José L. CASTIGLIONI and Hernán J. SAN MARTÍN

ON THE VARIETY OF HEYTING ALGEBRAS WITH SUCCESSOR GENERATED BY ALL FINITE CHAINS

A b s t r a c t. Contrary to the variety of Heyting algebras, finite Heyting algebras with successor only generate a proper subvariety of that of all Heyting algebras with successor. In particular, all finite chains generate a proper subvariety, SLH_ω, of the latter.

There is a categorical duality between Heyting algebras with successor and certain Priestley spaces. Let X be the Heyting space associated by this duality to the Heyting algebra with successor H.

If there is an ordinal κ and a filtration on X such that $X = \bigcup_{\lambda \leq \kappa} X_\lambda$, the height of X is the minimun ordinal $\xi \leq \kappa$ such that $X_\xi^c = \emptyset$. In this case, we also say that H has height ξ. This filtration allows us to write the space X as a disjoint union of antichains. We may think that these antichains define levels on this space.

We study the way of characterize subalgebras and homomorphic images in finite Heyting algebras with successor by means of their Priestley spaces. We also depict the spaces associated to the free algebras in various subcategories of SLH_ω.

1. Introduction

In this paper we study the successor function in Heyting algebras. This operation was introduced by Kusnetsov in [10] and studied by Caicedo and Cignoli in [3]. A set $E(f)$ of equations in the signature of Heyting algebras augmented with the unary function symbol f is said to define an implicit operation of Heyting algebras if for any Heyting algebra H there is at most one function $f_H : H \to H$. The function f is an implicit compatible operation provided all f_H are compatible.

The system $E(S)$ consisting of the following equations [3] defines an implicit compatible operation S (called successor) of Heyting algebras:

(S1) $x \leq S(x)$,

(S2) $S(x) \leq y \vee (y \to x)$,

(S3) $S(x) \to x = x$.

Since S does not exist in the Heyting algebra $[0,1]$, we get that it is not a term in the language of Heyting algebras. We say that a Heyting algebra endowed with its successor function is a S-Heyting algebra. We write (H, S) for a S-Heyting algebra.

We recall that Heyting duality (see [13] or [9]) establishes a dual equivalence between the category **HA** of Heyting algebras and Heyting algebra homomorphisms and the category **HS** of Heyting spaces and p-continuous morphisms,

$$\mathcal{PF} : \mathbf{HA} \leftrightarrows \mathbf{HS}^{op} : \mathcal{CU}$$

Morphisms of **SH** are called Heyting morphisms of Heyting spaces. Here, for every Heyting algebra H we write $\mathcal{PF}(H)$ for the set of prime filters of H. For every Heyting space $(X, \leq)$, $\mathcal{CU}(X, \leq)$ denotes the set of clopen upsets of $(X, \leq)$. We have that $\varphi_H(x) = \{P \in \mathcal{PF}(H) : x \in P\}$ defines an isomorphism of Heyting algebras between H and $\mathcal{CU}(\mathcal{PF}(H), \subseteq)$ and $G_X(x) = \{U \in \mathcal{CU}(X, \leq) : x \in U\}$ defines an isomorphism of Heyting spaces between $(X, \leq)$ and $\mathcal{PF}(\mathcal{CU}(X, \leq))$.

We write **SHA** for the category whose objets are S-Heyting algebras and whose morphisms are Heyting algebras homomorphisms that commute

with the successor. A Heyting space $(X, \leq)$ is a S-Heyting space if for every $U \in \mathcal{CU}(X, \leq)$ the set $U \cup (U^c)_M$ is clopen, where $(U^c)_M$ is the set of maximal elements in U^c. We observe that $(X, \leq)$ is a S-Heyting space if and only if it is a Heyting space such that for every clopen downset V the set V_M is clopen. Let $(X, \leq)$ and $(Y, \leq)$ be S-Heyting spaces and $g : (X, \leq) \to (Y, \leq)$ a Heyting morphism of Heyting spaces. We say that g is a S-Heyting spaces morphism if for every downset V in $(X, \leq)$ it holds that $g^{-1}(V_M) = [g^{-1}(V)]_M$. We denote by $\mathbf{SH}_S$ the category whose objects are S-Heyting spaces and whose morphisms are S-Heyting space morphisms. We recall that in [4] it was proved that there exists a dual equivalence between the category $\mathbf{SHA}$ and the category $\mathbf{SH}_S$ and that if $(X, \leq)$ is a S-Heyting space then in $\mathcal{CU}(X, \leq)$ the successor function takes the form

$$S(U) = U \cup (U^c)_M$$

In section 2 we give some general properties of the successor function. In section 3 we study some particular cases of the duality: linear Heyting algebras with successor and algebras of finite and ω height. In section 4 we characterize the S-subalgebras and homomorphic images. For the case of finite S-Heyting algebras we give some pictorial examples of their determination. Finally in section 5 we study the free algebras in finite generators of subvarieties of the variety generated by all finite chains.

2. Some general properties of S

Let H be a Heyting algebra and E_x the filter $\{y \in H : y \to x \leq y\}$. It was proved in [4] that the function S can be defined as $S(x) = min\ E_x$.

For example, the successor operation exists in any Boolean algebra, where it is the constant function 1. It also exists in any finite Heyting algebra and in the free Heyting algebra with one generator. In the chain of natural numbers with a top ω, $S(n) = n + 1$, for every natural number n, and $S(\omega) = \omega$. For more examples see [7].

Proposition 2.1. *Let (H, S) be a S-Heyting algebra.*

(i) $S(x \wedge y) = S(x) \wedge S(y)$, *for every $x, y \in H$.*

(ii) *The equations* $(S1)$ *and* $(S3)$ *are equivalent to the equation* $S(x) \to x \leq S(x)$.

(iii) $S(x) = x$ *if and only if* $x = 1$.

Proof. (i) It is a consequence of ([4], Lemma 2.2).
(ii) It is a consquence of the fact that, $y \to x \leq y$ if and only if $y \to x = x$, $x \leq y$.
(iii) It follows from $(S1)$ and $(S3)$. $\square$

We shall say that M is a subalgebra of (H, S) (or a S-subalgebra) if M is a subalgebra of H such that for every $x \in M$, $S(x) \in M$.

Lemma 2.2. *Let* (H_1, S) *and* (H_2, S) *be S-Heyting algebras, and let f :* $H_1 \to H_2$ *be a Heyting algebras homomorphism. The following conditions are equivalent:*

(a) f *is a morphism in* **SHA***.*

(b) *For every* $x \in H_1$ *we have that* $f(S(x)) \leq S(f(x))$.

(c) *For every* $x, y \in H_1$ *the following equations hold:*

$$f(S(x)) \leq y \vee (y \to f(x)) \,, \tag{1}$$

$$f(S(x)) \to f(x) = f(x) \tag{2}$$

(d) $Im(f)$ *is a S-subalgebra of* H_2.

Proof.
$(a) \Leftrightarrow (b)$. We only need to prove $(b) \Rightarrow (a)$. We have that $S(f(x)) \leq f(S(x)) \vee (f(S(x)) \to f(x)) = f(S(x)) \vee f(S(x) \to x) = f(S(x)) \vee f(x) = f(x \vee S(x)) = f(S(x))$.

$(a) \Leftrightarrow (c)$. We suppose that the equations (1) and (2) hold. Taking $y = S(f(x))$ in (1), $f(S(x)) \leq S(f(x)) \vee (S(f(x)) \to f(x)) = S(f(x)) \vee f(x) = S(f(x))$. Hence (b) holds, and then (a) holds.
Conversely we suppose that (a) holds. Using $(S2)$ we conclude that $f(S(x)) = S(f(x)) \leq y \vee (y \to f(x))$. Now using $(S3)$ we conclude that $f(S(x)) \to f(x) = S(f(x)) \to f(x) = f(x)$.

$(a) \Leftrightarrow (d)$. We suppose that f commute with S. Let $y \in Im(f)$. Thus there exists $x \in H_1$ such that $f(x) = y$. Then $S(y) = S(f(x)) = f(S(x))$. Therefore $S(y) \in Im(f)$.

Conversely, let $Im(f)$ be a Heyting subalgebra of H_2 closed by S. If $x \in H_1$ we have that $S(f(x)) \in Im(f)$. Then there exists $z \in H_1$ such that $S(f(x)) = f(z)$. By $(S2)$ we conclude that $S(x) \leq z \vee (z \to x)$. Hence $f(S(x)) \leq f(z) \vee (f(z) \to f(x)) = S(f(x)) \vee (S(f(x)) \to f(x)) = S(f(x)) \vee f(x) = S(f(x))$. Then (b) holds, so (a) holds. $\qquad\square$

Lemma 2.3. (i) *Let (H_1, S) and (H_2, S) be S-Heyting algebras, and let $f : H_1 \to H_2$ be a Heyting algebras homomorphism. If f is a surjective function then it is a morphism in* **SHA**.

(ii) *Let $(X, \leq)$ and $(Y, \leq)$ be S-Heyting spaces and $g : (X, \leq) \to (Y, \leq)$ a continuous p-morphism. If g is an injective function then it is a morphism in* **SH**$_S$.

Proof.
(i) It is immediate from Lemma 2.2.d.

(ii) It follows from (i) and the duality between **SH**$_S$ and **SHA**. $\qquad\square$

3. Particular cases of the duality

In this section we characterize the Heyting spaces asociated to linear Heyting algebras with successor. Then we define the notion of height of a Heyting space and of a Heyting algebra.

3.1 Linear S-**Heyting algebras**

Linear Heyting algebras were considered by Horn in [8] as an intermediate step between the classical calculus and intuitionistic one and were studied also by Monteiro [12], G. Martínez [11] and others. This is the subvariety of Heyting algebras generated by the class of totally ordered Heyting algebras and can be axiomatized by the usual equations for Heyting algebras plus the linearity law $(x \to y) \vee (y \to x) = 1$. In ([1], ch.IX) and in [12] there are characterizations for linear Heyting algebras.

If H is a Heyting algebra, for every $P \in \mathcal{PF}(H)$ we define $C_P = \{Q \in \mathcal{PF}(H) : P \subseteq Q\}$. We say that $\mathcal{PF}(H)$ is a root system if for every $P \in \mathcal{PF}(H)$ the set C_P is totally ordered in $(\mathcal{PF}(H), \subseteq)$. Horn showed in [8] (although it was in fact proved by Monteiro, see [12]) that linear Heyting algebras can be characterized among Heyting algebras in terms of the prime filters. Specifically, a Heyting algebra is a linear Heyting algebra if and only if $(\mathcal{PF}(H), \subseteq)$ is a root system.

Let SLH be the full subcategory of **SHA** whose objects are linear Heyting algebras with S. A SL-Heyting space is a S-Heyting space such that for every $x \in X$ the set $[x) = \{y \in X : y \geq x\}$ is totally ordered. The category SLS is the full subcategory of **SHS** whose objects are SL-Heyting spaces.

For the duality given between **SHA** and **SH**$_S$ we have the following

Theorem 3.1. *There is a dual equivalence between the categories* SLH *and* SLS.

We now give an equivalent way to describe S in the algebra of clopen upsets of an object of SLS.

Let $(X, \leq)$ be an object in SLS and V a clopen downset in $(X, \leq)$. For every $x \in V$ we write x_V for the maximum of the set $[x) \cap V = \{y \in V : y \geq x\}$.

Proposition 3.2. *Let* $(X, \leq)$ *be an object in* SLS *and* V *a non empty clopen downset. Then for every* $x \in V$ *there exists* x_V. *Morever,* $V_M = \bigcup_{x \in V}\{x_V\}$.
In particular, if $U \in \mathcal{CU}(X, \leq)$ *and* $U \neq X$ *then*

$$S(U) = U \cup (\bigcup_{x \in U^c} \{x_{U^c}\})$$

Proof. Let V be a non empty clopen downset. First we prove that for every $x \in V$ the set $[x) \cap V$ has maximal elements. It is equivalent to prove that if $H = \mathcal{CU}(X, \leq)$ then for every $P \in \mathcal{PF}(H)$ and V clopen downset in $\mathcal{PF}(H)$ such that $P \in V$ we have that the set $\{Q \in \mathcal{PF}(H) : P \subseteq Q, Q \in V\}$ has maximal elements. The latter is a consequence of Zorn's Lemma (see

the proof of Lemma 4.4 in [4]).

Let $y \in [x) \cap V$ and y be maximal in this set. If $z \in [x) \cap V$ then $x \leq z$ and $x \leq y$. Thus $y \leq z$ or $z \leq y$. Since $y, z \in [x) \cap V$ and y is maximal in this set, if $y \leq z$ then $y = z$. In any case $z \leq y$, so $[x) \cap V$ has a maximum.

We now prove that $V_M = \bigcup_{x \in V}\{x_V\}$. If $x \in V_M$ then $x \in [x) \cap V$. Let $y \in [x) \cap V$, so $x \leq y$ with $y \in V$. Using that $x \in V_M$ we conclude that $x = y$. Hence $[x) \cap V = \{x\}$ and then $x \in \bigcup_{x \in V}\{x_V\}$.

Conversely, let $y \in \bigcup_{x \in V}\{x_V\}$. Thus $y \in V$ and there exists $x \in V$ such that $x \leq y$, and if $x \leq z$ and $z \in V$ then $z \leq y$. We want to prove that $y \in V_M$. Let $y \leq z$, with $z \in V$. In particular it holds that $x \leq z$ and then $z \leq y$. Hence $z = y$.   □

3.2 S-Heyting algebras of height ω

Let $(X, \leq)$ be a S-Heyting space. Let κ be an ordinal. We define a filtration on $(X, \leq)$ by $X_{\emptyset} = X_M$ and for $\lambda \leq \kappa$, $X_{\kappa} = \left(\bigcup_{\lambda < \kappa} X_{\lambda}\right) \cup \left(\left(\bigcup_{\lambda < \kappa} X_{\lambda}\right)^c\right)_M$. We say that $(X, \leq)$ is a κ-*filtered space* if

$$X = \bigcup_{\lambda \leq \kappa} X_{\lambda} \tag{3}$$

We call the *height* of a κ-filtered space $(X, \leq)$ to the minimun ordinal $\xi \leq \kappa$ such that $X_{\xi}^c = \emptyset$. We say that a S-Heyting algebra H has *height* ξ if its associated Heyting space does.

Remark 3.3. Note that for the finite ordinals we have that $X_{\emptyset} =:$ $X_1 \subseteq X_2 \subseteq \ldots \subseteq X_n \subseteq X_{n+1} \subseteq \ldots$. Hence, if we write ω for the first non finite ordinal, and we assume that $(X, \leq)$ is ω-filtered, then (3) can be written as

$$X = \bigcup_{n \leq \omega} X_n = \bigcup_{n \in \mathbb{N}} X_n \cup \left(\bigcup_{n \in \mathbb{N}} X_n\right)^c_M \tag{4}$$

Let $\mathcal{K}$ be a class of S-Heyting algebras of height less equal to a fixed ordinal ξ. Using the categorical duality between S-Heyting algebras and S-Heyting spaces, it can be shown that the elements of classes $\mathbf{H}(\mathcal{K})$, $\mathbf{S}(\mathcal{K})$ and $\mathbf{P}(\mathcal{K})$ have also height less or equal to ξ. Here $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{P}$ are the

class operators of universal algebra [2]. Hence for each ordinal ξ, the class of S-Heyting algebras of height less or equal to ξ is a variety.

For every natural number $n \geq 1$ we write $\mathrm{SH_n}$ for the variety of S-Heyting algebras of height less or equal to n. This variety can be characterized by the equation

$$S^{(n)}(0) = 1$$

We write SH_ω for the variety of S-Heyting algebras of height ω.

Note that $\mathrm{SH_1}$ is exactly the variety of Boolean algebras, and that we have that

$$\mathrm{SH_1} \subseteq \mathrm{SH_2} \subseteq \ldots \mathrm{SH_n} \subseteq \ldots \subseteq \mathrm{SH}_\omega$$

Example 1. Let L_n be the chain $\{0, 1, \ldots, n\}$ seen as Heyting algebra with successor. The height of this algebra is $h(\mathrm{L}_n) = n$, and in consequence $\mathrm{L}_n \in \mathrm{SH_n}$. Let us now consider the product algebra

$$\mathbf{L} := \prod_{n \in \mathbb{N}} \mathrm{L}_n$$

Take $\mathbf{C}$ as the S-subalgebra of $\mathbf{L}$ generated by 0. It can be seen that $\mathbf{C}$ is isomorphic as a chain to the first infinite ordinal. Hence $\mathbf{C} \in \mathrm{SH}_\omega$. In fact, $\mathbf{C}$ is in the subvariety of SH_ω, $\mathrm{SLH}_\omega = \mathrm{SH}_\omega \cap \mathrm{SLH}$.

Remark 2. Since the chain $\mathbf{C}$ of previous Example lies in SLH_ω, we have on one hand that the variety $\mathbf{V}$ generated by $\mathbf{C}$ is a subvariety of SLH_ω. On the other hand, since each finite chain is homomorphic image of $\mathbf{C}$, we have that SLH_ω is a subvariety of $\mathbf{V}$. Hence, $\mathbf{V} = \mathrm{SLH}_\omega$.

4.   Subalgebras and homomorphic images

4.1   Subalgebras

Let L be a bounded distributive lattice and M a sublattice of L. We define the binary relation

$$R_M = \{(P, Q) \in \mathcal{PF}(L) \times \mathcal{PF}(L) : Q \cap M \subseteq P\}$$

For each binary relation R in $\mathcal{PF}(L)$ we define the subset of clopen upsets of $\mathcal{PF}(L)$

$$M_R = \{U \in \mathcal{CU}(\mathcal{PF}(L), \subseteq) : R^{-1}(U) \subseteq U\}.$$

It was shown in [6] that M_R is a bounded sublattice of $\mathcal{CU}(\mathcal{PF}(L), \subseteq)$ and that the relation R_M is reflexive, transitive and when $X = \mathcal{PF}(L)$ it verifies that

(l) If for $P, Q \in \mathcal{PF}(L)$ such that $(P, Q) \notin R_M$ there exists $U \in M_R$ such that $P \in U$ and $Q \notin U$ then $(G_X(P), G_X(Q)) \notin R_{M_R}$.

It was also shown that the correspondence $M \longmapsto R_M$ establishes an anti-isomorphism between the lattice of bounded sublattices of a bounded distributive lattice L and the lattice of binary relations defined in the Priestley space $\mathcal{PF}(L)$ which are reflexive, transitive and satisfies the condition (l).

Let X be a set, R_1 and R_2 binary relations on X. We define the binary relation $R = R_1 \circ R_2$ in the following way:

$(x, y) \in R \Leftrightarrow$ *there exists* $z \in X$ *such that* $(x, z) \in R_1$ *and* $(z, y) \in R_2$.

Let H be a Heyting algebra. We consider the following binary relation in $\mathcal{PF}(H)$:

$(P, Q) \in R_H$ iff for all $x, y \in H$, if $x \to y \in P$ and $x \in Q$, then $y \in Q$.

This relation is the inclusion (Theorem 4.24 of [5]).

Let H a Heyting algebra with successor. We define the following binary relation in $\mathcal{PF}(H)$:

$$(P, Q) \in R_S \text{ iff } S^{-1}(P) \subseteq Q$$

Theorem 4.1. *Let* (H, S) *be a S-Heyting algebra. The correspondence* $M \longmapsto R_M$ *establishes an anti-isomorphism from the lattice of subalgebras of* (H, S) *and the lattice of binary relations defined in the Heyting space* $\mathcal{PF}(H)$ *which are reflexive, transitive, satisfies the condition (l) and such that*

(1) $R_M^{-1} \circ R_H \subseteq R_H \circ (R_M^{-1} \cap R_M)$,

(2) $R_M^{-1} \circ R_S \subseteq R_S \circ R_M^{-1}$.

Proof. It follows from ([4], Corolary 4.3).   □

Lemma 4.2. *If (H, S) is a non trivial S-Heyting algebra of finite height then*

$$(P, Q) \in R_S \Leftrightarrow P \subset Q$$

Proof. If $(P, Q) \in R_S$ then $S^{-1}(P) \subseteq Q$. Thus by $(S1)$ we conclude that $P \subseteq Q$. Suppose that $P = Q$, then $S^{-1}(P) = P$. On the other hand, there is $n \in N$ such that $S^{(n)}(0) = 1$. Thus $0 \in P$, a contradiction. Then $P \subset Q$. The converse is a consequence of condition (**RF3**) and Theorem 4.4 in [4].   □

Corolary 4.3. *Let (H, S) be a S-Heyting algebra of finite height and M a subalgebra of H. The following conditions are equivalent:*

(a) *M is a S-subalgebra.*

(b) *If $P, Q, Z \in \mathcal{PF}(H)$ satisfy that $P \cap M \subseteq Z \subset Q$ then there is $W \in \mathcal{PF}(H)$ such that $P \subset W$ and $W \cap M \subseteq Q$.*

Proof. It follows from Lemma 4.2 and Theorem 4.2 of [4].   □

Let H be a finite Heyting algebra. In the last part of this section we shall build up a bijection between the subalgebras of (H, S) and some equivalence relations defined in $(\mathcal{PF}(H), \subseteq)$. We start with some preliminary lemmas.

Lemma 4.4. *If H is a finite Heyting algebra then $(\mathcal{PF}(H), \subseteq)$ has finite height.*

Proof. As H is finite, by (iii) of Proposition 2.1 and the fact that $x \leq S(x)$, there is a natural number n such that $S^{(n)}(0) = 1$. Then $(\mathcal{PF}(H), \subseteq)$ has finite height.   □

Let $X_0 = \emptyset$. We define $\widehat{X}_i = (X_{i-1}^c)_M$, for $i = 1, ..., n$.

Lemma 4.5. *Let $(X, \leq)$ be a S-Heyting space of height n. Then $X = \bigcup_{i=1}^{n} \widehat{X}_i$.*

Proof. It follows from the definitions of $\{X_i\}_i$ and $\{\widehat{X}_i\}_i$.   □

Note that previous lemma allows us to write the space X as a disjoint union of antichains $\widehat{X}_i$. We may think that these antichains define levels

on the space.

Let $(X, \leq)$ be a S-Heyting space of height n and R an equivalence relation defined in X. We write X/R for the quotient topological space. For every $x \in X$ we denote by $[x]_R$ the equivalence class of x with respect to R. Then we can define in X/R the following partial order:

$$(C) \quad [x]_R \leq_R [y]_R \text{ iff } [x]_R \subseteq \downarrow [y]_R$$

i.e., if for every $z \in [x]_R$ there is $w \in [y]_R$ such that $z \leq w$.

We write $(R1)$ and $(R2)$ for the following conditions:

(R1) $R = \bigcup_{i=1}^{n} R_i$, where $R_i = R \cap (\widehat{X}_i \times \widehat{X}_i)$ (for $i = 1, ..., n$).

(R2) Let $x, y \in X$. If $x \leq y$ then $[x]_R \leq_R [y]_R$.

Let H be a S-Heyting algebra and M a subalgebra of (H, S). We define the following equivalence relation on $\mathcal{PF}(H)$:

$$P R^M Q \Leftrightarrow P \cap M = Q \cap M.$$

We can define in $\mathcal{PF}(H)/R^M$ the following partial order:

$$[P]_{R^M} \leq_M [Q]_{R^M} \Leftrightarrow P \cap M \subseteq Q \cap M$$

Lemma 4.6. *Let H be a finite Heyting algebra and M be a subalgebra of (H, S). The following conditions hold:*

(i) *The aplication $f : (\mathcal{PF}(H), \subseteq) \to (\mathcal{PF}(M), \subseteq)$ given by*

$$f(P) = P \cap M$$

   is an epimorphism in the category $\mathbf{SH}_S$.

(ii) *If $\leq_{R^M}$ is the partial order in $\mathcal{PF}(H)$ given in (C) then $\leq_M = \leq_{R^M}$. Morever, R^M satisfies the conditions $(R1)$ and $(R2)$.*

(iii) *$(\mathcal{PF}(H)/R^M, \leq_{R^M})$ is a S-Heyting space.*
   Morever, the aplication $g_M : (M, S) \to (\mathcal{CU}(\mathcal{PF}(H)/R^M, \leq_{R^M}), S)$ given by

$$g_M(x) = \{[P]_{R^M} : x \in P \cap M\}$$

   is an isomorphism in the category $\mathbf{SHA}$.

Proof. (i) The inclusion $i : M \to H$ is a monomorphism in **SHA**. Then, if $f = \mathcal{CU}(i)$ we have that $f : (\mathcal{PF}(H), \subseteq) \to (\mathcal{PF}(M), \subseteq)$ is given by $f(P) = P \cap M$ and it is an epimorphism in $\mathbf{SH}_S$.

(ii) Let $P, Q \in \mathcal{PF}(H)$. We must prove the following:

$$[P]_{R^M} \leq_M [Q]_{R^M} \Leftrightarrow [P]_{R^M} \leq_{R^M} [Q]_{R^M}.$$

Let $[P]_{R^M} \leq_M [Q]_{R^M}$, so $P \cap M \subseteq Q \cap M$. Let $T \in [P]_{R^M}$. We consider the function f given in (i). As $T \cap M \subseteq Q \cap M$ then $f(T) \subseteq f(Q)$. By (i) f is p-morphism, so there exists $Z \in \mathcal{PF}(H)$ such that $T \subseteq Z$ and $Z \cap M = f(Z) = f(Q) = Q \cap M$. Then $[P]_{R^M} \leq_{R^M} [Q]_{R^M}$.

Conversely, let $[P]_{R^M} \leq_{R^M} [Q]_{R^M}$. By definition we have that $[P]_R \leq_M [Q]_R$.

The condition $(R2)$ holds because $\leq_M = \leq_{R^M}$. By Lemmas 4.4 and 4.5, to prove condition $(R1)$ is enough to prove the following fact:

If $P \in (\mathcal{PF}(H))_M$ and $(P, Q) \in R^M$ then $Q \in (\mathcal{PF}(H))_M$.

By (i) we conclude that $f^{-1}[(\mathcal{PF}(M))_M] = (\mathcal{PF}(H))_M$, so

$$Q \in (\mathcal{PF}(H))_M \Leftrightarrow f(Q) \in (\mathcal{PF}(M))_M \Leftrightarrow Q \cap M \in (\mathcal{PF}(M))_M.$$

But as $P \in (\mathcal{PF}(H))_M$ we conclude that $P \cap M \in (\mathcal{PF}(M))_M$. As $P \cap M = Q \cap M$ then $Q \cap M \in (\mathcal{PF}(M))_M$.

(iii) $(\mathcal{PF}(H)/R^M, \leq_{R^M})$ is a S-Heyting space because H is finite. Now we can define the bijection $g : (\mathcal{PF}(H)/R^M, \leq_{R^M}) \to (\mathcal{PF}(M), \subseteq)$ given by $g([P]_{R^M}) = f(P)$ (it is surjective by (i) and injective by definition of R^M). Besides, as g is an isomorphism in **SH**, by (ii) of Lemma 2.3 it is an isomorphism in $\mathbf{SH}_S$ too. Then $g_1 = \mathcal{CU}(g) : (\mathcal{CU}(\mathcal{PF}(H), \subseteq), S) \to (\mathcal{CU}(\mathcal{PF}(H)/R^M, \leq_{R^M}), S)$ given by $g_1(U) = g^{-1}(U)$ is also an isomorphism in **SHA** and $\varphi_M : (M, S) \to (\mathcal{CU}(\mathcal{PF}(M), \subseteq), S)$. Thus $g_M : (M, S) \to (\mathcal{CU}(\mathcal{PF}(H))/R^M, \leq_{R^M}), S)$, given by $g_M(x) = g_1(\varphi_M(x))$, is an isomorphism in **SHA**. Then we have that $g_1(\varphi_M(x)) = \{[P]_{R^M} : x \in P \cap M\}$. $\square$

If X is a topological space and R is an equivalence relation in X we define the application $\rho_R : X \to X/R$ given by $\rho_R(x) = [x]_R$.

Lemma 4.7. *Let H be a finite Heyting algebra, R an equivalence relation in $\mathcal{PF}(H)$ that satisfies the conditions $(R1)$ and $(R2)$, and $\leq_R$ the partial order in $\mathcal{PF}(H)/R$ given by (C). Then $\mathcal{CU}(\mathcal{PF}(H)/R, \leq_R)$ is isomorphic to some subalgebra M of (H,S). Morever, $R = R^M$.*

Proof. Since H is a finite Heyting algebra we have that $(\mathcal{PF}(H), \leq)$ is a S-Heyting space. The quotient map $\rho_R : (\mathcal{PF}(H), \subseteq) \to (\mathcal{PF}(H)/R, \leq_R)$ is a morphism in $\mathbf{SH}_S$. The function ρ_R is continue because H is finite. By $(R2)$ we have that ρ_R preserves order. Let now $P, Q \in \mathcal{PF}(H)$ such that $\rho_R(P) \leq_R [Q]_R$, so $[P]_R \leq_R [Q]_R$. Therefore there is $F \in \mathcal{PF}(H)$ such that $P \subseteq F$ and $[F]_R = [Q]_R$. By $(R2)$ we have that $\rho_R(P) \leq_R \rho_R(F) = \rho_R(Q) = [Q]_R$, so ρ_R is a p-morphism.

Let $U \in \mathcal{CU}(\mathcal{PF}(H), \subseteq)$. We will prove that

$$\rho_R^{-1}[(U^c)_M] = [\rho_R^{-1}(U^c)]_M.$$

If $P \in [\rho_R^{-1}(U^c)]_M$ then $[P]_R \in U^c$. Let $[P]_R \leq_R [Q]_R$, with $[Q]_R \in U^c$. Then there is $F \in \mathcal{PF}(H)$ such that $P \subseteq F$ and $[F]_R = [Q]_R$. In consequence $[F]_R \in U^c$, so $P = F$. Then $[P]_R = [F]_R = [Q]_R$. Therefore $[\rho_R^{-1}(U^c)]_M \subseteq \rho_R^{-1}[(U^c)_M]$. Conversely, let $P \in \rho_R^{-1}[(U^c)_M]$ and hence $[P]_R \in (U^c)_M$. Let $P \subseteq Q$ with $[Q]_R \in U^c$. By $(R2)$, $\rho_R(P) \leq_R \rho_R(Q)$. Then $[P]_R = [Q]_R$. By $(R1)$ there is some natural number k such that $(P,Q) \in \widehat{X}_k$. Since $P \subseteq Q$, $P = Q$, and we get that $\rho_R^{-1}[(U^c)_M] \subseteq [\rho_R^{-1}(U^c)]_M$. In consequence ρ_R is a surjective S-morphism.

Let $(X, \leq) = (\mathcal{PF}(H), \subseteq)$. We have that ρ_R is an epimorphism in $\mathbf{SH}_S$, so $h_1 = \mathcal{CU}(\rho_R) : (\mathcal{CU}(X/R, \leq_R), S) \to (\mathcal{CU}(X, \leq), S)$ is a monomorphism in $\mathbf{SHA}$ and for this and Lemma 2.2 the function $g_R = \varphi_H^{-1} h_1 : (\mathcal{CU}(X/R, \leq_R), S) \to (\varphi_H^{-1} h_1(\mathcal{CU}(X/R, \leq_R)), S)$ is an isomorphism in $\mathbf{SHA}$. Morever, $g_R(U) = \varphi_H^{-1}(\rho_R^{-1}(U))$. Define now the following subalgebra of (H,S):

$$M = \varphi_H^{-1} h_1(\mathcal{CU}(X/R, \leq_R))$$

We have that

$$x \in M \Leftrightarrow \exists U \in \mathcal{CU}(X/R, \leq_R) : \varphi_H(x) = \rho_R^{-1}(U) \qquad (5)$$

On the other hand the applications $h_2 = \mathcal{PF}(g_R) : (\mathcal{PF}(M), \subseteq) \rightarrow \mathcal{PF}(\mathcal{CU}(X/R, \leq_R))$ and $G_{X/R} : (X/R, \leq_R) \rightarrow \mathcal{PF}(\mathcal{CU}(X/R, \leq_R)$ are isomorphisms in $\mathbf{SH}_S$, so $h_3 = G_{X/R}^{-1} h_2 : (\mathcal{PF}(M), \subseteq) \rightarrow (X/R, \leq_R))$ is also an isomorphism. By (5) we have that $h_3(P \cap M) = [P]_R$. By similar arguments to those of the proof (iii) of Lemma 4.6 we conclude that $g : (X/R^M, \leq_{R^M}) \rightarrow (\mathcal{PF}(M), \subseteq)$ given by $g([P]_{R^M}) = P \cap M$ is an isomorphism in $\mathbf{SH}_S$ and then $h = h_3 g : (X/R^M, \leq_{R^M}) \rightarrow (X/R, \leq_R)$, given by $h([P]_{R^M}) = [P]_R$, is an isomorphism. Therefore we have that

$$(P, Q) \in R \Leftrightarrow [P]_R = [Q]_R \Leftrightarrow [P]_{R^M} = [Q]_{R^M} \Leftrightarrow (P, Q) \in R^M$$

Hence $R = R^M$. $\square$

Theorem 4.8. *Let H be a finite Heyting algebra. There is a bijection $M \longmapsto R^M$ between the subalgebras of (H, S) and the equivalence relations in $\mathcal{PF}(H)$ which satisfy $(R1)$ and $(R2)$.*

Proof. By (ii) of Lemma 4.6 the bijection is well defined. Let M and N be subalgebras of (H, S) such that $R^N = R^M$. Then $\mathcal{CU}(\mathcal{PF}(H)/R^M, \leq_{R^M})$ coincides with $\mathcal{CU}(\mathcal{PF}(H)/R^N, \leq_{R^N})$. Let $x \in M$. By (iii) of Lemma 4.6 we have that there is $y \in N$ such that $g_M(x) = g_N(y)$. We want to prove that $x = y$. We suppose that $x \not\leq y$. Then by the Prime Filter Theorem there is $P \in \mathcal{PF}(H)$ such that $x \in P$ and $y \notin P$, so $x \in P \cap M$ and $y \notin P \cap N$. In consequence $[P]_{R^M} \in g_M(x)$ and $[P]_{R^N} \notin g_N(y)$. Since $[P]_{R^M} = [P]_{R^N}$ we have that $g_M(x) \neq g_N(y)$, a contradiction. Then $x = y \in N$, so $M \subseteq N$. The other inclusion is proved in the same way. Therefore $M = N$.
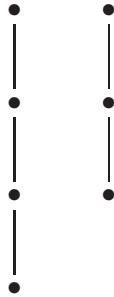
The surjectivity is a consequence of Lemma 4.7. $\square$

Remark 4.9. If M is a S-subalgebra then $R^M = R_M \cap R_M^{-1}$.

Example 3. The description given above for the S-subalgebras of finite algebras allows us to give a simple pictorial procedure to determine them.

Consider the following concrete example: Find all the S-subalgebras of

$L_5 \times L_4$. In order to so doing, let us draw the associated space

Note that we have taken care of preserving the levelwise structure of the space. As a consequence of Theorem 4.8, we have that the possible relations associated to some S-subalgebra are the trivial one and those depicted bellow
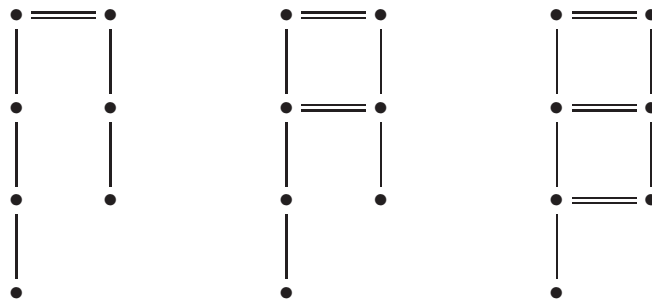
Here we are identifying the equalized dots. Thus the spaces associated to the nontotal S-subalgebras are
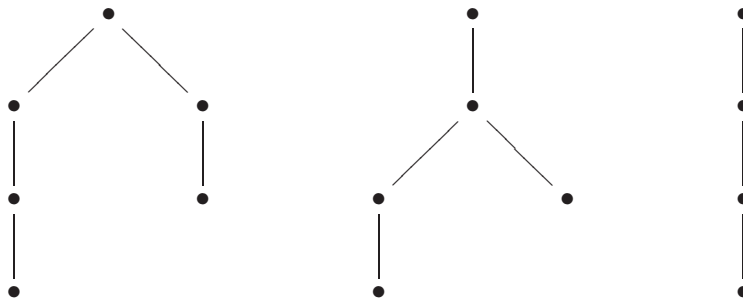
Observe that, in the finite case, the levels corresponds exactly to the fibers of the continuous map associated to the canonical inclusion of the subalgebra generated by the constants.

4.2 Homomorphic images

Let X be a S-Heyting space, Y a topological subspace of X and $i : Y \to X$ the inclusion function. For every $A \subseteq Y$ we define $\downarrow_Y A = \{y \in Y : (\exists a \in A : y \leq a)\}$. We write $\downarrow$ for $\downarrow_X$.

Lemma 4.10. *The following conditions hold:*

(a) *For every $U \subseteq X$, if Y is an upset then $U_M \cap Y = (U \cap Y)_M$.*

(b) *For every $U \subseteq X$, if Y is an upset then $\downarrow_Y (U \cap Y) = (\downarrow U) \cap Y$.*

(c) *If Y is an upset, then i is a momomorphism in $\textbf{SH}$ if and only if it is a monomorphism in $\textbf{SH}_S$.*

(d) *i is a monomorphism in $\textbf{SH}$ if and only if Y is a closed upset.*

Proof.

(a) Let $x \in (U \cap Y)_M$, then $x \in U \cap Y$. It suffices to show that $x \in U_M$. Let $x \leq u$ with $u \in U$. Since Y is an upset, $u \in U \cap Y$. Then $x = u$. Conversely, let $x \in U_M \cap Y$. In particular, $x \in U \cap Y$. Let $x \leq u$ with $u \in U \cap Y$. Since $x \in U_M$ we have that $x = u$.

(b) Let $x \in \downarrow_Y (U \cap Y)$. Thus $x \in Y$ and there exists $u \in U \cap Y$ such that $x \leq u$. Then $x \in (\downarrow U) \cap Y$. Conversely let $x \in (\downarrow U) \cap Y$, so $x \in Y$ and there exists $u \in U$ such that $x \leq u$. Since Y is an upset we have that $u \in Y$, so $u \in U \cap Y$; therefore $x \in \downarrow_Y (U \cap Y)$.

(c) Let i be a monomorphism in $\textbf{SH}$. First we will prove that Y is an object of $\textbf{SH}_S$. Let V a clopen downset in Y. Then there are an open set A and a closed set B in X such that $V = A \cap Y = B \cap Y$. As V is closed in Y and Y is compact, V is compact. In particular we have that $V = A \cap V$. On the other hand, A is union of clopens $\{A_i\}_{i \in I}$, and as V is compact we have that $V = (\bigcup_{i=1}^n A_i) \cap V$, for some natural n. We define $C = (\bigcup_{i=1}^n A_i)$, so C is clopen and $V = C \cap Y$. Then, by (b) we have that $V = \downarrow_Y V = (\downarrow C) \cap Y$. As X is a S-Heyting space, $\downarrow C$ is a clopen downset in X, so by (a) we conclude that $V_M = (\downarrow C)_M \cap Y$. Therefore we have that V_M is clopen in Y. Besides by (a) i is a S-morphism. The converse is inmediate.

(d) Let i be a monomorphism in **SH**. As Y is compact and X Hausdorff we have that Y is closed. Let now $y \leq x$ with $y \in Y$, so $i(y) \leq x$. Since i is a $p-$morphism there exists $z \in Y$ such that $y \leq z$ and $i(z) = i(x)$. Hence $z = x$. Then $x \in Y$ and Y is an upset. Conversely, let Y be a closed upset of X. Besides Y is a Priestley space. Let A be an open set in Y. Then there is an open set U in X such that $A = U \cap Y$. By (b) we conclude that $\downarrow_Y A = (\downarrow U) \cap Y$. Since X is a S-Heyting space and $\downarrow U$ is open in X we have that $\downarrow_Y A$ is open in Y.   □

Theorem 4.11. *i is a monomorphism in $\boldsymbol{SH}_S$ if and only if Y is a closed upset of X.*

Proof. It is a consequence of (c) and (d) of Lemma 4.10.   □

5. Free S-Heyting algebras in varieties generated by finite chains

Let $n \geq 0$. Write SLH_n for the subvariety of SH_n generated by L_n. It is shown in ([3], Theorem 6.1), that all implicit connectives of L_n are terms.

The proof of this theorem also shows that the class of implicit connectives of L_n coincides with the set of Heyting polynomials on it; identical, by affine completeness, to the set of compatible functions of L_n. Hence, the unary implicit connectives of L_n, which constitute the elements of the free algebra in one generator of SLH_n, may be explicitly described as those functions $f : \mathrm{L}_n \to \mathrm{L}_n$ satisfying for some $a \in \mathrm{L}_n$ that $f(x) \geq x$, for every $x \in [0, a]$ and $f(x) = a$ on $(a, 1]$.

A diagram of the free algebra in one generator of SLH_3 is depicted in ([3], figure 2).

In this section we shall completely characterize the free algebras in one generator for the varieties SLH_n, for $n \geq 1$ and the free algebra in one generator for the variety SLH_ω. In order to get free algebras in more generators we give at the end of this section a brief description of the product of finite S-Heyting spaces

5.1 Free algebras for SLH_n

For any two distributive lattices M, N, we write $M \oplus N$ for the ordinal sum (as posets) of these lattices (see [1], p. 39). If M and N are finite, then $M \oplus N$ is also finite, and hence a Heyting algebra. We write $\mathbf{0}$ for the lattice with only one element. Let us also write F_n for the free algebra in one generator, x_n, in the variety SLH_n.

Lemma 5.1. *The cardinal of the universe of* F_n *is*

$$|\mathrm{F}_n| = \sum_{j=0}^{n-1} \frac{n!}{j!} \tag{6}$$

Proof. A direct counting argument based on the explicit description of the elements of the free algebra in one generator of SLH_n as functions $f : \mathrm{L}_n \to \mathrm{L}_n$, given above. $\qquad\square$

Observe that for any $n \geq 0$ we have that $\mathbf{0} \oplus \mathrm{F}_n \in \mathrm{SLH}_{n+1}$; hence we have a unique SLH_{n+1}-morphism $\alpha_{n+1} : \mathrm{F}_{n+1} \to \mathbf{0} \oplus \mathrm{F}_n$ such that $\alpha_{n+1}(x_{n+1}) = x_n$. We also have that $\mathrm{L}_{n+1} \in \mathrm{SLH}_{n+1}$, and hence a unique morphism $\beta_{n+1} : \mathrm{F}_{n+1} \to \mathrm{L}_{n+1}$ with $\beta_{n+1}(x_{n+1}) = 0$. By the universal property of the product in SLH_{n+1}, we have a unique morphism δ making the following diagram commute:

$$\tag{7}$$

Since F_{n+1} and $(\mathbf{0} \oplus \mathrm{F}_n) \times \mathrm{L}_{n+1}$ both have the same finite cardinal, in order to prove they are SLH_{n+1}-isomorphic, it will suffice to see that δ is onto.

Lemma 5.2. $(1, 0) \in \mathrm{im}\, \delta$.

Proof. By definition of δ, $\delta(x_{n+1}) = (x_n, 0)$. Consider the S-Heyting algebra term $\tau(x) = S(0) \to x$. A straightforward computation allow us to

see that

$$\tau^{\mathbf{0} \oplus \mathrm{F}_n}(x_n) = 1, \text{ and} \tag{8}$$

$$\tau^{\mathrm{L}_{n+1}}(0) = 0 \tag{9}$$

Consider now the element $y_0 = \tau^{\mathrm{F}_{n+1}}(x_{n+1}) \in \mathrm{F}_{n+1}$. By (8) and (9), this element is such that $\delta(y_0) = \delta(\tau(x_{n+1})) = \tau^{(\mathbf{0} \oplus \mathrm{F}_n) \times \mathrm{L}_{n+1}}(x_n, 0) = (1, 0)$. $\quad\square$

Lemma 5.3. *The morphism δ of* (7) *is onto.*

Proof. Take $(x, n) \in (\mathbf{0} \oplus \mathrm{F}_n) \times \mathrm{L}_{n+1}$. Since $(x, n) = (x, 0) \vee (0, n)$, we shall consider separately the case $x = 0$ and $n = 0$.

If $x = 0$, take $y_1 = S^n 0 \wedge (x_{n+1} \to 0)$. We get that $\delta(y_1) = S^n(0, 0) \wedge [(x_n, 0) \to (0, 0)] = S^n(0, 0) \wedge (x_n \to 0, 1) = S^n(0, 0) \wedge (0, 1) = (0, S^n 0) = (0, n)$.

On the other hand, suppose that $n = 0$ and $x \geq S0$. Since F_n is free, there is an S-Heyting term t such that $x = t(x_n)$. Take $\bar{t}$ the term we get by replacing by $S0$ any appearence of 0 in t. We have that $x = \bar{t}^{\ \mathbf{0} \oplus \mathrm{F}_n}(x_n)$. Taking now $y_2 = \bar{t}(x_{n+1})$ and y_0 as in Lemma 5.2, we get that $u = y_0 \wedge y_2$ is such that $\delta(u) = (x, 0)$.

Thus, we have proved that im $\delta = (\mathbf{0} \oplus \mathrm{F}_n) \times \mathrm{L}_{n+1}$. $\quad\square$

As an immediate consequence of Lemmas 5.1 and 5.3, we have the following

Theorem 5.4. *There is an S-isomorphism of Heyting algebras between F_{n+1} and $(\mathbf{0} \oplus \mathrm{F}_n) \times \mathrm{L}_{n+1}$.*

5.2   Free algebra for SLH_ω

Let us write F_ω for the free algebra in one generator, x_ω, in SLH_ω. Since for each finite n, we have that $\mathrm{F}_n \in \mathrm{SLH}_\omega$, there is a unique surjective S-morphism of Heyting algebras $\Omega_n : \mathrm{F}_\omega \to \mathrm{F}_n$, applying x_ω on x_n. In fact, $\mathrm{F}_n \cong \mathrm{F}_\omega / \langle S^n 0 \rangle$. Let us also write for $n \geq m$, $\Omega_{mn} : \mathrm{F}_n \to \mathrm{F}_m$ for the unique S-morphism applying x_n on x_m. The system $\Omega_{mn} : \mathrm{F}_n \twoheadrightarrow \mathrm{F}_m$, $m \leq n$ is directed.

Theorem 5.5. *Let $\Omega_{mn} : \mathrm{F}_n \twoheadrightarrow \mathrm{F}_m$, $m \leq n$ be as before. Then*

$$\mathrm{F}_\omega = \mathrm{colim}\ \mathrm{F}_n \tag{10}$$

Proof. Call $H = \text{colim } F_n$. Let us verify that H has the universal property of the free algebra in one generator in the variety SLH_ω.

Let us first give an explicit construction of H. Take $H' = \prod_{j \geq 1} F_j$. We take H as the subalgebra of H' whose elements $a = (a_1, a_2, ...)$ are such that $\Omega_{ij}(a_j) = a_i$ for $i \leq j$. We have in particular that $x = (x_1, x_2, ...) \in H$.

Let $u \in H$, hence $u = (u_1, u_2, ...)$ with $\Omega_{ij}(u_j) = u_i$ for $i \leq j$. Since $u_i \in F_i$ there must be an SLH_ω-term t such that $u_i = t^{F_i}(x_i)$, for every $i \geq 1$. By the finiteness of the length of terms, there must be some $m \geq 1$ such that t is a term in SLH_m. Hence $u = t^H(x)$, showing that H is generated by x. It is straightforward to verify that $H = \text{colim } F_n$.

Let A be any algebra in SLH_ω, and $a \in A$. Let $\varphi_j : H \to A/ < S^j 0 = 1 >$ be the S-morphism given by $\varphi_j(u) := \overline{t^A(a)}_j$, the class modulo the filter generated by $S^j(0)$ of $t^A(a)$, being t the SLH_ω-term that defines u. For each term t there is a j such that φ_j is completely determined by the condition of applying x_j to a. Thus, the condition $\varphi(x) = a$ defines a unique SLH_ω-morphism $\varphi : H \to A$, that given by $\varphi(x) := t^A(a)$. $\square$

5.3   Priestley spaces for F_n

We have seen that there are dualities between the categories SLH_m, $m = 1, 2, ..., \omega$ and certain categories of Heyting spaces. In this section we use the recursive nature of the definition of F_n and the "good" properties of any categorical duality to make explicit constructions for these spaces.

Condition $S(0) = 1$ forces an S-Heyting algebra to be boolean. Hence $\text{SLH}_1 = \text{Boole}$, and hence F_1 is the free boolean algebra in one generator, whose Stone space is
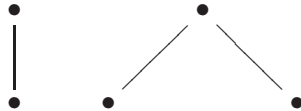
$$\bullet \qquad \bullet$$

We have seen that $F_2 \cong (\mathbf{0} \oplus F_1) \times L_2$. Since $\mathcal{PF}$ is part of a categorical duality, it sends products into coproducs, and $\mathcal{PF}(F_2) \cong \mathcal{PF}(\mathbf{0} \oplus F_1) \sqcup \mathcal{PF}(L_2)$, which is simply the coproduct of the topological spaces. On the
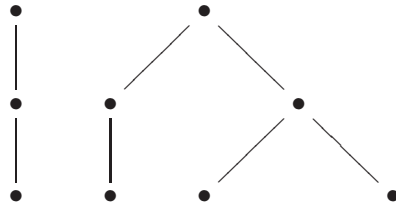
other hand, $\mathcal{PF}(\mathbf{0} \oplus \mathrm{F}_1)$ can be seen to be
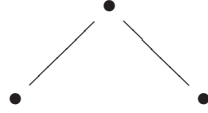
Thus, $\mathcal{PF}(\mathrm{F}_2)$ is

Observe that in general $\mathcal{PF}(\mathbf{0} \oplus \mathrm{F}_n)$ is built up from $\mathcal{PF}(\mathrm{F}_n)$ by adding a new point over it. Then, for instance, we get that $\mathcal{PF}(\mathrm{F}_3)$ is

This procedure allows us to effectively built up the space of any one of the free algebras in one generator of the varieties generated by one finite chain.

Since $\mathrm{F}_\omega = \operatorname{colim} \mathrm{F}_n$ as we have previously seen, $\mathcal{PFF}_\omega$ is the forest X_ω of infinite height depicted above, whose topology is given by the subbases of upper clopens of the form $U_x = \{u \in X_\omega \mid u \geq x\}$ where x is any element of X_ω.
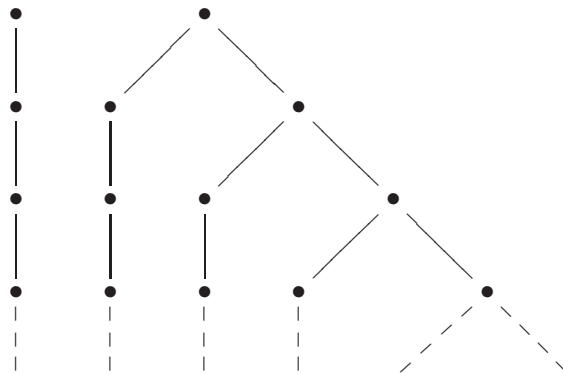
5.4   Product of finite Priestley spaces in $\mathbf{SH}_S$

Since $\mathbf{SH}_S$ is dually categorically equivalent to a variety, it has arbitrary products. In this subsection we want to give a simple explicit description for the product of two finite spaces in $\mathbf{SH}_S$.

As we have already noted at the end of Example 3, the levelwise structure of finite spaces in $\mathbf{SH}_S$ plays an important role in their description. Let X and Y be two finite spaces in $\mathbf{SH}_S$. Since they are finite, they both have finite height. Suppose that $h(X) = m \leq n = h(Y)$. Write $\mathcal{L}_n$ the Priestley space of L_n. Hence we have morphisms $i_X : X \to \mathcal{L}_n$, $i_Y : Y \to \mathcal{L}_n$ and $i : X \times Y \to \mathcal{L}_n$ induced by the unique SH-maps from L_n to $\mathcal{CU}(X)$, $\mathcal{CU}(Y)$ and $\mathcal{CU}(X \times Y)$ respectively. The following diagram must commute in $\mathbf{SH}_S$:

$$
\begin{array}{ccc}
& X \times Y & \\
{\scriptstyle p_X} \swarrow & \downarrow {\scriptstyle i} & \searrow {\scriptstyle p_Y} \\
X & & Y \\
{\scriptstyle i_X} \searrow & \downarrow & \swarrow {\scriptstyle i_Y} \\
& \mathcal{L}_n &
\end{array}
\tag{11}
$$

Thus, the projections must send fibers in fibers, i.e, the product must preserve the levelwise structure.

Proposition 5.6. *Let X and Y be two finite spaces in $\mathbf{SH}_S$. Suppose that $X = \bigcup_{i=1}^{m} \hat{X}_i$ and $Y = \bigcup_{i=1}^{n} \hat{Y}_i$ and $\hat{X}_i$ and $\hat{Y}_i$ are as in Lemma 4.5. Assume that $m \leq n$, and write $X = \bigcup_{i=1}^{n} \hat{X}_i$, with $\hat{X}_i = \emptyset$, if $i > m$. Then, we have that*

$$
X \times Y = \bigcup_{i=1}^{n} (\hat{X}_i \times \hat{Y}_i)
$$

the union is disjoint and the order is the induced by the usual product order. Since we are considering finite spaces, the topology is the discrete one.

Example 4. Let us illustrate the product of two finite spaces in $\mathbf{SH}_S$ by calculating the Priestley space of the free algebra in two generators in SLH$_2$, $_2\mathrm{F}_2$. Since $_2\mathrm{F}_2 \cong \mathrm{F}_1 \coprod \mathrm{F}_1$ and $\mathcal{PF}$ is part of a duality, $\mathcal{PF}(_2\mathrm{F}_2) \cong \mathcal{PF}(\mathrm{F}_1) \times \mathcal{PF}(\mathrm{F}_1)$. A straightforward calculation shows that $\mathcal{PF}(_2\mathrm{F}_2)$ may

be depicted as
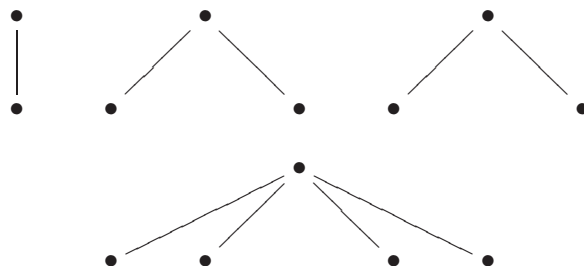
Acknowledgements

References

[1] R. Balbes and P. Dwinger, Distributive Lattices, University of Missouri Press, Columbia, Miss. (1974).

[2] H. Burris and H. P. Sankappanavar, A Course in Universal Algebra, Springer Verlag, New York (1981).

[3] X. Caicedo and R. Cignoli, *An algebraic approach to intuitionistic connectives*, Journal of Symbolic Logic **66**, No.4 (2001), pp. 1620–1636.

[4] J. L. Castiglioni, M. Sagastume and H. J. San Martin, *On frontal Heyting algebras*, Reports on Mathematical Logic **45** (2010), pp. 201–224..

[5] S. Celani and R. Jansana, *Bounded distributive lattices with strict implication*, Mathematical Logic Quarterly **51** (2005), pp. 219–246.

[6] R. Cignoli, S. Lafalce and A. Petrovich, *Remarks on Priestley duality for distributive lattices*, Orden, **8** (1991), pp. 183–197.

[7] L. Esakia, *The modalized Heyting calculus: a conservative modal extension of the Intuitionistic Logic*, Journal of Applied Non-Classical Logics **16**, No.3-4 (2006), pp. 349–366.

[8] A. Horn, *Logic with truth values in a linearly ordered Heyting algebra*, J. Symbolic Logic **34** (1969), pp. 395–405.

[9] P. Jonstone, Stone Spaces, Cambridge University Press, 1982.

[10] A. V. Kusnetzov, *On the Propositional Calculus of Intuitionistic Provability*, Soviet Math. Dokl. **32** (1985), pp. 18–21.

[11] G. Martínez and H. Priestley, *On Priestley Spaces of Lattice-Ordered Algebraic Structures*, Orden **15** (1998), pp. 297–323.

[12] A. Monteiro, *Sur les algebres de Heyting symétriques*, Portugal. Math. **39** (1980).

[13] P. Morandi, *Dualities in Lattice Theory*, Mathematical Notes http://sierra.nmsu.edu/morandi/.

Departamento de Matemática,

Facultad de Ciencias Exactas, UNLP.

Casilla de correos 172,

La Plata (1900)

Argentina.

jlc@mate.unlp.edu.ar

hsanmartin@mate.unlp.edu.ar