R\rangle_{\mathfrak{H}} - \langle f^\epsilon_R, f^\epsilon_R\rangle_{\mathfrak{H}} \\ &= \int_{\mathbb{R}^{d-1}} \frac{d^{d-1}p}{2\omega_{\bar{p}}}\left[\left(\hat{f}^{-\frac{s}{2},\epsilon}_{\varphi,R} + i\omega_{\bar{p}}\hat{f}^{-\frac{s}{2},\epsilon}_{\pi,R}\right)^*\left(\hat{f}^{\frac{s}{2},\epsilon}_{\varphi,R} + i\omega_{\bar{p}}\hat{f}^{\frac{s}{2},\epsilon}_{\pi,R}\right) - (\hat{f}^\epsilon_{\varphi,R} + i\omega_{\bar{p}}\hat{f}^\epsilon_{\pi,R})^*(\hat{f}^\epsilon_{\varphi,R} + i\omega_{\bar{p}}\hat{f}^\epsilon_{\pi,R})\right], \end{aligned} \tag{A35}$$

where in the penultimate line we have used that $f^{s_1+s_2,\epsilon}_R = u(\Lambda^{s_2}_1)f^{s_1,\epsilon}_R$ for all $s_1, s_2 \in \mathbb{R}$. For a moment, let assume that this last expression converges to

$$N(s) = \int_{\mathbb{R}^{d-1}} \frac{d^{d-1}p}{2\omega_{\bar{p}}}\left[\left(\hat{f}^{-\frac{s}{2}}_{\varphi,R} + i\omega_{\bar{p}}\hat{f}^{-\frac{s}{2}}_{\pi,R}\right)^*\left(\hat{f}^{\frac{s}{2}}_{\varphi,R} + i\omega_{\bar{p}}\hat{f}^{\frac{s}{2}}_{\pi,R}\right) - (\hat{f}_{\varphi,R} + i\omega_{\bar{p}}\hat{f}_{\pi,R})^*(\hat{f}_{\varphi,R} + i\omega_{\bar{p}}\hat{f}_{\pi,R})\right], \tag{A36}$$

when $\epsilon \to 0^+$. We prove this in the next subsection. The second term of the above integrand is independent on s and hence its analytic continuation is trivial. Let us then focus on the first term. Using the Poincaré covariance and causality of the Klein-Gordon equation, it is not difficult to show that

$$\hat{f}^s_{\varphi,R}(\bar{p}) + i\omega_{\bar{p}}\hat{f}^s_{\pi,R}(\bar{p}) = \hat{f}_{\varphi,R}(\Lambda^s_1\bar{p}) + i\Lambda^s_1\omega_{\bar{p}}\hat{f}_{\pi,R}(\Lambda^s_1\bar{p}), \tag{A37}$$

where $\Lambda^s_1\bar{p} = (p^1\cosh(s) - \omega_{\bar{p}}\sinh(s), \bar{p}_\perp)$ and $\Lambda^s_1\omega_{\bar{p}} = \omega_{\bar{p}}\cosh(s) - p^1\sinh(s)$. Then, the first integrand term of (A36) becomes

$$\begin{aligned} &(\hat{f}^{-\frac{s}{2},\epsilon}_{\varphi,R}(\bar{p}) + i\omega_{\bar{p}}\hat{f}^{-\frac{s}{2},\epsilon}_{\pi,R}(\bar{p}))^*(\hat{f}^{\frac{s}{2},\epsilon}_{\varphi,R}(\bar{p}) + i\omega_{\bar{p}}\hat{f}^{\frac{s}{2},\epsilon}_{\pi,R}(\bar{p})) \\ &\quad = \int_{\mathbb{R}^{2(d-1)}} d^{d-1}x d^{d-1}y (f_{\varphi,R}(\bar{x}) - i\omega_{\bar{p}}f_{\pi,R}(\bar{x}))(f_{\varphi,R}(\bar{y}) + i\omega_{\bar{p}}f_{\pi,R}(\bar{y}))\mathrm{e}^{i\Lambda^{-\frac{s}{2}}(\bar{p})\cdot\bar{x}}\mathrm{e}^{-i\Lambda^{\frac{s}{2}}(\bar{p})\cdot\bar{y}}, \end{aligned} \tag{A38}$$

where $-i\Lambda^{\frac{s}{2}}(\bar{p})\cdot\bar{y} = -i(-\sinh(\frac{s}{2})\omega_{\bar{p}} + \cosh(\frac{s}{2})p^1)y^1 - i\bar{p}_\perp\cdot\bar{y}_\perp$, and equivalently for $i\Lambda^{-\frac{s}{2}}(\bar{p})\cdot\bar{x}$. Then

$$\begin{aligned} &-i\left(-\sinh\left(\frac{s}{2}\right)\omega_{\bar{p}} + \cosh\left(\frac{s}{2}\right)p^1\right)y^1 \\ &\quad \underset{s\to s+i\sigma}{\to} -i\left(-\sinh\left(\frac{s+i\sigma}{2}\right)\omega_{\bar{p}} + \cosh\left(\frac{s+i\sigma}{2}\right)p^1\right)y^1 \\ &\quad = -i\left(-\sinh\left(\frac{s}{2}\right)\omega_{\bar{p}} + \cosh\left(\frac{s}{2}\right)p^1\right)y^1\cos\left(\frac{\sigma}{2}\right) - \underbrace{\left(\cosh\left(\frac{s}{2}\right)\omega_{\bar{p}} - \sinh\left(\frac{s}{2}\right)p^1\right)}_{\ge m}y^1\sin\left(\frac{\sigma}{2}\right), \end{aligned} \tag{A39}$$

where the second term provides an exponential dumping in Eq. (A38) when $\sigma \in (0, 2\pi)$ because $\mathrm{supp}(f_{\varphi,R})$, $\mathrm{supp}(f_{\pi,R}) \subset \Sigma$. Equivalently it can be shown that $i\Lambda^{-\frac{s}{2}}(\bar{p}) \cdot \bar{x}$ also provides an exponential dumping for $\sigma \in (0, 2\pi)$. Hence we have that

$$\tilde{N}(s+i\sigma) \text{ is an analytic function for } s+i\sigma \in \mathbb{R}+i(0,2\pi). \tag{A40}$$

Looking at expressions (A36) and (A38), it is easy to determine that

$$\lim_{\sigma\to 2\pi^-, s=0} \tilde{N}(s+i\sigma) = 0. \tag{A41}$$

a. Convergence of $N^\epsilon(s)$

In order to show that expression (A36) holds, we need to prove the following two limits:

$$N^\epsilon(s) \underset{\epsilon\to 0^+}{\to} N(s) = \frac{i}{2}(Q(0,s) - R(0,s)) - \frac{1}{2}\|g_R^s\|^2_{\mathfrak{H}}, \tag{A42}$$

$$N^\epsilon(s) \underset{\epsilon\to 0^+}{\to} \int_{\mathbb{R}^{d-1}} \frac{d^{d-1}p}{2\omega_{\bar{p}}} \left[\left(\hat{f}^{-\frac{s}{2}}_{\varphi,R} + i\omega_{\bar{p}} \hat{f}^{-\frac{s}{2}}_{\pi,R} \right)^* \times \left(\hat{f}^{\frac{s}{2}}_{\varphi,R} + i\omega_{\bar{p}} \hat{f}^{\frac{s}{2}}_{\pi,R} \right) - (\hat{f}_{\varphi,R} + i\omega_{\bar{p}} \hat{f}_{\pi,R})^* \times (\hat{f}_{\varphi,R} + i\omega_{\bar{p}} \hat{f}_{\pi,R}) \right]. \tag{A43}$$

To do this, we must be precise in which sense the functions $f^{s,\epsilon}_{\varphi,R}, f^{s,\epsilon}_{\pi,R}$ converge in (A33). To begin we choose the following smooth step function (A32)

$$\Theta_\varepsilon(t) = \begin{cases} 0 & \text{if } t \le \frac{\varepsilon}{2}, \\ \left[\exp\left(\frac{\epsilon(t-\frac{3\epsilon}{4})}{(t-\frac{3\epsilon}{4})^2 - (\frac{\epsilon}{4})^2}\right) + 1\right]^{-1} & \text{if } \frac{\varepsilon}{2} < t < \varepsilon, \\ 1 & \text{if } t \ge \varepsilon. \end{cases} \tag{A44}$$

First we focus on the limit (A43). Looking back to (A35), we can rewrite the rhs of that expression as

$$\begin{aligned} N^\epsilon(s) &= \langle f_R^{-\frac{s}{2},\epsilon}, f_R^{\frac{s}{2},\epsilon} \rangle_{\mathfrak{H}} - \langle f_R^\epsilon, f_R^\epsilon \rangle_{\mathfrak{H}} = \langle f_R^{-\frac{s}{2},\epsilon}, f_R^{\frac{s}{2},\epsilon} - f_R^\epsilon + f_R^\epsilon \rangle_{\mathfrak{H}} - \langle f_R^\epsilon, f_R^\epsilon \rangle_{\mathfrak{H}} \\ &= \langle f_{\varphi,R}^{-\frac{s}{2},\epsilon}, f_{\varphi,R}^{\frac{s}{2},\epsilon} - f_{\varphi,R}^\epsilon \rangle_{\mathfrak{H}} + \langle f_{\varphi,R}^{-\frac{s}{2},\epsilon}, f_{\pi,R}^{\frac{s}{2},\epsilon} - f_{\pi,R}^\epsilon \rangle_{\mathfrak{H}} + \langle f_{\varphi,R}^{-\frac{s}{2},\epsilon} - f_{\varphi,R}^\epsilon, f_{\varphi,R}^\epsilon \rangle_{\mathfrak{H}} \\ &\quad + \langle f_{\pi,R}^{-\frac{s}{2},\epsilon} - f_{\pi,R}^\epsilon, f_{\varphi,R}^\epsilon \rangle_{\mathfrak{H}} + \langle f_{\pi,R}^{-\frac{s}{2},\epsilon}, f_{\varphi,R}^{\frac{s}{2},\epsilon} - f_{\varphi,R}^\epsilon \rangle_{\mathfrak{H}} + \langle f_{\varphi,R}^{-\frac{s}{2},\epsilon} - f_{\varphi,R}^\epsilon, f_{\pi,R}^\epsilon \rangle_{\mathfrak{H}} \qquad \text{(A45)} \\ &\quad + \underbrace{\langle f_{\pi,R}^{-\frac{s}{2},\epsilon}, f_{\pi,R}^{\frac{s}{2},\epsilon} - f_{\pi,R}^\epsilon \rangle_{\mathfrak{H}}}_{\circledast} + \underbrace{\langle f_{\pi,R}^{-\frac{s}{2},\epsilon} - f_{\pi,R}^\epsilon, f_{\pi,R}^\epsilon \rangle_{\mathfrak{H}}}_{\circledast}. \qquad \text{(A46)} \end{aligned}$$

It is not difficult to see that

$$f^{s,\epsilon}_{\varphi,R} \underset{\epsilon\to 0^+}{\to} f^s_{\varphi,R} \quad \text{and} \quad f^{s,\epsilon}_{\pi,R} \underset{\epsilon\to 0^+}{\to} f^s_{\pi,R}, \quad \text{in } L^2(\mathbb{R}^{d-1}), \tag{A47}$$

which implies that all terms in (A46) are convergent, except perhaps those pointed by $\circledast$. Now we concentrate in those remaining terms, e.g.,

$$\langle f_{\pi,R}^{-\frac{s}{2},\epsilon}, f_{\pi,R}^{\frac{s}{2},\epsilon} - f_{\pi,R}^\epsilon \rangle_{\mathfrak{H}} = \frac{1}{2} \int_{\mathbb{R}^{d-1}} d^{d-1}p \hat{f}_{\pi,R}^{-\frac{s}{2},\epsilon}(\bar{p}) (\hat{f}_{\pi,R}^{-\frac{s}{2},\epsilon}(\bar{p}) - \hat{f}_{\pi,R}^\epsilon(\bar{p})) \omega_{\bar{p}}. \tag{A48}$$

The convergence of (A48) is guaranteed by the fact that

$$f^\epsilon_{\pi,R} - f^{-\frac{s}{2},\epsilon}_{\pi,R} \underset{\epsilon\to 0^+}{\to} f_{\pi,R} - f^{-\frac{s}{2}}_{\pi,R} \quad \text{in } H^1(\mathbb{R}^{d-1}), \tag{A49}$$

$$\Downarrow$$

$$(\hat{f}^\epsilon_{\pi,R} - \hat{f}^{-\frac{s}{2},\epsilon}_{\pi,R})\omega_{\bar{p}} \underset{\epsilon\to 0^+}{\to} (\hat{f}_{\pi,R} - \hat{f}^{-\frac{s}{2}}_{\pi,R})\omega_{\bar{p}} \quad \text{in } L^2(\mathbb{R}^{d-1}). \tag{A50}$$

In order to probe (A49) we remember that $f_{\pi,R}(\bar{x}) - f^s_{\pi,R}(\bar{x}) = g^s_\pi(\bar{x})\Theta(x^1)$ with $g^s_\pi \in \mathcal{S}(\mathbb{R}^{d-1}, \mathbb{R})$ and $g^s_\pi|_{x^1=0} = 0$. Then the following lemma ensures (A49).

Lemma A.3: Let $g \in \mathcal{S}(\mathbb{R}^n)$ with $g|_{x^1=0} = 0$, $g_R(\bar{x}) = g(\bar{x})\Theta(x^1)$ and $g_R^\epsilon(\bar{x}) = g(\bar{x})\Theta_\epsilon(x^1)$ with Θ_ϵ as (A44). Then $g_R \in H^1(\mathbb{R}^n)$ and $g_R^\epsilon \to_{\epsilon\to 0^+} g_R$ in $H^1(\mathbb{R}^n)$.

Proof.—The fact that $g_R \in H^1(\mathbb{R}^n)$ is guaranteed by lemma 5. 10. Then we prove the convergence for $n = 1$. The generalization to $n > 1$ is straightforward. Since g_R, g_R^ϵ satisfies the hypothesis of the lemma A.2, their weak derivatives coincide with theirs pointwise derivatives and hence

$$\begin{aligned}\|g_R^\epsilon - g_R\|'^2{}_{H^1} &= \int_{-\infty}^{+\infty} dx|g(x)\Theta_\epsilon(x) - g(x)\Theta(x)|^2 + \int_{-\infty}^{+\infty} dx|\partial_x[g(x)\Theta_\epsilon(x) - g(x)\Theta(x)]|^2 \\ &\leq \int_{-\infty}^{+\infty} dx|g(x)|^2|\Theta_\epsilon(x) - \Theta(x)|^2 + \int_{-\infty}^{+\infty} dx|g'(x)|^2|\Theta_\epsilon(x) - \Theta(x)|^2 \\ &\quad + \int_{-\infty}^{+\infty} dx|g(x)|^2|\Theta'_\epsilon(x)|^2 + 2\int_{-\infty}^{+\infty} dx|g(x)||g'(x)||\Theta_\epsilon(x) - \Theta(x)||\Theta'_\epsilon(x)| \\ &\leq \int_{\frac{\epsilon}{2}}^{\epsilon} dx(|g(x)|^2 + |g'(x)|^2) + \int_{\frac{\epsilon}{2}}^{\epsilon} dx|g(x)|^2|\Theta'_\epsilon(x)|^2 + 2\int_{\frac{\epsilon}{2}}^{\epsilon} dx|g(x)||g'(x)||\Theta'_\epsilon(x)|. \end{aligned} \tag{A51}$$

We notice that since $g \in C^\infty(\mathbb{R})$ and $g(0) = 0$, by the Taylor theorem we have that $g(x) = g'(0)x + r(x)x$ with $r(x)\to_{x\to 0}0$ and $r \in C^\infty(\mathbb{R})$. We also have that $\max_{x\in\mathbb{R}}|\Theta'_\epsilon(x)| = \frac{4}{\epsilon}$, which follows from the definition of that function. Then using the above properties and assuming $0 < \varepsilon \leq 1$,

$$\begin{aligned}\|g_R^\epsilon - g_R\|'^2{}_{H^1} &\leq \max_{x\in[0,1]}(|g(x)|^2 + |g'(x)|^2)\int_{\frac{\epsilon}{2}}^{\epsilon} dx + \max_{x\in[0,1]}|g'(0) + r(x)|^2\frac{16}{\epsilon^2}\int_{\frac{\epsilon}{2}}^{\epsilon} dxx^2 \\ &\quad + \max_{x\in[0,1]}|g'(x)|\max_{x\in[0,1]}|g'(0) + r(x)|\frac{8}{\epsilon}\int_{\frac{\epsilon}{2}}^{\epsilon} dxx \\ &\leq \max_{x\in[0,1]}(|g(x)|^2 + |g'(x)|^2)\frac{\epsilon}{2} + \max_{x\in[0,1]}|g'(0) + r(x)|^2\frac{14}{3}\epsilon \\ &\quad + \max_{x\in[0,1]}|g'(x)|\max_{x\in[0,1]}|g'(0) + r(x)|3\epsilon \underset{\epsilon\to 0^+}{\to} 0. \end{aligned} \tag{A52}$$

■

Then we have that all terms in (A46) converge. By continuity of the scalar product, the limit of (A46) is just this same expression but evaluated at $\epsilon = 0$, which coincides with the lhs of (A47).

We use the same arguments to prove the limit (A42). The first two terms of (A34) are convergent due to (A47), and the remaining term is also convergent due to (A49) and (A50). Then by continuity of the scalar product we have that

$$N^\epsilon(s) \underset{\epsilon\to 0^+}{\to} N(s) = \frac{i}{2}(Q(0,s) - R(0,s)) - \frac{1}{2}\|g_R^s\|^2_{\mathfrak{H}}. \tag{A53}$$

Finally, expression (A36) holds.

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