A general-purpose tool for Reliability and Availability Analysis of Repairable Systems

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Dissertation presented to Escola Superior de Tecnologia e Gestão of Instituto Politécnico de Bragança to obtain the Master Degree in Industrial Engineering

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Abstract

This thesis covers general mathematical and simulation models for the reliability and availability analysis of repairable systems along with estimation methods and model selection criterion. A combined mathematical and simulation model called the Failure-Repair Process is proposed, based on the trend-renewal process. This model is based on a binary state system, where the system may only be in one of two states: working or failed. This model is then integrated into a general-purpose tool, for automated modelling of repairable systems. The classical Akaike information criterion is used to automate the choice of failure and repair models that best fit the available data. Estimators for different performance measures of the systems are studied, such as point and mean availability, rate of occurrence of failures and a first order reliability estimator based on the Kaplan-Meier estimator. Numerical studies are conducted in the proposed non-analytical estimators for the availability, leading to a robust mean availability estimator and a intuitive but sample demanding point availability estimator. Furthermore, a complete quantitative study is conducted on real data from the food industry together with a presentation of the implemented tool functionalities. Overall, the proposed model is able to adapt very well to real data with different characteristics, and, consequently, the resulting performance indicators are befitting to practice.

Keywords: repairable systems; trend-renewal process; simulation; reliability analysis; availability.

Resumo

Esta tese aborda modelos matemáticos e de simulação para a análise de confiabilidade e disponibilidade de sistemas reparáveis, juntamente com métodos de estimação e critério de seleção de modelos. Um modelo matemático e de simulação combinados denominado *Failure-Repair Process* é proposto, baseado no *trend-renewal process*. Este modelo consiste em um sistema de caracterização binária, onde o sistema pode estar em apenas um de dois estados: em funcionamento ou falha. Este modelo é então integrado em uma ferramenta de uso geral, para modelagem automatizada de sistemas reparáveis. O clássico critério de informação de Akaike é usado para automatizar a escolha dos modelos de falha e reparo que melhor se ajustam aos dados disponíveis. São estudados estimadores para diferentes medidas de desempenho dos sistemas, tais como disponibilidade pontual e média, taxa de ocorrência de falhas e um estimador de confiabilidade de primeira ordem baseado no estimador Kaplan-Meier. Estudos numéricos são conduzidos nos estimadores não-analíticos propostos para a disponibilidade, levando a um estimador de disponibilidade média robusto e um estimador de disponibilidade puntual intuitivo, mas que demanda grandes amostras. Além disso, é realizado um estudo quantitativo completo sobre dados reais da indústria de alimentos juntamente com uma apresentação das funcionalidades da ferramenta implementada. De maneira geral, o modelo proposto é capaz de se adaptar muito bem a dados reais com diferentes características e, consequentemente, os indicadores de desempenho resultantes são adequados à prática.

Palavras-chave: sistemas reparáveis; trend-renewal process; simulação; análise de confiabilidade; disponibilidade.

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Listings

Acronyms

AIC Akaike information criterion.

CLT Central Limit Theorem.

CM Corrective Maintenance.

EDF Empirical Distribution Function.

ERF Empirical Reliability Function.

FRP Failure-Repair Process.

i.i.d. independent and identically distributed.

IEC International Eletrotechnical Comission.

K-L Kullback-Leibler.

KPI Key Performance Indicators.

ML Maximum Likelihood.

MLE Maximum Likelihood Estimator.

PI Performance Indicators.

PM Preventive Maintenance.

r.v. random variable.

ROCOF rate of occurrence of failures.

Chapter 1

Introduction

Reliability and availability analysis are useful not only for predicting system behaviour, but also to verify how the system is responding or not to external factors, like proper maintenance. The performance of a repairable system may usually be measured with its availability, which is the proportion of time that the system is functioning. As repairable system are complex in nature, we need proper mathematical tools to able to treat the data.

Traditionally, when modelling complex repairable systems, we divide the failure processes into perfect, minimal and imperfect repair processes. The first one represents the case where after each repair the system is set to an as-good-as-new state. Such behaviour occurs naturally when considering individual parts or when a system is comprised of different parts, but only a few of them are responsible for the majority of maintenance interventions. On the other hand, the minimal repair policy represents systems made out of several different parts, where each part repair does not influence significantly the overall reliability of the system. In other words, under minimal repair, the reliability of the system does not change with each repair action, namely, the system after the repair is as-bad-as-old. Last but not least, imperfect repair processes are the ones which fall between perfect and minimal repair. In general, these models are more complex and can capture more subtle system behaviours.

A general repairable system has a characteristic degradation curve, backed up by

empirical evidence. Such a curve is called the bathtub curve and has 3 main portions, which are characterised by the 3 types of failure that may happen: early failures, random failures and wear-out failure. The curve is defined by a decreasing rate of occurrence of failures for the first stage, then a seemingly constant rate in the second stage; and, lastly, an increasing rate for the last stage.

To model each of these behaviours, the standard mathematical tools are the stochastic processes. Stochastic processes are well defined mathematical entities, with a very strong theoretical background and a great range of applicability. Arguably, the most used ones are the Poisson processes, which may used to model constant and varying arrival rates with its counterparts the Homogeneous Poisson Process and the Nonhomogeneous Poisson Process, respectively. Another common model is the Renewal Process.

In the case of a renewal process, we are dealing with the perfect repair policy. Mathematically, under the RP, the inter-arrival times are independent and identically distributed, which means that they can be statistically treated by standard analytical tools. When in the NHPP, on the other hand, we are under the minimal repair policy and the inter-arrival times are neither independent nor identically distributed; therefore, one cannot use standard tools, and must then seek the specific tools available for the analysis of point processes.

For the imperfect repair cases, there has been plenty of proposed models. One of which is the so-called trend-renewal process (TRP). Its main property is the fact that it is a simple generalisation of NHPP properties that also expand the model to the renewal processes domain. Formally, the TRP is a time-transformed renewal process. That means it has both the RP and the NHPP as special cases, which makes this model a very powerful one. One of the most known TRP is the Weibull-Power Law TRP (WPLP). The reason why it is very interesting is that it is a generalisation of two very important and used models, the Weibull Renewal Process and the Power Law NHPP. Furthermore, the model has some analytical properties that make a good choice for a general-purpose tool. One of which is the fact that both the expected number of failures and its derivative, the rate of occurrence of failures, are analytically defined.

When it comes to modelling the repair times of a system, the common approach is to consider that the data are independent and identically distributed, which simplifies the analysis. In the classical literature, the most common model for the repair times are the exponential, normal and lognormal distributions. Arguably, the most fit is the lognormal distribution, given some of its properties.

To model the data, the model uses the Maximum Likelihood Method in the WPLP model, since it allows for the construction of well defined functions, which maximises the probability of a set of data to be generated by a given model. Also, as several models are presented and discussed, a strong and reliable framework for model selection must be in place. Such requirement is fulfilled by the classical Akaike information criterion, which a well known information-theoretic approach to model selection, which minimises the information loss among a available model set. The use of the Maximum Likelihood method also allows us to calculate the confidence intervals of each parameter, and those are available in the final tool.

The limitations with the counting processes is that they do not take into account the repair times of the system. That is, we are only modelling how the system is degrading and not seeing the complete picture. A way around that is with the use of simulation. A simulation model is a computational statistical experiment, that allows us to run several possible outcomes of the probabilistic event of interest, and draw conclusions from that.

By utilising the correct equation, we are able to simulate different sample paths for the system and from that we may calculate an estimator of the availability of the system. This is possible under the assumption that the mean of all the replications for each inter-failure time, falls in the Central Limit Theorem (CLT), which states that the expected value of a random variable approaches the sample mean, as the sample size tends to infinity.

The use of simulation models relies heavily on the applicability of the Central Limit Theorem (CLT). That said, simulation models must be designed with this in mind, since it simplifies the treatment of the output data, and also gives a strong validation sense to the model.

Another important measure is the rate of occurrence of failures (ROCOF), which is

closely related to the expected number of failures, given that the ROCOF is simply the rate at which the failure occurs in time. For the TRP, the expected number of failures is well defined, and may be calculated using classical numerical methods for renewal-type equations. After that, we may apply simple finite difference methods to end-up with the ROCOF.

The purpose of this thesis is primarily to develop a general-purpose computational tool that calculates several reliability and availability metrics for a general repairable system. To do so, we propose a general model that combines mathematical and simulation techniques in a way that the tool becomes completely automated.

The model must be robust, and be able to give good indicatives of the behaviour of the system. It must allow for the calculation of some important performance indicators: the rate of occurrence of failures, the reliability, and both the point and mean availability of the system.

A thorough compilation of the standard terms and definitions pertinent to the general field of Reliability Engineering is presented in Chapter 2.

Chapter 3 consists of an extensive literature review concerning repairable system failure models along with general models for the analysis of repair time data. Parameter estimation for all the models, using the maximum likelihood method are also presented. Lastly, we explore the Akaike information criterion, which sets the model selection framework that the implement tool works in.

Chapter 4 reviews general concepts for simulation models, along with the pertinent properties and techniques that are exploited to calculate the more complex measures of a repairable system.

Chapter 5 explores the proposed model, along with defining the different estimators used for each metric and also give more deep description of the general algorithm that runs on the background of the implemented tool.

Chapter 6 is dedicated to a practical application of all the methods described in previous chapters as well as a guide for using the implemented general-purpose tool. The data comes from an enrobing machine used in a food industry in the South region of

Brazil. A general description of the data is presented, along with its analysis. The final reliability and availability estimators efficiency are also discussed.

Chapter 2

Reliability Theory

2.1 General Concepts

Every engineered object is unreliable in the sense that it degrades and eventually fails [1]. This degradation means that the reliability of an engineered object will naturally decrease with time and is up to the maintenance crew to slow down this decrease. In a broad perspective, reliability is associated with the successful operation of an item in the absence of failures [2]. This is embedded in the more general definition of reliability given by the International Eletrotechnical Comission (IEC) 60050-192:2015 [3], which states:

Definition 2.1 (Reliability - Qualitative)
The ability to perform as required, without failure, for a given time interval, under given conditions [3].

Where the failure of an engineered object is defined as [3]:

Definition 2.2 (Failure)
The termination of the ability of an item to perform a required function [3].

The term item refers to the engineered object being considered and that can be seen as an entity [3], [4]. Further, 'the item may be an individual part, component, device,

functional unit, equipment, subsystem, or system' [3]. Whenever items are combined into a system, this is called a *system reliability analysis* [4].

Any system may have one or more functions, which may be active or passive, and are necessary to deliver a service [4]. When performing a reliability analysis, it is important to specify the required functions of your object of study, as this will determine what is considered to be a failure for that system.

Traditionally, there are two types of items: non-repairable and repairable. Their definition, according to IEC 60050-192:2015, follows:

Definition 2.3 (Non-repairable items)
Item that cannot, under given conditions, after a failure, be returned to a state in which it can perform as required [3].

Definition 2.4 (Repairable items)
Item that can, under given conditions, after a failure, be returned to a state in which it can perform as required [3].

An important point is that 'the "given conditions" may include technical, economic and other considerations' [3] which leads to the idea that 'an item that is repairable under some conditions may be non-repairable under other conditions' [3].

Following [5], it is important to understand that successful reliability basically depends on the operation of an engineered object under certain conditions arising from the design phase. Now, when performing a failure study in a system, there are a few factors that must be taken into account, which may influence on the good operation of the system during its life cycle, be it during design or operation phases. Those are: *errors*, *failures* and *faults*.

Definition 2.5 (Fault)
The inability to perform as required due to an internal state. Results from a failure,

either of the item itself, or from a deficiency in an earlier stage of the life cycle [3].

Definition 2.6 (Error)

Discrepancy between a computed, observed or measured value or condition, and the true, specified or theoretically correct value or condition. An error within a system may be caused by failure of one or more of its components, or by the activation of a systematic fault. [3].

Therefore, in short, an anomaly in an engineered object happens in a cycle, which starts with an error, which may evolve to a failed state when nothing is done, and lastly turn into a fault state - the last state of functional failure [5].

2.2 Maintenance

Environmental and operational conditions, along with the age and/or usage of an item, influence how the object degrades [1]. These will also affect the performance of an item, in addition with design and implemented maintenance. The latter one is of great concern 'since proper functioning over an extended time requires effective maintenance' [1]. The definition of maintenance is presented as follows:

Definition 2.7 (Maintenance)

Combination of all technical and management actions intended to retain an item in, or restore it to, a state in which it can perform as required [3].

To avoid misconceptions, some terminology is then in place. In [6], the authors propose definitions for three terms in the context of maintenance management, those are: maintenance action, maintenance concept and maintenance policy. However, in [3], the terms maintenance concept and policy are interchangeable. For consistency, the IEC 60050-192:2015 is used.

Definition 2.8 (Maintenance Action)

Sequence of elementary maintenance activities [3].

Definition 2.9 (Maintenance Policy/Concept)

Definition of the maintenance objectives, line of maintenance, indenture levels, maintenance levels, maintenance support, and their interrelationships [3].

Further, maintenance policy 'provides the basis for maintenance planning, determining supportability requirements, and developing logistic support' [3].

The improvement of system performance (reliability) may be achieved by adopting appropriate maintenance policies [7]. According to [7], the three policies generally used are: (1) repair of failed units; (2) provision of redundant units; and (3) maintenance of units before failure. All three policies are described below:

(1) **Repair of failed units**: The first one is named *Corrective Maintenance* (CM) and is 'adopted when units can be repaired but their failures do not adversely affect the whole system' [7]. The IEC [3] defines it as the 'maintenance carried out after fault detection to effect restoration'.

(2) **Provision of redundant units**: The second policy is adopted when 'system reliability can be improved by providing redundant and spare units' [7].

(3) **Maintenance of units before failure**: The last policy comes from the fact that 'maintenance of unit after failure may be costly' [7], therefore it is important to 'determine when and how to maintain preventively [items] before failure' [7]. This is termed *Preventive Maintenance* (PM) defined as the 'maintenance carried out to mitigate degradation and reduce the probability of failure' [3]. Additionally, it can be further distinguished between: condition-based PM and planned PM [8]. The first one defined as 'preventive maintenance based on the assessment of physical condition' [3], and the latter as 'maintenance carried out in accordance with a specified time schedule' [3].

Given the above discussion, for the present thesis, the definition 2.4 for repairable items is extended, adding that the item 'may be restored [...] by any method, other than the replacement of the entire item' [9] and also 'include the possibility of additional maintenance actions which aim at servicing the system for better performance [referred] to as Preventive Maintenance' [8].

2.3 System Performance Measures

The performance of an engineered object is a complex entity characterised by a group of measurable properties of the item [1]. Utilising the division proposed by [1], there are two types of performance measures: (i) non-reliability performance measures and (ii) reliability performance measures. The first one include metrics like the fuel efficiency of an vehicle or emissions of an oil plant. The latter embraces, for example, the number of failures and the availability of a system for a given time interval, under given conditions.

In [1], the authors note that "the performance of an item degrades due to the degradation the material and components of the item". Following [3], the degradation of an item is defined as

Definition 2.10 (Degradation)

Detrimental change in ability to meet requirements [3].

Also, note that degradation "may occur with storage or use, brought about by internal processes or effects of the environment" [3], and when is beyond specified limits, it may constitute a degraded state or failure [3].

In a system reliability analysis, one is interested in finding reliability performance measures. To derive such measures, models that capture the behaviour of the system are needed. This is the field of *quantitative reliability analysis*, which 'uses real failure data in conjunction with suitable mathematical models to produce quantitative estimates of product or system reliability' [1].

Under the general reliability performance measures, one could create and combine any

of them to create new ones, which may be more indicative of the system performance. Therefore, in general, theese are referred to as *performance indicators* (PI) [1].

Definition 2.11 (Performance Indicator)
Variable characterising the performance of a task or activity. It can be at the business, engineered object or component level [1].

Nonetheless, it is common to refer to the important relevant measures as *key performance measures* (KPI), and an important part of any reliability study is to determine the ones which actually capture the behaviour of the system.

Definition 2.12 (Key Performance Indicator)
Performance indicator that is deemed to be important and is used for assessing performance [1].

2.3.1 Two-state Characterisation

Following [1], the performance of an item may be characterise by a variable, $Y(t)$, which indicates the state or condition of such item, as a function of age. A common characterisation is the *two-state characterisation*.

In the two-state characterisation, the state $Y(t)$ is binary valued such that:

- $Y(t) = 1$ when the item is in the *working state*.
- $Y(t) = 0$ when the item is in the *failed state*.

Under the IEC 60050-192:2015, the working and failure state are referred to as up state and down state, respectively, as illustrated in Figure 2.1. Their definitions are as follows:

Definition 2.13 (Up State)
State of being able to perform as required [3].

Definition 2.14 (Down State)

State of being unable to perform as required, due to internal fault, or preventive maintenance [3].

However, care must be taken as 'an item may be considered to be in an up state for some functions and in a down state for others, concurrently' [3].

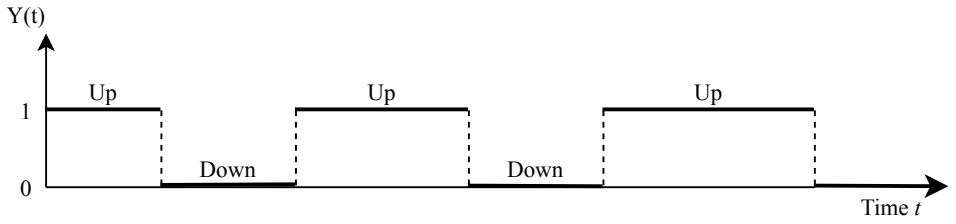

Figure 2.1: Two-state characterisation.

2.3.2 Reliability

The first idea is how long an item can operate without failure, i.e., *reliability* [7]. Previously, the qualitative definition of reliability was presented, now, for a more measurable definition:

Definition 2.15 (Reliability - Quantitative)

The probability of performing as required for the time interval (t_1, t_2)*, under given conditions [3].*

The term 'time' is very general and represents any sort of scale that is able to capture the characteristics of the item. This scale can be the mileage of a vehicle, the operating time of a machine, the on-off cycles of a lamp, etc [3].

2.3.3 Repair Times and Downtime

Given that the interest revolves around the behaviour of system failure 'it is of great importance to know how long and how many times the system is down during a time

interval' [7]. The two reasons for such interest is because system down is sometimes costly and/or dangerous [7].

Definition 2.16 (Downtime)
Time interval for which the item is in a down state. Also, downtime excludes disabled time due to lack of external resources, but includes maintenance time [3].

Definition 2.17 (Repair Time)
Part of active corrective maintenance time taken to complete repair action. Excluding technical, administrative and logistic delays [3].

Since 'the downtime of an item can usually be regarded as a sum of elements like access time, diagnosis time, active repair time, checkout time, etc.' [4] downtime will usually be longer than the actual repair time [1]. While repair times and downtimes are two different concepts, in the context of this thesis the two terms will be used interchangeably.

2.3.4 Availability

When items are replaced upon failure or preventively maintained, one is interested with the ratio at which units can operate, i.e., *availability* [7]. The definition of availability, according to [3] is:

Definition 2.18 (Availability - Qualitative)
The ability to be in a state to perform as required [3].

The availability of an item may be quantified by a different number of measures, namely [1]: (1) point availability, $A(t)$; (2) mean availability, $\overline{A}(t_1, t_2)$, and; (3) steady state availability, A.

Definition 2.19 (Point Availability)
Probability that an item is in a state to perform as required at a given instant [3].

Definition 2.20 (Mean Availability)

Average value of the instantaneous availability over a given time interval (t_1, t_2) *[3]. Related to the point availability as:*

$$\overline{A}(t_1, t_2) = \frac{1}{t_1 - t_2} \int_{t_1}^{t_2} A(t)dt$$

Definition 2.21 (Steady State Availability)

Limit, if it exists, of the point availability when the time tends to infinity [3]. Related to the point and mean availability as:

$$A = \lim_{t \to \infty} A(t) = \overline{A}(0, \infty)$$

An important note to the steady state availability is that 'under certain conditions the steady state availability may be expressed as the quotient of the mean up time to the sum of the mean up time, to mean down time' [3]. However, one should be careful, as in general, for repairable systems, this is not true [9].

Chapter 3

Repairable Systems

3.1 General Concepts

In a repairable system reliability analysis, the interest is in finding reliability performance measures such as 'the availability of the system, the mean number of failures during a specified time interval, the mean time to the first system failure, and the mean time between system failures' [4]. The traditional tool is through the use of stochastic processes. Following the definition of Ross [10], a *stochastic process* $\underline{\mathbf{X}} = \{X(t) : t \in T\}$ is a collection of random variable (r.v.). That is, for each t in the *index set* T, $X(t)$ is random variable. If T is a countable set, then $\underline{\mathbf{X}}$ is a *discrete-time* stochastic process, and if T is a continuum, it is named a *continuous-time* process.

The simplest and most common stochastic processes used in reliability analysis are the *point processes*, which 'is a mathematical model for a physical phenomenon characterised by highly localised events distributed randomly in a continuum' [9]. In order to simplify the models, events are said to be failures. Consider a repairable system put into operation at time $t = 0$. The ith failure will happen at time T_i. As per Definition 2.4, after failure the system will be restored to a functioning state. The repair time is considered to be negligible. That results in a sequence of failure times $T_1, T_2, ...$ Also, a sequence of time between failures $X_1, X_2, ...$, where $X_i = T_i - T_{i-1}$, $i = 1, 2, 3, ...$, and $T_0 \equiv 0$, is present.

In general, the sequence of inter-arrival times $X_1, X_2, ...$ will not be i.i.d..

The key difference between T_i and X_i is illustrated in Figure 3.1. The random variable T_i measures the *total time from 0* - a convenient fixed origin - to the ith failure, while X_i measures the *time between failures*, such that the time to the kth failure is $T_k \equiv X_1 + X_2 + ... + X_k$. As the X_i's do not have their origin in $t = 0$ (except for X_1), they are said to be *chronologically ordered* [9].

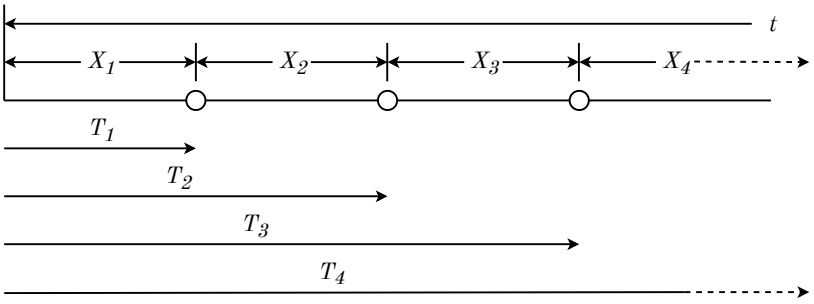

Figure 3.1: Realisation of a stochastic point process [9].

The particular stochastic process $\mathbf{N} = \{N(t) : t \geq 0\}$, with

$$N(t) = \sum_{i=1}^{\infty} \mathbb{1}(T_i \leq t) \text{ with } \mathbb{1}(T_i \leq t) = \begin{cases} 1 & \text{if } T_i \leq t \\ 0 & \text{otherwise} \end{cases}$$

is called a *counting process*. It keeps track of the number of failures in $(0, t]$ and the instants T_i, $i = 1, 2, 3, \ldots$, at which they occur. Following Ross [10], the formal definition of a counting process is as follows:

Definition 3.1 (Counting Process)

A stochastic process $\mathbf{N} = \{N(t) : t \geq 0\}$ *is said to be a counting process if* $N(t)$ *satisfies:*

i. $N(t) \geq 0$.

ii. $N(t)$ *is integer valued.*

iii. *If $s \leq t$, then $N(s) \leq N(t)$.*

iv. *For $s \leq t$, $[N(t) - N(s)]$ represents the number of failures that have occurred in the interval $(s, t]$.*

The counting process $\mathbf{N}$ can be completely characterised by its *intensity function* [11]. Let $\mathcal{H}_{t-}$ denote the history of the process up to, but not including, time t. This history is also called a *filtration* and 'contains information about the components, the number of failures up to time t, covariate information and type of interventions performed' [12]. The intensity function of a process is then defined as [13]

$$\gamma(t \mid \mathcal{H}_{t-}) = \gamma(t) \equiv \lim_{\Delta t \to 0} \frac{\Pr\{N(t + \Delta t) - N(t) = 1 \mid \mathcal{H}_{t-}\}}{\Delta t} \tag{3.1}$$

In words, the above equation is stating that the intensity function 'gives the instantaneous probability of an event occurring at t, conditional on the process history' [13].

To keep consistency throughout, for the random variable $N(t)$, denote its expected value as $\mu(t)$, i.e., $\mu(t) \equiv E[N(t)]$. This is called the *mean function*. Assume $\mu(t)$ is absolutely continuous, and denote its derivative as $\rho(t)$. This is the time rate of change of an expected number of failures and will be termed the rate of occurrence of failures (ROCOF), i.e.,

$$\rho(t) \equiv \frac{d}{dt} E[N(t)] = \text{ROCOF} \tag{3.2}$$

The ROCOF, $\rho(t)$, can be interpreted as follows: '$\rho(t)\delta t$ is the probability that a failure, not necessarily the first, occurs in $(t, t + \delta t]$' [9]. Although it may have many different shapes [1], it is generally drawn as a *bathtub curve* [7], shown in Figure 3.2. The lifetime of an item is then divided into three intervals [4]: the *burn-in period* (region A), the *useful life period* (region B), also called the *chance failure period*, and the *wear-out period* (region C).

In region A (decreasing rate), the failure is due to manufacturing and/or assembly errors; in region B (constant rate), the failure is random and independent of age; in region C (increasing rate), failure is due to ageing effects [1]. Also, 'for the majority of

mechanical items the [ROCOF] function will show a slightly increasing tendency in the useful life period' [4].

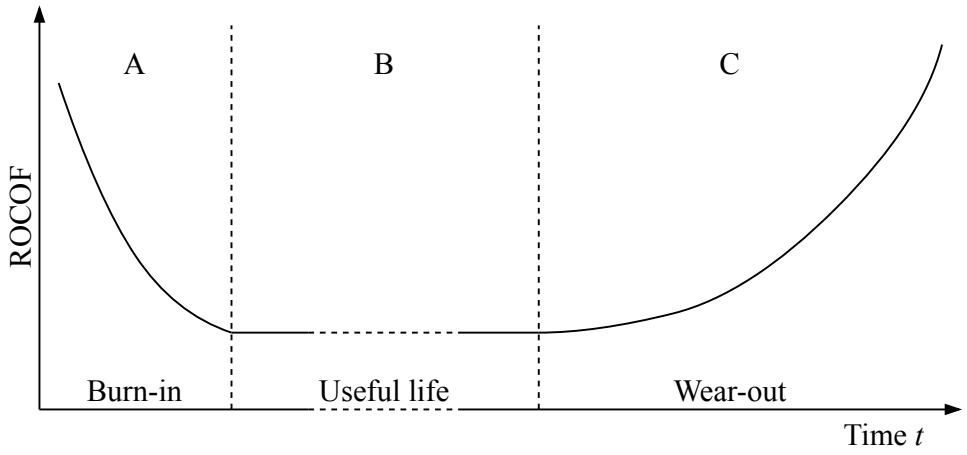

Figure 3.2: Bathtub curve. Adapted from [1], [4].

As in general the inter-arrival times will not be i.i.d., the *mean time between failures*, $MTBF_i = E[X_i]$, will be a function of i and $X_1, X_2, ...$ [4], [9]. Following Ascher and Feingold [9], the reciprocal of the ROCOF is the *instantaneous mean time between failures*.

For a point process, the probability of failure free operation over a time interval $(t, t+s]$ given the history $\mathcal{H}_{t-}$ is defined as [9]

$$R(t, t+s \mid \mathcal{H}_{t-}) \equiv \Pr\{N(t, t+s) = 0 \mid \mathcal{H}_{t-}\} \tag{3.3}$$

and is called the *reliability function*.

3.2 Failure Process Models

A lot of different models applicable to repairable systems have been proposed in the literature. According to Lindqvist [8], 'the most commonly used models for the failure process of a repairable system are renewal processes (RP) [...] and non-homogeneous Poisson processes (NHPP)'. The first one is used to model a repair situation where the item is replaced or restored to a "as good as new" condition. This is termed a *perfect*

repair. The latter is appropriate when the repair does not affect the reliability of the system, that is, the system after repair is "as bad as old" and it is termed a *minimal repair*.

Of course, the RP and NHPP are two extreme types of repair. A general repair action should lay between this two extremes, this is termed *imperfect repair*. A lot of imperfect repair models exist in the literature, here the focus is on the trend-renewal process (TRP), proposed and studied by Lindqvist, Elvebakk and Heggland [14]. The trend-renewal process have both the renewal processes and the non-homogeneous Poisson processes as special cases and therefore is classified as an imperfect repair model.

Due to the dynamic nature in which recurrent events appear and the necessity of incorporating the effects of performed interventions, the use of hazard functions are more appropriate than that of density functions in modelling [12]. Therefore, let $z(t)$ represent the failure rate, also called the hazard rate function of a system in a reliability setting, such that for a continuous random variable $T \geq 0$ with density function $f(t)$ and distribution function $F(t)$, thus [2]

$$z(t) = \frac{f(t)}{1 - F(t)} \text{ and } \mathcal{Z}(t) = -\log(1 - F(t)) = \int_0^t z(u)du \tag{3.4}$$

3.2.1 Homogeneous Poisson Process

The most simple way of defining the HPP is as a 'nonterminating sequence of independent and identically distributed exponentially distributed X_i's' [9]. Combining the definitions given by Ross [10] and Ascher and Feingold [9]:

Definition 3.2 (Homogeneous Poisson Process)
A stochastic process $\mathbf{N} = \{N(t) : t \geq 0\}$ *is said to be a homogeneous Poisson process if* $N(t)$ *satisfies:*

i. $N(0) = 0$.

ii. The process has independent increments.

iii. The number of events in any interval of length $t_2 - t_1$ is Poisson distributed with mean $\lambda(t_2 - t_1)$. That is, for all $t_2 > t_1 \geq 0$,

$$\Pr\{N(t_2) - N(t_1) = n\} = \frac{1}{n!}\left[\lambda(t_2 - t_1)\right]^n \exp\left[-\lambda(t_2 - t_1)\right] \tag{3.5}$$

And is denoted HPP(λ).

From the definition, one can see that

$$E[N(t_2 - t_1)] = \lambda(t_2 - t_1) \tag{3.6}$$

where the constant λ is the rate of occurrence of failures (ROCOF), given $\mu'(t) = z(t) = \lambda$. Thus, for the HPP(λ), the ROCOF equals the intensity function. This is true for any Poisson process [13]. As per Equation 3.3, one can see that the reliability function, $R(t_1, t_2)$ of a HPP is

$$R(t_1, t_2) = \exp\{-\lambda(t_2 - t_1)\} \tag{3.7}$$

3.2.2 Nonhomogeneous Poisson Process

The nonhomogeneous Poisson process (NHPP) is a generalisation of the HPP. Let the rate of occurrence of failure vary with time rather than being a constant and denote it NHPP($\lambda(\cdot)$).

Definition 3.3 (Nonhomogeneous Poisson Process)

A stochastic process $\{N(t), t \geq 0\}$ is said to be an NHPP($\lambda(\cdot)$) if $N(t)$ satisfies:

i. $N(0) = 0$.

ii. The process has independent increments.

iii. The number of events in any interval (t_1, t_2) is Poisson distributed with mean

$\int_{t_1}^{t_2} \lambda(t)dt$. *That is, for all* $t_2 > t_1 \geq 0$,

$$\Pr\{N(t_2) - N(t_1) = n\} = \frac{1}{n!}\left[\int_{t_1}^{t_2} \lambda(t)dt\right]^n \exp\left[-\int_{t_1}^{t_2} \lambda(t)dt\right] \tag{3.8}$$

The same way as for the HPP,

$$E[N(t_2) - N(t_1)] = \int_{t_1}^{t_2} \lambda(t)dt \tag{3.9}$$

The general form of the reliability function in the time interval $[t_1, t_2]$ for a NHPP with intensity function $\lambda(t)$ is given by [9]

$$R(t_1, t_2) = \exp\left\{-\int_{t_1}^{t_2} \lambda(t)dt\right\} \tag{3.10}$$

The major difference between the HPP(λ) and NHPP($\lambda(\cdot)$) models is that, 'under the latter model the $X_i's$ are neither independent nor identically distributed' [9]. Therefore, 'statistical techniques [...] based on the assumption that the data are independent and identically distributed cannot be [...] applied to the NHPP' [4], [9].

The NHPP is often used to model trends in the inter-arrival times, meaning, modelling improvement or deterioration of a system [4], [9]. Thanks to the independent increments property of the NHPP, system deterioration/improvement can be defined straightforwardly. A system modelled by a NHPP is deteriorating (improving) in the interval $(0, t_0]$ if [9]

$$\int_0^{t_1} \lambda(y)dy + \int_0^{t_2} \lambda(y)dy \leq (\geq) \int_0^{t_1+t_2} \lambda(y)dy,\ 0 < t_1 + t_2 \leq t_0 \tag{3.11}$$

This property is termed the semistrict *supperadditive* (*subadditive*) property.

An improving (deteriorating) system is also called a *happy* (*sad*) system. Therefore, a happy (sad) system will have a decreasing (increasing) rate of occurrence of failures [4]. The two shapes for the intensity function that have been extensively used in the context of repairable systems reliability are [1], [4], [9]:

1. Power-law intensity function.

$$\lambda(t) = abt^{b-1},\ a > 0,\ b > 0,\ t \geq 0 \tag{3.12}$$

2. Log-linear intensity function.

$$\lambda(t) = \exp(a + bt),\ a > -\infty,\ b < \infty,\ t \geq 0 \tag{3.13}$$

Some conclusions - which comes from property 3.11 - concerning these functional forms are in order [1]:

- If $b = 1$ ($b = 0$), the power-law (log-linear) NHPP becomes the HPP.
- If $b < 1$ ($b < 0$), the power-law (log-linear) intensity is strictly decreasing with age. Hence, it is characteristic of a happy/improving system.
- If $b > 1$ ($b > 0$), the power-law (log-linear) intensity is strictly increasing with age. Hence, it is characteristic of a sad/deteriorating system.

In general, there are two situations where the NHPP may be an adequate model [4], namely:

(a) The items put into service are identical to the old ones.

(b) The system consists of a large number of components and repair only replaces a small fraction of the system.

When in situation (a) it means that the items should have aged outside the system under identical conditions for the same period of time as the replaced ones. On the other hand, in situation (b) a traditional assumption is that the reliability of the system does not change. In this approach the system is treated as a *black-box*, 'in that there is no concern about how the system looks inside' [4].

3.2.3 Renewal Process

As established before, the HPP is a process where the times between failures are independent and exponentially distributed. A natural generalisation is to allow the inter-failure times to be independent and identically distributed with an arbitrary distribution function [10]. This is called a Renewal Process (RP), denoted as RP(F), where F is the underlying distribution of the renewal process [4]. Formally [4],

Definition 3.4 (Renewal Process)
A counting process $\mathbf{N} = \{N(t), t \geq 0\}$ *is called a RP(F) when the inter-occurrence times* $X_1, X_2, \ldots$ *are independent and identically distributed with distribution function*

$$F(t) = \Pr\{X_i \leq t\} \text{ for } t \geq 0,\ i = 1, 2, \ldots$$

And $E[X_i] = \mu$ *and* $\text{var}[X_i] = \sigma^2 < \infty$ *for* $i = 1, 2, 3, \ldots$.

This is means that the process intensity is of the form [13]:

$$\gamma(t \mid \mathcal{H}_{t-}) = z(t - T_{N(t-)}) \tag{3.14}$$

where $z(\cdot)$ is the hazard function associated with F, as per Equation 3.4.

The renewal process is the expected process for a 'single-component system, where the replacements are always new items from the same population as the items being replace' [15]. Also, 'a system containing a single repairable component is a simple but frequently useful model. A system may have a weak part that accounts for practically all failure during a specified period of use' [15].

Although the basic concepts for a general counting process are applicable for a renewal process, many of them have been given specific names. Using the same notation of before, the expected value of $N(t)$, denoted $\mu(t) = E[N(t)]$, is referred to as the *renewal function* and its derivative, $\rho(t) = \frac{d}{dt}E[N(t)]$, is the *renewal density*.

The probability distribution and the expectation of $N(t)$ are related to the distribution

function F of the inter-failure times by an integral equation

$$\mu(t) = F(t) + \int_0^t \mu(t-x)f(x)dx \tag{3.15}$$

This is what is referred to as the *fundamental renewal equation* and can sometimes be solved for $\mu(t)$ [4]. To find the renewal density $\rho(t)$ one needs to differentiate equation 3.15. Most of the times, however, the analytical solution is very difficult or even impossible to obtain [1], [4]. Therefore, approximations and/or numerical methods should be used [4].

Renewal Function Estimation One of such methods is the one presented in [16] called the RS-Method, which is a discritization algorithm for the renewal equation (3.15) [17]. The idea is as follows: for a given time t, let t_i satisfy the condition $0 = t_0 < t_1 < \ldots < t_n = t$, then $\mu(t_i)$ may be calculated recursively through [16],

$$\mu(t_i) = \frac{F(t_i) - S_i - F(t_i - t_{i-\frac{1}{2}})\mu(t_{i-1})}{1 - F(t_i - t_{i-\frac{1}{2}})}$$

$$S_i = \sum_{j=1}^{i-1} F(t_i - t_{j-\frac{1}{2}})(\mu(t_j) - \mu(t_{j-1}))$$

where,

$$x_{i-\frac{1}{2}} = \frac{x_i + x_{i-1}}{2}$$

In the same paper [16], the authors also develop the algorithm with the simplifying assumption of equal lengths, i.e., when $t_i = it/n$. Under this assumption and following the formulation presented in [17], the time interval $[0, t]$ is divided by n (> 0) line segments with equal length d (> 0). Defining the values of the renewal function and the underlying distribution function at each point $i = 0, d, 2d, \ldots, nd = t$ by $\mu_i = \mu(id)$ and $K_i = F(id)$, respectively, where $F_i = F((i - 1/2)d)$. Then the method becomes,

$$\mu_i = \frac{K_i + \sum_{j=1}^{i-1} F_{i-j+1}(\mu_j - \mu_{j-1}) - \mu_{i-1}F_1}{1 - F_1} \tag{3.16}$$

where $\mu(0) = 0$ and $i = 1, \ldots, n$.

It is important to note that the accuracy of the method 'depends heavily on the choice of the grid size d' [17]. That is, 'the suitable grid size d is related to the shape of the underlying distribution F and the time scale t' [17]. Also, although the method results in smaller errors for a sufficiently larger time scale t, for smaller time scale, the method does not always performs well [17]. However, the main advantage of the method, which is also its goal, is to find a accurate estimate that is also easy to program [16].

Renewal Density Estimation Now, one is also interested in an estimate for the renewal density $\rho(t)$. This may be done by numerical differentiation, specifically with finite differences [18]. Let $\mu(t_i)$ be the values for the renewal function, $\mu(t)$, calculated using the RS-method for uniformly distributed points in time $t_i = id$, $i = 0, 1, \ldots, n$. The derivatives are approximated at the interior points, $\rho(t_i)$, $i = 1, \ldots, n-1$, by the centered finite difference [18]:

$$\rho(t_i) = \frac{\mu(t_{i+1}) - \mu(t_{i-1})}{2d} \tag{3.17}$$

And at the end-points, one-sided differences are used [18]:

$$\rho(t_0) = \frac{\mu(t_1) - \mu(t_0)}{d}, \quad \rho(t_n) = \frac{\mu(t_n) - \mu(t_{n-1})}{d} \tag{3.18}$$

Reliability Measures Since the inter-arrival times under a renewal process are independent and identically distributed, some reliability measures may be analytically defined. Then, let the counting process $\{N(t) : t \geq 0\}$ be a $RP(F)$. That is, $X_1, X_2, \ldots \sim F$. The reliability function, expected value, variance and is then [1], [4], [19], [20]

$$R(t) = 1 - F(t) \tag{3.19}$$

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx \tag{3.20}$$

$$\mathrm{Var}[X] = \int_{-\infty}^{\infty} x^2 f(x) dx - E[X]^2 \tag{3.21}$$

where $f(x) = dF(x)/dx$. The use of equations 3.19, 3.20 and 3.21 requires the definition of a parametric distribution function F. This way of estimating the reliability measures is called the *parametric approach*. The second approach is to compute the so-called *empirical distribution function* EDF and plug its values into the definition of the reliability function (Equation 3.19) [4]. This is the *non-parametric approach*.

Following [4], let $X_1, X_2, \ldots, X_n$ be a complete data set of n independent observations. Also, let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the data arranged in ascending order. The empirical distribution function is defined as

$$F_n(x) = \frac{\text{Number of observations} \leq x}{n} \tag{3.22}$$

Thus, the corresponding *empirical reliability function* ERF is

$$R_n(x) = 1 - F_n(x) = \frac{\text{Number of observations} > x}{n} \tag{3.23}$$

The ERF may be written as

$$R_n(x) = \begin{cases} 1 & \text{for } x < x_{(1)} \\ 1 - \dfrac{i}{n} & \text{for } x_{(i)} \leq x < x_{(i+1)};\ i = 1, 2, \ldots, (n-1) \\ 0 & \text{for } x_{(n)} \leq x \end{cases} \tag{3.24}$$

Plotting the pairs $(x_{(i)}, R_n(x_{(i)}))$ gives a general idea of the behaviour of the reliability functions. A powerful generalisation of the ERF is the so-called *Kaplan-Meier estimator* [4]. Following [4], let $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$ denote the recorded inter-failure times ordered according to size. Let J_x denote the set of all indices j where $X_{(j)} \leq x$. Let n_j denote the number of items functioning and in observation immediately before time $x_{(j)}$, $j = 1, 2, \ldots, n$. The Kaplan-Meier estimator of $R(x)$ is defined as [4]:

$$\widehat{R}(x) = \prod_{j \in J_x} \frac{n_j - 1}{n_j} \tag{3.25}$$

According to [4], with a complete data set, the Kaplan-Meier estimator, given in equation 3.25, is equal to the empirical reliability function, $R_n(x)$. As generally the interested is in some specific values of the reliability function, confidence intervals for the Kaplan-Meier estimator are necessary [21]. Following [22] the *exponential Greenwood formula* is presented, which gives an asymmetric confidence interval

$$\exp(-\exp(c_+(x))) < R(x) < \exp(-\exp(c_-(x))) \tag{3.26a}$$

where

$$c_\pm(x) = \log(-\log(\widehat{R}(x))) \pm z_{\alpha/2}\sqrt{\widehat{V}} \text{ and} \tag{3.26b}$$

$$\widehat{V} = \frac{1}{(\log \widehat{S}(x))^2} \sum_{x_i \leq x} \frac{1}{n_i(n_i - 1)} \tag{3.26c}$$

In 3.26b, z_α is the αth quantile of the normal distribution.

3.2.4 Trend-Renewal Process

Both the renewal process and the nonhomogeneous Poisson process are useful in describing systems in the cases of perfect and minimal repair, respectively. The homogeneous Poisson process is a special case of both models. Realistic systems, however, will be somewhere between these two extremes. Therefore, 'some generalisations and extensions of the basic models, with the aim to arrive at more realistic models which give better fit to data' [23] are in place. Several models have been suggested and they are within the so-called class of *imperfect repair* models [4], [8].

The trend-renewal process (TRP), introduced and studied by Lindqvist, Elvebakk and Heggland [14], is a imperfect repair model which includes both the NHPP and RP as special cases, where its 'main new feature is to allow a trend in processes of non-Poisson (renewal) type' [23]. The main idea behind the TRP is to generalise the following property of the NHPP [23].

Property 3.1 (Relation of the NHPP to the HPP)
Let the cumulative intensity function of an intensity $\lambda(\cdot)$ be defined as $\Lambda(t) = \int_0^t \lambda(u)du$. If $T_1, T_2, ...$ is a NHPP($\lambda(t)$), the time-transformed process $\Lambda(T_1), \Lambda(T_2), ...$ is HPP(1).

This generalisation comes by allowing the above HPP(1) to be any renewal process RP(F). Then, the TRP($F, \lambda(\cdot)$) is formally defined as follows [23]

Definition 3.5 (Trend-Renewal Process)
Let $\lambda(t)$ be a nonnegative function defined for $t \geq 0$, and let $\Lambda(t) = \int_0^t \lambda(u)du$. The process $T_1, T_2, ...$ is called TRP($F, \lambda(\cdot)$) if the transformed process $\Lambda(T_1), \Lambda(T_2), ...$ is RP(F), that is if the $\Lambda(T_i) - \Lambda(T_{i-1})$; $i = 1, 2, ...$ are i.i.d. with distribution function F.

From definition 3.5, the function $\lambda(\cdot)$ is the trend function, and F is the renewal distribution. For uniqueness of the model, it is assumed the F has expected value 1. Therefore, one may simply state that the trend-renewal process is 'a time-transformed renewal process having both the ordinary renewal process and the nonhomogeneous process as special cases' [14].

Now, consider a system where failures correspond to replacement of a major part, while the rest of the system is not maintained. As discussed before, the renewal process would be a plausible model, given that the system is not subjected to wear. However, when wearing is present, an increased replacement frequency is to be expected. In the TRP context, this is achieved by accelerating the internal time of the renewal process according to a time transformation which represents the cumulative wear [8].

It can be shown [14] that the intensity function for the TRP($F, \lambda(\cdot)$) is

$$\gamma(t) = z(\Lambda(t) - \Lambda(T_{N(t-)}))\lambda(t) \tag{3.27}$$

where $z(t)$ is the hazard rate corresponding to F, as per 3.4, and $t - T_{N(t-)}$ is the time since the last failure strictly before time t.

3.3 Weibull-Power Law TRP

An important and very useful TRP is the so-called *Weibull-Power Law trend-renewal process* (WPLP) [24]. The usefulness of this model comes from its special cases for different values of its parameters. As one may guess from the name of the model, it involves the power law function and the Weibull distribution. The former being the one presented during the discussion about the NHPP, and given in Equation 3.12. The latter is a common probability distribution used to model independent and identically distributed inter-failure times [9].

Let X be a random variable distributed according to a Weibull distribution with parameters θ_1 and θ_2, denoted $X \sim \mathcal{W}(\theta_1, \theta_2)$. The probability density function, $f(x)$, is then [1]

$$f(x) = \frac{\theta_1}{\theta_2}\left(\frac{x}{\theta_2}\right)^{\theta_1 - 1} \exp\left\{-\left(\frac{x}{\theta_2}\right)^{\theta_1}\right\} \text{ for } x \geq 0,\ \theta_1 > 0,\ \theta_2 > 0 \tag{3.28}$$

where the parameters θ_1 and θ_2 are the shape and scale parameters, respectively. The cumulative density function, the hazard function, the expected value of X and its variance are also given below:

$$F(x) = 1 - \exp\left\{-\left(\frac{x}{\theta_2}\right)^{\theta_1}\right\} \text{ for } x \geq 0,\ \theta_1 > 0,\ \theta_2 > 0 \tag{3.29}$$

$$z(x) = \frac{\theta_1}{\theta_2}\left(\frac{x}{\theta_2}\right)^{\theta_1 - 1} \tag{3.30}$$

$$E[X] = \theta_2 \Gamma\left(1 + \frac{1}{\theta_1}\right) \tag{3.31}$$

$$Var[X] = \theta_2^2 \left[\Gamma\left(1 + \frac{2}{\theta_1}\right) - \Gamma^2\left(1 + \frac{1}{\theta_1}\right)\right] \tag{3.32}$$

Consider the TRP$(F, \lambda(\cdot))$ with

$$\lambda_W(t; a, b) = abt^{b-1},\ a > 0,\ b > 0,\ \Lambda_W(t; a, b) = at^b \tag{3.33}$$

and a renewal distribution F following the Weibull distribution in Equation 3.29. However, the renewal function needs to be parametrised into having an expectation equal 1. Setting the expectation equal to 1 in Equation 3.31

$$E[X] = \theta_2 \Gamma\left(1 + \frac{1}{\theta_1}\right) = 1$$
$$\theta_2 = \frac{1}{\Gamma\left(1 + \frac{1}{\theta_1}\right)} \tag{3.34}$$

which simply means that in order to get the expectation to be equal to 1, one needs to substitute the scale parameter according to Equation 3.3 into the cumulative distribution function given in Equation 3.29. This results in the following renewal distribution, $F_W(\cdot)$, where α is the shape parameter.

$$F_W(x) = F_W(x;\alpha) = 1 - \exp[-(\Gamma(1 + 1/\alpha)x)^\alpha] \ (\alpha > 0) \tag{3.35}$$

The hazard function corresponding to F_W is

$$z_W(x) = z_W(x;\alpha) = [\Gamma(1 + 1/\alpha)]^\alpha \alpha x^{\alpha-1} \tag{3.36}$$

Then, following [25],

Definition 3.6 (Weibull-Power Law TRP)
The TRP($F, \lambda(\cdot)$) with $\lambda(\cdot)$ and F given by equations 3.33 and 3.35, respectively, is called the Weibull-Power Law TRP, denoted WPLP(a, b, α).

The intensity function for the WPLP, $\gamma_W(t)$, is derived using equation 3.27, as follows:

$$\begin{aligned}\gamma_W(t) &= z_W(\Lambda(t) - \Lambda(T_{N(t-)}))\lambda_W(t) = z_W(at^b - aT^b_{N(t-)})abt^{b-1} \\ &= [at^b - aT^b_{N(t-)}]^{\alpha-1}\alpha[\Gamma(1 + 1/\alpha)]^\alpha abt^{b-1} \\ \gamma_W(t) &= [a\Gamma(1 + 1/\alpha)]^\alpha \alpha bt^{b-1}[t^b - T^b_{N(t-)}]^{\alpha-1}\end{aligned} \tag{3.37}$$

Alternatively, the equation 3.37 may be re-written,

$$\gamma_W(t) = \varphi\alpha b t^{b-1}[t^b - T^b_{N(t-)}]^{\alpha-1} \tag{3.38}$$

where, $\varphi = [a\Gamma(1+1/\alpha)]^\alpha$.

Equation 3.38 is preferred for the estimation of the parameters of the model. From equation 3.37 three special cases are identified: (i) $\text{WPLP}(a, b, 1) = \text{NHPP}(\lambda(t))$, i.e. it becomes the Powel-Law process; (ii) $\text{WPLP}(a, 1, 1) = \text{HPP}(a)$; and (iii) $\text{WPLP}(a, 1, \alpha) = \text{RP}(F_W(ax; \alpha))$, i.e. a renewal process with inter-arrival times Weibull distributed. The former two relations may be directly seen by substituting the parameter values into the general intensity function, γ_W, given in equation 3.37. The latter relation, concerning the pure renewal process model, may be derived as follows.

Finding the cumulative distribution function, F, for the resultant Weibull Renewal Process (WRP) when the time trend parameter b in the WPLP is set to 1, is a important matter. From the definition of a trend-renewal process, it is known that $\Lambda(T_i) - \Lambda(T_{i-1})$ are independent and identically distributed random variables with distribution F. In this specific case, the cumulative trend function is linear, $\Lambda(t) = at$, therefore $a(T_i - T_{i-1})$ are i.i.d. random variables, denote them $S_i = a(T_i - T_{i-1})$. The distribution function of S is

$$F(s) = 1 - \exp[-(\Gamma(1+1/\alpha)s)^\alpha]$$

Now, look at the original random variables $X_i = T_i - T_{i-1}$, it is clear that they are nothing but $X_i = S_i/a$. Let, $G(x)$ be the distribution function of the X_i's, then

$$G(x) = \Pr\{X \le x\} = \Pr\left\{\frac{S}{a} \le x\right\} = \Pr\{S \le ax\} = F(ax) \qquad \blacksquare$$

As under the WRP the random variables are independent and identically distributed, it makes sense to analytically define the expectation and variance of the inter-failure times.

Using the basic properties of the expectation and variance operators [19], [20],

$$E[X] = E\left[\frac{S}{a}\right] = \frac{1}{a}E[S] = \frac{1}{a} \tag{3.39}$$

$$\mathrm{Var}[X] = \mathrm{Var}\left[\frac{S}{a}\right] = \frac{1}{a^2}\mathrm{Var}[S] = \frac{1}{a^2}\left[\frac{\Gamma(1+2/\alpha)}{\Gamma^2(1+1/\alpha} - 1\right] \tag{3.40}$$

3.4 Downtime Models

As discussed before, the down time of an item is the time period where the item is not able to perform one or more of its intended functions [4]. Following [4], downtimes may be divided in *unplanned* and *planned downtimes*. The first one being when the downtime is caused by random external events. And the latter caused by planned preventive maintenance and events.

Following [1], for treating and modelling downtime data, assume the time to repair D is a random variable characterised by its repair distribution function $G(t)$, $G(t) = \Pr\{D \leq t\}$, and repair density function $g(t)$. The repair rate is is usually a decreasing function of repair time and is defined as

$$\rho_D = \frac{g(t)}{1 - G(t)} \tag{3.41}$$

When modelling the repair times, it is usually assumed that the repair rate is increasing in the first phase [4]. After a while it begins to decrease, 'indicating that the probability of a repair being completed in a short time increases with the duration that the service has been going on' [1].

The mean downtime, MDT, is the expected value of the random variable D,

$$\mathrm{MDT} = E[D] = \int_0^\infty t g(t) dt \tag{3.42}$$

Although the scheduled downtime may often be regarded as deterministic [4], care must be taken as this should be true only when the variability in the repair time is small

in relation to the mean time for repair [1]. The same is true for the unscheduled planned downtime, even though it is subject to random variations, usually it should be rather straightforward to estimate a mean value [4].

In the next sections, some commonly used distributions in modelling downtime data are explored, assuming they are caused by item failures [4].

3.4.1 Exponential Distribution

Let $T \geq 0$ be a continuous random variable with probability density function $g(t)$ [26]

$$g(t) = \begin{cases} \nu \exp\{-\nu(t-\tau)\} & \text{for } t \geq \tau,\ \nu > 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.43}$$

This distribution is called the 2-*parameter exponential distribution* with location parameter τ and scale parameter $1/\nu$, denoted $T \sim \exp(\tau, \nu)$. It is the simplest downtime distribution one can choose. The mean downtime is MDT $= \tau + 1/\nu$. Although not realistic, the exponential distribution is often chosen for its tractability [4].

3.4.2 Normal Distribution

A random variable $T \geq 0$ is said to be normally distributed with mean τ and standard deviation ν, denoted $T \sim \mathcal{N}(\tau,\ \nu^2)$, when its probability density function is

$$g(t) = \frac{1}{\sqrt{2\pi}\nu} \exp\left\{-\frac{(t-\tau)^2}{2\nu^2}\right\} \text{ for } -\infty < t < \infty \tag{3.44}$$

Here, τ is also the location parameter, while ν is the scale parameter. Estimation of the MDT and the standard deviation is straightforward in the normal model [4].

3.4.3 Lognormal Distribution

A random variable $T \geq 0$ is said to be lognormally distributed with parameters ν and κ^2, denoted $T \sim \text{lognormal}(\nu,\ \kappa^2)$, if $Y = \ln T$ is normally distributed, i.e. $Y \sim \mathcal{N}(\ln \nu,\ \kappa^2)$. Thus, the probability distribution function is [27]

$$g(t) = \begin{cases} \dfrac{1}{\sqrt{2\pi}\kappa t} \exp\left\{ -\dfrac{(\ln t - \ln \nu)^2}{2\kappa^2} \right\} & \text{for } t > 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.45}$$

For the lognormal distribution, ν is the scale parameter, while κ is shape parameter. The mean downtime is given by [27]

$$\text{MDT} = \nu \exp\left\{ \frac{1}{2}\kappa^2 \right\} \tag{3.46}$$

According to [1], [4], [28] the lognormal distribution is appropriate for modelling the downtimes for many different products. That is because when using the lognormal distribution, the repair rate increases up to a maximum, and thereafter decreases asymptotically down to zero as a function of the elapsed downtime, which makes practical sense.

Following [27], a more general 3-parameter form of the lognormal includes the additional location parameter τ. To obtain it, simply replace t by $(t - \tau)$ in equation 3.45.

3.5 Statistical Inference

The approach to estimating model parameters is *via* the maximum likelihood principle [12]. It revolves around constructing a *likelihood function*, $L(\boldsymbol{\theta}; \boldsymbol{x})$, which describes how likely the observed sample, $\boldsymbol{x}$, is as a function of the possible parameter values, $\boldsymbol{\theta}$ [19]. The maximum likelihood estimator (MLE) $\widehat{\boldsymbol{\theta}}$ of the parameters is the set of values that maximises this function [1], [13]. The likelihood function based on available data vector, $\boldsymbol{x}$, is [13]

$$L(\theta) \propto \Pr\{\boldsymbol{\theta}; \boldsymbol{x}\} \tag{3.47}$$

The *log-likelihood* function is

$$\ell(\theta) = \log L(\theta) \tag{3.48}$$

and because the log function is monotonic, maximising $\ell(\theta)$ is equivalent to maximising $L(\theta)$ [13].

The derivative of the log-likelihood function is the *score function*, $s(\boldsymbol{\theta}; \boldsymbol{x})$ [13]:

$$\boldsymbol{s}(\boldsymbol{\theta}; \boldsymbol{x}) = \frac{\partial}{\partial \boldsymbol{\theta}} \ell(\boldsymbol{\theta}; \boldsymbol{x}) \tag{3.49}$$

where $\boldsymbol{\theta} = \{\theta_1, \ldots, \theta_p\}^T$, $\boldsymbol{x} = \{x_1, \ldots, x_n\}^T$ and $\boldsymbol{s}(\boldsymbol{\theta}; \boldsymbol{x}) = \{s_1(\boldsymbol{\theta}; \boldsymbol{x}), \ldots, s_p(\boldsymbol{\theta}; \boldsymbol{x})\}^T$, with

$$s_i(\boldsymbol{\theta}; \boldsymbol{x}) = \frac{\partial}{\partial \theta_i} \ell(\boldsymbol{\theta}; \boldsymbol{x}), \ i = 1, 2, \ldots, p \tag{3.50}$$

Each $s_i(\boldsymbol{\theta}; \boldsymbol{x})$ has expectation zero. The *observed information matrix*, $I(\boldsymbol{\theta})$, is the negative of the $p \times p$ Hessian matrix of the log-likelihood so that [13]

$$I_{ij}(\boldsymbol{\theta}) = -\frac{\partial^2}{\partial \theta_i \theta_j} \ell(\boldsymbol{\theta}) = -\frac{\partial}{\partial \theta_j} s_i(\boldsymbol{\theta}; \boldsymbol{x}), \ i, j = 1, 2, \ldots, p \tag{3.51}$$

And the *Fischer information matrix*, $\mathcal{I}(\boldsymbol{\theta})$, is both the expectation of the observed information matrix, and the covariance matrix of the score vector, thus

$$\mathcal{I}_{ij}(\boldsymbol{\theta}) = E\left[-\frac{\partial}{\partial \theta_j} s_i(\boldsymbol{\theta}; \boldsymbol{x})\right] = E[\boldsymbol{s}(\boldsymbol{\theta}; \boldsymbol{x}) \boldsymbol{s}^T(\boldsymbol{\theta}; \boldsymbol{x})] \tag{3.52}$$

The maximisation process 'is straightforward if the parameter space is unbounded and the distribution is differentiable with respect to θ' [1]. Also, as stated by [19], 'most statisticians recommend this method, at least when the sample size is large, since the resulting estimators have certain desirable efficiency properties'.

Consider the estimated parameter set, $\widehat{\boldsymbol{\theta}}$. This will be asymptotically normal, with [13]

$$E[\widehat{\theta}_i] = \theta_i; \ \mathrm{Var}[\widehat{\theta}_i] = \mathcal{I}_{ii}^{-1}(\boldsymbol{\theta}) \tag{3.53}$$

Similarly, Equation 3.53 may be rewritten with the observed information matrix, that is easier to obtain. Thus, $\widehat{\boldsymbol{\theta}}$ will be approximately normal with

$$E[\widehat{\theta}_i] = \theta_i;\ \mathrm{Var}[\widehat{\theta}_i] = I_{ii}^{-1}(\boldsymbol{\theta}) \tag{3.54}$$

Therefore, the so-called *Wald type confidence intervals* that gives a $1-\alpha$ confidence interval may be constructed [13],

$$\widehat{\theta}_i \pm z_{\alpha/2}\sqrt{\mathcal{I}_{ii}^{-1}(\boldsymbol{\theta})} \text{ or } \widehat{\theta}_i \pm z_{\alpha/2}\sqrt{I_{ii}^{-1}(\boldsymbol{\theta})} \tag{3.55}$$

$i = 1, 2, \ldots, p$, where z_α is the standard normal upper α-quantile.

3.5.1 Failure Process Model Estimation

Considering a single repairable system observed from time 0 to time σ, resulting in observations $T_1, T_2, \ldots, T_{N(\sigma)}$ the likelihood function is [23]

$$L(\sigma) = \left\{\prod_{i=1}^{N(\sigma)} \gamma(T_i)\right\} \exp\left\{-\int_0^\sigma \gamma(u)du\right\}$$

The time σ is the *stopping time* which, in practice, may be of two censoring schemes: *time truncation* or *failure truncation*. The first is when σ is a prespecified point in time, while the latter the process is observed until a given number n of failures have occurred [29].

For the general TRP$(F, \lambda(\cdot))$ model observed in the time interval $[0, \sigma]$, applying the substitution $v = \Lambda(T_i) - \Lambda(T_{i-1})$, the likelihood function takes the form [24]

$$\begin{aligned} L(\sigma) = &\left\{\prod_{i=1}^{N(\sigma)} z(\Lambda(T_i) - \Lambda(T_{i-1}))\lambda(T_i) \times \exp\left(-\int_0^{\Lambda(t_i)-\Lambda(t_{i-1})} z(v)dv\right)\right\} \\ &\times \exp\left\{-\int_0^{\Lambda(\sigma)-\Lambda(t_{N(\sigma)})} z(v)dv\right\} \end{aligned} \tag{3.56}$$

and the log-likelihood function is defined by [24]

$$\begin{aligned}\ell(\sigma) &= \log L(\sigma) \\ &= \sum_{i=1}^{N(\sigma)} \left\{ \log(z(\Lambda(T_i) - \Lambda(T_{i-1}))) + \log(\lambda(T_i)) - \int_0^{\Lambda(T_i)-\Lambda(T_{i-1})} z(v)dv \right\} \\ &\quad - \int_0^{\Lambda(\sigma)-\Lambda(t_{N(\sigma)})} z(v)dv \end{aligned} \tag{3.57}$$

The maximum-likelihood (ML) estimates of the parameters of the model are obtained by direct maximisation of either the likelihood function [12] or the log-likelihood functions defined in equations 3.56 and 3.57, respectively.

In the case where the observation is finished at the n-th failure time point, that is, in failure truncation, $\sigma = T_{N(\sigma)}$ and $N(\sigma) = n$ [24]. On the other hand, for a time truncation censoring at time t_0, $\sigma = t_0$ and $N(\sigma) = n$.

Consider the Weibull Power-Law trend-renewal process denoted WPLP$(F_W, \lambda_W(\cdot))$, where F_W is the Weibull distribution with expectation equal to 1 and $\lambda_W(\cdot)$ is the power law intensity function. The log-likelihood function for the WPLP$(F_W, \lambda_W(\cdot))$ becomes [24],

$$\begin{aligned}\ell(\sigma) &= \log L(\sigma; \varphi, b, \alpha) \\ &= N(\sigma)(\ln\varphi + \ln b + \ln\alpha) + (b-1)\sum_{i=1}^{N(\sigma)} \ln t_i + (\alpha - 1)\sum_{i=1}^{N(\sigma)} \ln(t_i^b - t_{i-1}^b) \\ &\quad - \varphi\left[\sum_{i=1}^{N(\sigma)} (t_i^b - t_{i-1}^b)^\alpha + (\sigma^b - t_{N(\sigma)}^b)^\alpha\right] \end{aligned} \tag{3.58}$$

In [24], the authors developed the ML-estimators for the WPLP for any stopping time σ, based on equation 3.58. They end-up with a non-linear system of equations that may be solved for the ML estimators $\widehat{\varphi}$, $\widehat{b}$ and $\widehat{\alpha}$ of the parameters φ, b and α, where $\varphi = [a\Gamma(1 + 1/\alpha)]^\alpha$.

3.5.2 Downtime Model Estimation

A common assumption with downtimes is that they are independent from one another. Let $D_1, D_2, \ldots, D_n$ be a random sample from a probability distribution $g(t)$, the likelihood then is simply a product of the individuals probability density functions [19]:

$$L(d_i;\ \theta) = \prod_{i=1}^{n} g(d_i;\ \theta) \tag{3.59}$$

Exponential Distribution Suppose $D_1, D_2, \ldots, D_n$ is a random sample from a 2-parameter exponential distribution. Then, by equation 3.59 along with equation 3.43, it results in [19]:

$$L(d_i;\ \theta) = L(d_i;\ \tau, \nu) = \nu^n \exp\left(-\nu \sum_{i=1}^{n} (d_i - \tau)\right) \tag{3.60}$$

The log-likelihood function then becomes:

$$\ell(d_i;\ \tau, \nu) = n \ln(\nu) - \nu \sum_{i=1}^{n} (d_i - \tau) \tag{3.61}$$

The direct maximisation of equation 3.61 with respect to parameters ν and τ yields the MLE of the parameters, $\widehat{\nu}$ and $\widehat{\tau}$. The **SciPy** package [30] allows the user to easily obtain this parameters for the 2-parameter exponential distribution. In Listing 3.1, a skeleton for fitting an exponential distribution to a set of data $D_1, D_2, ..., D_n$ with **SciPy** is presented.

```
from scipy.stats import expon
import numpy as np

x = np.array([D_1, D_2, ..., D_n]) #data
loc, scale = expon.fit(x) #tau and nu parameters, respectively
```

Listing 3.1: Pseudo-code for the MLE of the 2-parameter exponential distribution

Normal Distribution Suppose $D_1, D_2, \ldots, D_n$ is a random sample from a normal distribution. The likelihood function, based on equations 3.59 and 3.44 is [19]

$$L(d_i;\ \nu,\ \tau^2) = \left(\frac{1}{2\pi\tau^2}\right)^{n/2} \exp\left\{-\sum_{i=0}^{n}\frac{(d_i-\nu)}{2\tau^2}\right\} \tag{3.62}$$

Thus,

$$\ell(d_i;\ \nu,\ \tau^2) = -\frac{n}{2}\ln(2\pi\tau^2) - \frac{1}{2\tau^2}\sum_{i=1}^{n}(d_i-\nu)^2 \tag{3.63}$$

Therefore, maximising equation 3.63 results in

$$\widehat{\nu} = \frac{1}{n}\sum_{i=1}^{n} D_i \text{ and } \widehat{\tau}^2 = \frac{1}{n}(D_i-\nu)^2 \tag{3.64}$$

In Listing 3.2 a skeleton for fitting an lognormal distribution to a set of data $D_1, D_2, ..., D_n$ with **`SciPy`** is presented.

```
from scipy.stats import norm
import numpy as np

x = np.array([D_1, D_2, ..., D_n]) #data
loc, scale = norm.fit(x) #tau and nu parameters, respectively
```

Listing 3.2: Pseudo-code for the MLE of the normal distribution

Lognormal Distribution Let $D_1, D_2, \ldots, D_n$ be a random sample from a lognormal distribution. The likelihood function, based on equations 3.59 and 3.45 takes the form of [31]

$$L(d_i;\ \kappa, \tau, \nu) = (2\pi\kappa^2)^{(-n/2)}\prod_{i=1}^{n}(d_i-\tau)^{-1}\exp\left\{-\frac{(\ln(d_i-\tau)-\ln\nu)^2}{2\kappa^2}\right\} \tag{3.65}$$

Therefore,

$$\ell(d_i;\ \kappa,\tau,\nu) = -\frac{n}{2}\ln(2\pi\kappa^2) - \sum_{i=1}^{n}\ln(d_i-\tau) - \frac{1}{2\kappa^2}\left[\sum_{i=1}^{n}\ln(d_i-\tau)^2 + 2\sum_{i=1}^{n}\ln\nu\ln(d_i-\tau) - n\ln^2\nu\right] \quad (3.66)$$

The direct maximisation of equation 3.66 with respect to parameters κ, τ and ν yields the MLE of the parameters, $\widehat{\kappa}$, $\widehat{\tau}$ and $\widehat{\nu}$. The **SciPy** package [30] allows the user to easily obtain this parameters for the 3-parameter lognormal distribution. In Listing 3.3 a skeleton for fitting an lognormal distribution to a set of data $D_1, D_2, ..., D_n$ with **SciPy** is presented.

```
from scipy.stats import lognorm
import numpy as np

x = np.array([D_1, D_2, ..., D_n]) #data
shape, loc, scale = lognorm.fit(x) #kappa, tau and nu parameters,
    respectively
```

Listing 3.3: Pseudo-code for the MLE of the 3-parameter lognormal distribution

3.6 Model Selection

Selecting a suitable model from a set of candidates that is able to characterise the underlying data is a challenge [32]. In this context, assume that there are data, an appropriate selection of models and that statistical inference is to be model based [33]. By appropriate selection of models it is meant the ones where 'careful development and formulation of the candidate collection' [32] has been taken. Also, 'classical inference often involves a data-based search, over the model set, for [...] that single correct model [...] with estimated parameters' [33].

Model selection should be based on a well-justified criterion [33] that assesses whether a fitted model offers an optimal balance between goodness-of-fit and parsimony [32]. Such

criterion must also be 'estimable from the data and must fit into a general statistical inference framework' [33]. The criterion must be 'generalisable, in that it should effectively describe or predict new data arising from the same phenomenon' [32].

One of the most used approaches is the 'information-theoretic selection based on Kullback-Leibler (K-L) information loss' [33]. Such approach is represented by the so-called Akaike information criterion (AIC) [34] which, according to [32], 'was the first model selection criterion to gain widespread attention in the statistical community, and continues to be one of the most widely known and used [...] in statistical practice'.

3.6.1 Akaike information criterion

Akaike [34] proposed a generalisation of the maximum likelihood principle. Such generalisation 'allowed model selection to be firmly based on a fundamental theory' [33]. The classical maximum likelihood framework allows the estimation of model parameters, given that it has specified dimension and structure [32]. What Akaike did was to 'find a formal relationship between K-L information [...] and likelihood theory' [33] which allowed 'both model estimation and selection [to] be simultaneously achieved' [32]. That is, 'combine estimation [...] and model selection under a unified optimisation framework' [33].

The following discussion is based on [32]. Let f denote the truth, conceptually. Then, let g represent an approximating model, a probability distribution. Kullback-Leibler (K-L) information $I(f,\ g)$ is the information lost when model g is used to approximate f.

$$I(f,\ g) = \int f(x) \log \left(\frac{f(x)}{g(x \mid \theta)} \right) dx \tag{3.67}$$

Therefore, the best model is the one with the least information loss relative to other models in the collection. The criterion $I(f,\ g)$ cannot be used directly in model selection because one would need knowledge of the full truth and the parameters θ in the approximating models, g_i. An approach to go around such problem is that of minimising the

expected estimated K-L information rather than minimising known K-L information.

$$I(f,\ g) = E_f[\log(f(x))] - E_f[\log(g(x \mid \theta))] \tag{3.68}$$

However, the above equation may be rewritten as,

$$I(f,\ g) = C - E_f[\log(g(x \mid \theta))] \tag{3.69}$$

where,

$$C = \int f(x) \log(f(x)) dx \tag{3.70}$$

since the term $E_f[\log(f(x))]$ does not depend on the approximating models in the set. This means that only the term $E_f[\log(g(x \mid \theta))]$ needs to be estimated for each model in the set.

Akaike found that the maximised log-likelihood value was a biased estimate of the relative expected K-L information, and that this bias was approximately equal to K, an asymptotic bias correction term. He then multiplied this result by -2, and proposed it as a model selection criterion, known as the Akaike information criterion (AIC). The AIC is given by [1]:

$$\text{AIC} = -2(\text{maximum loglikelihood}) + 2(\text{number of model parameters})$$

Formally written as [33]:

$$\text{AIC} = -2\log(L(\widehat{\theta} \mid \text{data})) + 2K \tag{3.71}$$

The best model is identified by the minimum value of AIC [32]. Acording to [32], in AIC context, the term based on the empirical log-likelihood $-2\log(L(\widehat{\theta} \mid \text{data}))$ is called the *goodness-of-fit term* while the bias correction factor $2K$ is called the *penalty term*.

Even though the AIC is one of the most known and utilised model selection criterion [32] there is also a small-sample (second-order bias correction) version called AIC_C, that

is recommended when K is large relative to sample size n [33],

$$\mathrm{AIC}_C = -2\log(L(\widehat{\theta})) + 2K + \frac{2K(K+1)}{n-K-1} \tag{3.72}$$

The rule of thumb states that this should be used unless $n/K \gtrapprox 40$ for the model with the largest value of K. According to [33], the AIC values can not be interpreted, given that they contain arbitrary constants and are very much affected by sample size. Thus, lets rescale AIC as

$$\Delta_i = \mathrm{AIC}_i - \mathrm{AIC}_{\min} \tag{3.73}$$

where $\mathrm{AIC}_{\min}$ is the minimum of the R different AIC_i values which means that the best model will have $\Delta_i = 0$. For statistical inference on the other $R-1$ possible models, some rules of thumb are in place [33]: Models having $\Delta_i \leq 2$ have substantial evidence; models in which $4 \leq \Delta_i \leq 7$ have considerable less support; and models with $\Delta_i > 10$ have essentially no support.

Following [33], a convenient way to visualise the Δ_i's is to normalise them into the *Akaike weights*, w_i, which, under a heuristic interpretation, are interpreted as the probability that model i is the K-L best model for the data.

$$w_i = \frac{\exp(-\Delta_i/2)}{\sum_{r=1}^{R}\exp(-\Delta_r/2)} \tag{3.74}$$

Chapter 4

Simulation

4.1 General Concepts

Most real-world systems are too complex to allow realistic models to be evaluated analytically [35] since analysing the interplay of the many different factors affecting the behaviour of repairable systems becomes too complicated [36]. Therefore, such complex models must be studied by means of *simulation* [35].

Intuitively, a 'simulation model is a computer representation of a system that mimics the time history of changes taking place in the system' [1] and is 'one of the most widely used operations-research and management-science techniques' [35]. These time history data describes a *sample path* which is 'a record of the time-dependent behaviour of a system' [37]. As general repairable systems are highly stochastic, with each simulation run (replication) what is seen is just one sample outcome of the underlying process [1].

When it comes to maintenance, the time history are events such as failures and maintenance actions, and one is interested in evaluating their impact on the system performance measures [1]. Simulation then allows us to generate new sample paths without building new systems [37] given that the outcome changes when the simulation is repeated [1]. Lastly, *sample-path analysis* extracts the desired performance measures from sample paths [37].

4.2 Discrete-event Simulation

Following [35], a *discrete-event simulation* concerns the modelling of a system as it evolves over time by a representation in which the state variables change instantaneously at separate points in time. Those models are discrete, dynamic, and stochastic, meaning their states may change based on highly localised events distributed across time [9], the system evolves over time and at least some random input are in place [35]. In this models, the objective is to simulate lifetime scenarios for a system on a computer, and deduct estimates of the performance measures of interest [4].

Consider a simulation model, the standard way of keeping track of the current value of simulated time is the next-event approach [35], which advances the *simulation clock* by, according to [1], generating the next event (using random number generators) and updating the clock and the other counts in a cyclic fashion. The simulation is continued until the simulation clock reaches a predefined time or some event happens [4].

Following [37], a generic stochastic-process model will consist of:

- $\{S_n : n = 0, 1, 2, \ldots\}$. The *state-change* process, which represents all relevant information about the state of the system.
- $\{T_n : n = 0, 1, 2, \ldots\}$. The *event-epoch* process, where T_n is the time of the nth system event.
- $\{Y(t) : t \geq 0\}$. The *output* process, defined by $Y(t) \leftarrow S_n$ when $T_n \leq t < T_{n+1}$, which contains information about the state of the system and how long it visited a particular state.

The following discussion comes from [35] where the author presents a more general framework to understand the next-event approach. The simulation clock is initialised to zero and the times of occurrence of future events are determined. Then, the simulation clock is advanced to the time of occurrence of the most imminent of these future events, at which point the state of the system is updated along with the times of occurrence of future events. The simulation clock is advanced to the time of the (new) most imminent

event, the state of the system is updated, and future event times are determined, etc. Since all state changes occur only at event times for a discrete event simulation model, periods of inactivity are skipped over by jumping the clock from event time to event time.

Then, according to [4], the computer creates a chronological log file where all event and the time for each event is recorded. From this, performance measures are calculated. To obtain estimates of satisfactory accuracy, a rather high number of life histories of the system are simulated. For complex systems, several thousands of replications may be needed.

4.3 Random-Variate Generation

Often uncertainty is a significant factor [1], therefore it is necessary to generate random observations in order to determine future event times [35]. Such generated observations are referred to as *random variate*. According to [37], any random variable can be defined as a function of another random variable U that has the uniform distribution on $[0, 1]$. This allows us to generate random variables $T_1, T_2, \ldots$ with a specified distribution function $F_T(t)$ on a computer [4].

A random variable U is said to have a *uniform distribution*, denoted $U \sim \mathcal{U}(a, b)$, if $\Pr\{a \leq U \leq b\}$ depends only on the width $b - a$ of the interval [19]. Its probability distribution function, $f_U(u)$ on the interval $[a, b]$ will be [19]

$$f_U(u) = \begin{cases} \dfrac{1}{b-a} & a \leq u \leq b \\ 0 & \text{otherwise} \end{cases} \tag{4.1}$$

Following [4], [37], let T denote a random variable with with a continuous, increasing cumulative distribution function (cdf), $F_T(t)$. The solution of $F_T(T) = U$ in terms of T is denoted by $T = F_T^{-1}(U)$, the *inverse cdf*. Variables $U_1, U_2, \ldots$ which are uniformly distributed over $[0, 1]$, may be generated by a pseudo-random number generator. The variables $T_i = F_T^{-1}(U_j)$ for $i = 1, 2, \ldots$, will then have distribution function $F_U(t)$. This

is called the *inverse-transform method* [35]. And although it is not necessarily the best one for all cdfs, it is sufficient to develop intuition [37].

4.3.1 Weibull-Power Law Process

As per definition 3.6, the Weibull-Power Law Process (WPLP) is a trend-renewal process with a power-law trend function (equation 3.33), $\lambda_W(\cdot)$, and a Weibull renewal function (equation 3.35), F_W. According to [24], the WPLP(a, b, α) process may be generated through the following formula for the jump times:

$$T_i = \left\{ T_{i-1}^b + \frac{1}{a\Gamma(1+1/\alpha)} \left[\ln\left(\frac{1}{1-U_i}\right) \right]^{\frac{1}{\alpha}} \right\}^{\frac{1}{b}} \tag{4.2}$$

Where $T_0 = 0$ and U_i are random numbers from the uniform distribution $\mathcal{U}(0, 1)$. Equation 4.2 is derived as follows. From the definition of a TRP, the process $\Lambda(T_1)$, $\Lambda(T_2)$, ... is a renewal process on a time axis different than t, let it be $s > 0$. Then applying the inverse-cdf method, such that

$$\begin{aligned} F_W(s) = 1 - \exp\{-[s\Gamma(1+1/\alpha)]^\alpha\} &= U \\ \exp\{-[s\Gamma(1+1/\alpha)]^\alpha\} &= 1 - U \\ [s\Gamma(1+1/\alpha)]^\alpha &= \ln\left(\frac{1}{1-U}\right) \\ s\Gamma(1+1/\alpha) &= \left[\ln\left(\frac{1}{1-U}\right)\right]^{\frac{1}{\alpha}} \\ \therefore\ s = \frac{1}{\Gamma(1+1/\alpha)} \left[\ln\left(\frac{1}{1-U}\right)\right]^{\frac{1}{\alpha}} & \end{aligned} \tag{4.3}$$

Now, the s-scale is related to the t-scale, for the WPLP(a, b, α) as

$$s_i = a[T_i^b - T_{i-1}^b] \tag{4.4}$$

Thus, substituting equation 4.4 into equation 4.3.1, results

$$a[T_i^b - T_{i-1}^b] = \frac{1}{\Gamma(1+1/\alpha)}\left[\ln\left(\frac{1}{1-U_i}\right)\right]^{\frac{1}{\alpha}}$$

$$T_i^b - T_{i-1}^b = \frac{1}{a\Gamma(1+1/\alpha)}\left[\ln\left(\frac{1}{1-U_i}\right)\right]^{\frac{1}{\alpha}}$$

$$T_i^b = T_{i-1}^b + \frac{1}{a\Gamma(1+1/\alpha)}\left[\ln\left(\frac{1}{1-U_i}\right)\right]^{\frac{1}{\alpha}}$$

$$\therefore\ T_i = \left\{T_{i-1}^b + \frac{1}{a\Gamma(1+1/\alpha)}\left[\ln\left(\frac{1}{1-U_i}\right)\right]^{\frac{1}{\alpha}}\right\}^{\frac{1}{b}} \qquad \blacksquare$$

4.3.2 Exponential Distribution

Let X be exponentially distributed, that is $X \sim \exp(\nu)$, with location parameter equal to zero, that is $\tau = 0$. From equation 3.43 one may see that the cumulative distribution function, $G(t)$, takes the form [19]

$$G(x) = \begin{cases} 1 - \exp(-\nu x) & \text{for } x \geq 0,\ \nu \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.5}$$

Thus, applying the inverse-cdf method $u = G(x)$ and solving for x [35]

$$F^{-1}(u) = -\frac{1}{\nu}\ln(1-u)$$

According to [35], $1-u$ may be substituted by u given that both terms have the same $\mathcal{U}(0,\ 1)$ distribution. Therefore, the generating equation becomes

$$X_i = -\frac{1}{\nu}\ln(U_i) \tag{4.6}$$

where U_i are uniformly distributed random variables, $U_i \sim \mathcal{U}(0,\ 1)$. Now, to get 2-parameter exponentially distributed random variables, D_i, simply add the location parameter, τ, for each generated value, that is

$$D_i = X_i + \tau \tag{4.7}$$

4.3.3 Normal Distribution

Unfortunately, sometimes the inverse cdf F^{-1} is not available in a closed form and this is, indeed, the case for the normal distribution [35]. Other methods have been proposed in order to get around this issue. One of such is the Marsaglia's Polar Method [38], which consists in generating a pair of independent standard normal random variables using a pair of uniform random numbers. The algorithm for the Marsaglia's Polar Method is:

Step 1. Generate $U_1,\ U_2 \sim \mathcal{U}(0,\ 1)$.

Step 2. Accept if $W_1^2 + W_2^2 < 1$, where $W_i = 2U_i - 1$ for $i = 1,\ 2$.

Step 3. Return two independent normal random variables $Z_1,\ Z_2$ where

$$Z_1 = W_1\sqrt{\frac{-2\ln(W_1^2 + W_2^2)}{W_1^2 + W_2^2}} \text{ and } Z_2 = W_2\sqrt{\frac{-2\ln(W_1^2 + W_2^2)}{W_1^2 + W_2^2}}$$

Both Z_1 and Z_2 are normal random variables coming from a standard normal distribution also called z-scores, that is $Z_1,\ Z_2 \sim \mathcal{N}(0, 1)$. In order to get the desired distribution with parameters $X \sim \mathcal{N}(\nu,\ \tau^2)$ use the formula for the z-scores [19]

$$Z = \frac{X - \nu}{\tau} \rightarrow X = Z\tau + \nu \tag{4.8}$$

4.3.4 Lognormal Distribution

As discussed before, if a random variable $X \sim \text{lognormal}(\nu,\ \kappa^2)$, then $Y = \ln X$ and $Y \sim \mathcal{N}(\mu_Y,\ \sigma_Y^2)$. Therefore, use random normal variables to generate lognormally distributed ones. In order to do so, use the following formulas to calculate the parameters $\mu_Y,\ \sigma_Y^2$ of the associated normal distributed, given the original lognormal distribution parameters $\nu,\ \kappa^2$

$$\mu_Y = \ln\left(\frac{\nu^2}{\sqrt{\kappa^2+\nu^2}}\right) \text{ and } \sigma_Y^2 = \sqrt{\ln\left(1+\frac{\kappa^2}{\nu^2}\right)} \tag{4.9}$$

Using the parameters from the associated normal distribution, simulate the data using the Marsaglia's Polar Method and then apply an exponential transformation to get the lognormal data, that is:

Step 1. Calculate the parameters $\mu_Y,\ \sigma_Y^2$ from the associated normal distribution.

Step 2. Use the methods to generate normally distributed data.

Step 3. Since $Y = \ln X$, to get the lognormal data, simply apply $X = \exp(Y)$.

Now, to get 3-parameter lognormally distributed random variables, D_i, add the location parameter, τ, for each generated value, that is

$$D_i = X_i + \tau \tag{4.10}$$

4.4 Output Data Analysis

Output analysis is the examination of data generated by a simulation [39]. On account of simulation being a computer-based statistical sampling experiment, appropriate statistical techniques must be used to analyse its output [35]. As uncertainty is usually a important factor, the simulation output data will be random [1] and one must be careful when drawing conclusions about the model's true characteristics [35].

Generally, one needs to perform several simulation runs to obtain proper estimates and confidence intervals [1]. In view of [39], let the performance of the system is measured by a parameter θ. The resulting output of a simulation experiment will be an estimator $\widehat{\theta}$ of θ. The estimator $\widehat{\theta}$ precision can then be measured by its standard error or by the width of a confidence interval for θ. Estimating this standard error/confidence interval along with the number of observations required to achieve a specific standard error/confidence interval is the purpose of output data analysis [39].

Following [35], let $\{T_i : i = 1, 2, \ldots\}$ be the output of a general stochastic process from a single simulation run. In general, the $T_i's$ will be neither independent nor identically distributed. Thus, general statistical techniques are not directly applicable. Now, suppose M independent replications with stopping time σ are performed. Since the inputs are random, the outputs will also be. Resulting in M different realisations (sample paths) of the $T_i's$ random variables.

Denote the complete simulation output as $\{T_{im} : i = 1, 2, \ldots; m = 1, 2, \ldots, M\}$, that is T_{im} is the ith event of the mth replication. According to [35], 'this *independence across runs* is the key to the relatively simple output-data-analysis methods' [35]. Furthermore, regarding output analysis, simulations may be divided into two groups: terminating and nonterminating simulations [35].

As per [35], a *terminating simulation* is one for which there is a clear event E that specifies the length of each replication. While a *nonterminating simulation* is one for which there is no clear event E to specify the length of a run. However, it is important to notice that 'a simulation for a particular system might be either terminating or nonterminating, depending on the objectives of the simulation study' [35]. In this thesis the techniques applicable to terminating simulations will be explored.

4.4.1 Terminating Simulations

Consider a terminating simulation over $[0,\ T_E]$ resulting in observations $\{T_i : i = 1, 2, \ldots, N;\}$, with T_i being the time for the ith event. Let the number of observations

be a random variable N. According to [39], a common goal is to get a point estimate of the mean $\mu = E[T]$ for some random variable T, as well as the confidence interval for such estimate. As per [35], make M independent replications of the simulation and let $T_1, T_2, \ldots, T_M$ be the resulting i.i.d. random variables. Thus,

$$\overline{T}(M) = \frac{1}{M}\sum_{i=1}^{M} T_i \tag{4.11}$$

is an unbiased estimator for the mean. An approximate $100(1-\varepsilon)$ percent $(0 < \varepsilon < 1)$ confidence interval for μ is

$$\overline{T}(M) \pm t_{M-1,1-\varepsilon/2}\sqrt{\frac{S^2(M)}{M}} \tag{4.12}$$

where $S^2(M)$ is the sample-variance given by

$$S^2(M) = \frac{1}{M-1}\sum_{i=1}^{M}[T_i - \overline{T}(M)]^2 \tag{4.13}$$

and $t_{M-1,1-\varepsilon/2}$ is the upper $1-\varepsilon/2$ critical point for the t distribution with $M-1$ degrees of freedom [35].

For the case of the output process $\{Y(t),\ 0 \leq t \leq T_E\}$ since it is a function of t, one might be interested in calculating its sample time average [37]

$$\overline{Y} = \frac{1}{T_E}\int_0^{T_N} Y(t)dt \tag{4.14}$$

Since $Y(t)$ will always be piecewise constant, the area under it will be composed of rectangles of height S_{N-1} and width $T_i - T_{i-1}$, thus [37]

$$\int_0^{T_N} Y(t)dt = \sum_{j=1}^{N}(T_j - T_{j-1})S_{j-1} \tag{4.15}$$

All previous estimator depend on the parameter M, the number of replications. Thus, to increase accuracy of the estimates sufficient replications need to be performed. So far,

the discussion focused on the so-called fixed-sample-size procedure [35], where the number of replications is fixed based on previous experience and analyst judgement. There are other types of dynamic procedures, where algorithms are used to determine the number of replications (see [35], [39], [40]). However, this is beyond the scope of this thesis.

4.4.2 Savitzky–Golay filter

Consider a set of observations resulting from M simulation runs of a particular performance measure calculated into equally spaced time points, $f(t)$. Resulting in the output process $\{f(t_i) : 0 = t_0 < t_1 < \ldots < t_n\}$. Now, following [41], say that this measure variable of interest is both slowly varying and also corrupted by random noise. One would then be interested in getting point measures with a reduced level of noise and without biasing the value obtained. This is the premise of *data smoothing* [41].

A common method of data smoothing based on local least-squares polynomial approximation is the *Savitzky-Golay filters* which are able reduce noise while maintaining the shape and height of waveform peaks [42]. They are particularly interesting as 'they derive directly from a particular formulation of the data smoothing problem in the time domain' [41], which is the one most repairable systems simulations are run.

The following is based on [41]. Recall the output process $f_i \equiv f(t_i)$, with $i = 0, 1, 2, \ldots$. Write $t_i \equiv t_0 + i\Delta$ for some constant sample spacing Δ. Replace each data value f_i by a linear combination g_i of itself and some number of nearby neighbours,

$$g_i = \sum_{n=-n_L}^{n_R} c_n f_{i+n} \tag{4.16}$$

where n_L is the number of points used "to the left" of a data point i and n_R is the number used to the right.

The idea of Savitzky-Golay filtering is to find filter coefficients c_n that approximate the underlying function within the moving window by a polynomial of higher order. That is, for each point f_i, a polynomial is fitted to all $n_L + n_R + 1$ points in the moving window, by the method of least-squares, and then set g_i to be the value of that polynomial at

position i.

The Scipy [30] package, available in Python [43], has a function for applying the Savitzky-Golay filter to a data set. The method has three main parameters: **x**, the data to be filtered; **window_length**, the length of the filter window, i.e., the number of coefficients; and **polyorder**, the order of the polynomial used to fit the samples. The following is a simple example of the use of the package, for some random data derived from a $\sin(x)$ wave.

```
from scipy.signal import savgol_filter
import numpy as np
np.random.seed(666)

x = np.linspace(0, 2*np.pi, 100)
y = np.sin(x) + np.random.uniform(-.1, .1, 100)
sgf_y = savgol_filter(y, 51, 3) # window size 51, polynomial order 3
```

Listing 4.1: Code for the Savitzky-Golay filter example

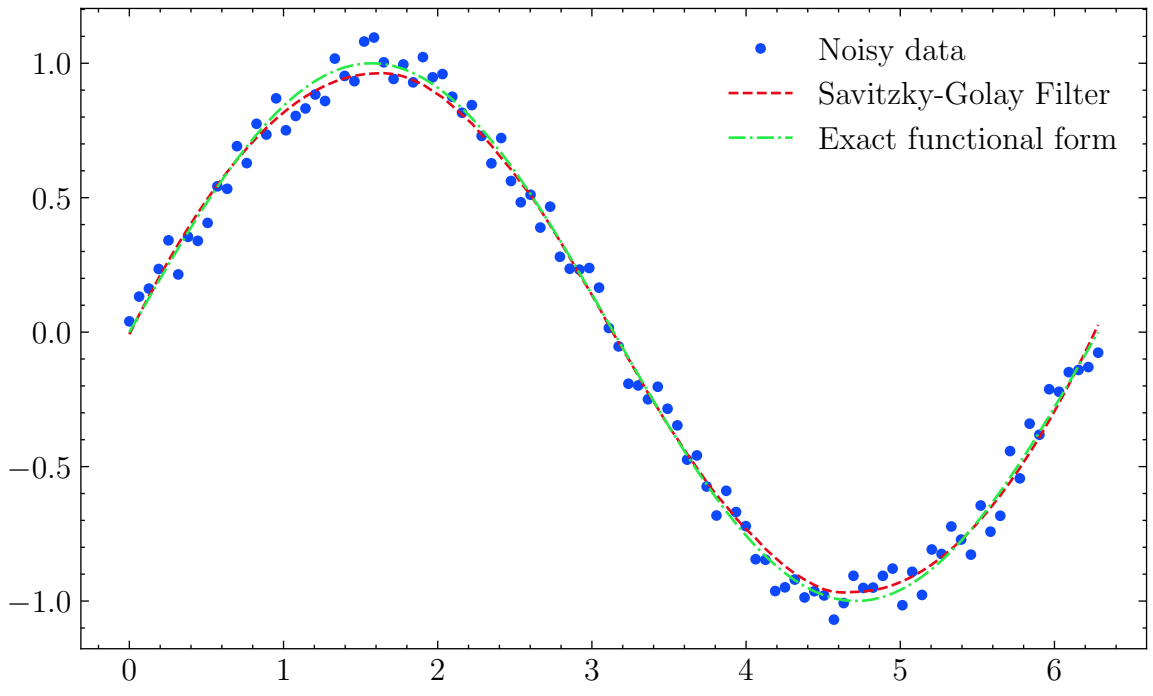

Figure 4.1: Savitzky-Golay filter example

4.5 Additional Comments

As previously discussed, statistical inference 'is concerned with making decisions about a population based on the information contained in a *random sample* from that population' [20]. Following [20], the definition of a random sample is:

Definition 4.1 (Random Sample)
The random variables $X_1, X_2, \ldots, X_n$ are a random sample of size n if (a) the X_i's are independent random variables and (b) every X_i has the sample probability distribution.

Now, consider a random sample $X_1, X_2, \ldots, X_n$ of size n. The *sample mean* $\overline{X}$, the *sample variance* S^2, and the *sample standard deviation* S,

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \tag{4.17}$$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2 \tag{4.18}$$

$$S = \sqrt{S^2} \tag{4.19}$$

are random variables. Therefore, it has a probability distribution [20]. Such distribution the is called *sampling distribution*, and in the case of the statistic $\overline{X}$ it is the *sampling distribution of the mean* [20]. The interest in such distribution is that, given n large enough, a suitable normal curve will approximate the actual distribution of $\overline{X}$ [19]. The formal statement of the theorem is as follows, as per [19]:

Definition 4.2 (Central Limit Theorem (CLT))
Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with mean μ and variance σ^2. Then if n is sufficiently large, $\overline{X}$ has approximately a normal distribution with $\mu_{\overline{X}} = \mu$ and $\sigma^2_{\overline{X}} = \sigma^2/n$. The larger the value of n, the better the approximation.

This result is one of the most important and useful theorem in probability and statistics [19], [20] and is the reason why many random variables in engineering and science are

normally distributed [20]. When it comes to knowing what is a sample size large enough to apply the CLT, it depends on the underlying distribution of the data. According to [19], [20], a general guideline is $n > 30$.

Chapter 5

Proposed Model and Implementation

5.1 General Concepts

Let repairable system be characterised by a binary state variable, $Y(t)$, which indicates the state or condition of such item, as a function of the age, t. This is the two-state characterisation, discussed in Section 2.3.1 and depicted in Figure 2.1. Therefore, at time t the system may be in one of two states: up or down state. The transition between the states are governed by appropriate models. The proposed model assumes that, when an item is in an up state, an imperfect repair process is the underlying degradation model which eventually takes the system to a down state. When in a down state, the system gradually returns to a functioning state. Such gradual return is governed by an appropriate probability distribution.

The imperfect repair process of choice is the Weibull-Power Law Process (WPLP), discussed in Section 3.3, since it is a generalisation of very common reliability models. The repair probability distribution, $G(t)$, responsible for the return to the up state is one of 3 possible choices, namely the exponential, normal and lognormal distributions, as per Section 3.4. A diagram representing the proposed model is presented in Figure 5.1, where WPLP$(\widehat{\alpha}, \widehat{a}, \widehat{b})$ is the WPLP model with parameter estimates $\widehat{\alpha}$, $\widehat{a}$ and $\widehat{b}$, and $\rho(t \mid \widehat{\theta})$ is the repair rate function with estimated parameters $\widehat{\theta}$, as per Equation 3.41. The

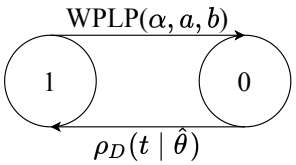

Figure 5.1: Proposed model diagram.

parameter estimates are all based on the Maximum-Likelihood (ML) Principle and the data is assumed to be collected under the failure truncation censoring scheme, while the selection method is using the Akaike information criterion (AIC). The proposed model shall be known as the Failure-Repair Process (FRP).

Note that this model is than composed of two "sub-models", a mathematical model and a simulation model. Thus, the performance measures of the FRP may be of two types: analytical and non-analytical measures. The analytical measures are the measures that are possible to derive purely from the mathematical model. While the the non-analytical measures are only possible through the use of a simulation model. A general way of referring to both types of measure is as Key Performance Indicators (KPI), as discussed in Section 2.3.

The implemented tool works based on the proposed FRP model. A flowchart representing the default algorithm along with different modules of the program is presented in Figure 5.2. The program is divided into 6 big phases: (1) input; (2) model fitting; (3) model selection; (4) simulation; (5) KPI's calculation; and (6) output. Each of those phases perform different tasks, and are mainly existent for better organisation and independence between the modules. The programming language that was used for the entirety of the project is Python 3 [43] with 3 main important packages: NumPy [44], SciPy [30] and statsmodels [45].

The flowchart in Figure 5.2 summarises the straightforward but complex flow of the implemented tool. On account of clarity, following the functions and assumptions in each module are condensed, keeping in mind that the following sections will explore the whole range of the model in depth. Also, Table 5.1 summarises the symbols used throughout

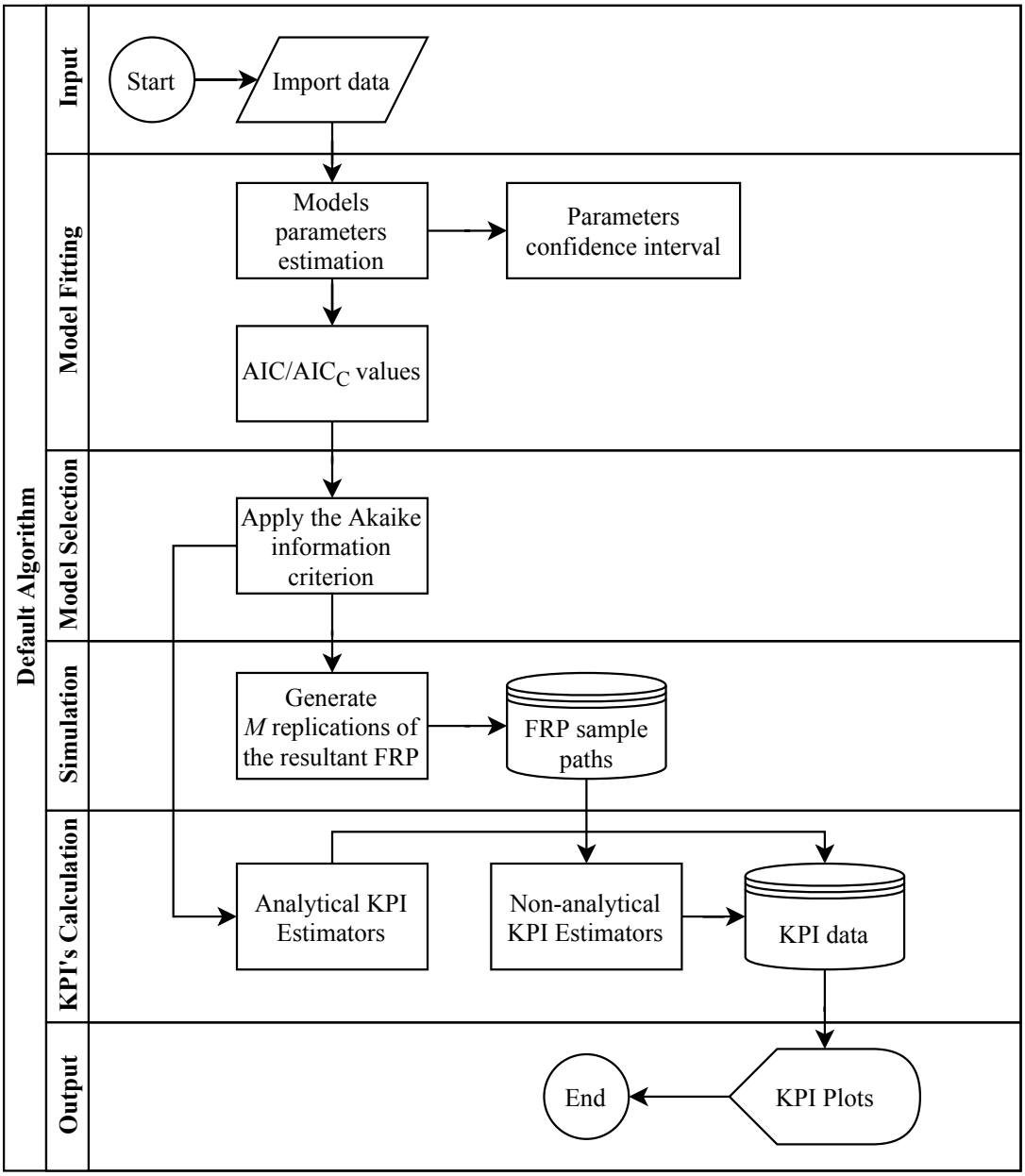

Figure 5.2: Default Algorithm Flowchart.

the chapter.

(1) **Input**. The tool is able to handle only CSV files, with a comma separator. The failure data must be chronologically ordered inter-failure times under failure truncation. The downtime data may be in any order, since the model is based on its order statistics.

(2) **Model Fitting**. The fitting process is based solely based on the ML Principle under the failure truncation censoring scheme. The model parameters Wald type confidence intervals are also calculated. And the resultant log-likelihood function values allows for the calculation of the AIC/AIC$_C$ values.

(3) **Model Selection**. The model selection process is based on the Akaike information criterion (AIC), since 'information-based criterions can be used to automate model selection' [27].

(4) **Simulation**. The tool generates M simulation runs, with the use of the inverse-cdf method for random-variate generation. Each replication is a terminating simulation with end-trigger events, E, such that each simulation runs, out of the M replications, stops when the simulation clock exceeds a maximum value, $T_E^{(G)}$, and the number of events of type j reaches a certain threshold, $N_{E,j}$, where $j = 0, 1$ for failure and downtime events, respectively. All of these sample paths are stored into an appropriate file for posterior treatment.

(5) **Performance Measures**. Using the resultant FRP model and the generated sample-paths, the key performance indicators of the system may be calculated, both analytical and non-analytical, in a time mesh ranging from $[0, T_E^{(G)}]$ divided into N_t points. In general, this means the calculation of the: (i) point availability, $A(t)$; (ii) mean availability in $[0, t]$, $\overline{A}(t)$; (iii) reliability function for the ith inter-failure time, $R(X_i > t)$; (iv) expected number of failures at time t, $E[N(t)]$; (v) rate of occurrence of failures of the system, $\rho(t)$; and (vi) next failure time predictors. The KPI's estimates are then stored into an appropriate file for posterior analysis.

(6) **Output**. The output simply puts together the KPI's into an appropriate graphical interface which is meant to display the data clearly for the user with the help of graphical plots.

5.2 Mathematical Model

Consider a repairable system starting at time $t = 0$ for which its failure times $T_1, T_2, \ldots$ are observed, where $T_0 \equiv 0$. The failure process is denoted as $\{T_i : i = 0, 1, 2, \ldots\}$. Define the time between failures as $X_i = T_i - T_{i-1}$, that is, the elapsed time between failures i and $i - 1$. Alternatively, the failure process may be characterised by the X_i's, denoted $\{X_i : i = 0, 1, 2, \ldots\}$. Lastly, let $N(t)$ be the number of failures up until time t, which results is the process $\{N(t) : t > 0\}$ with $N(t = 0) = 0$.

As before, define the mean function as the expected value of $N(t)$, $\mu(t) = E[N(t)]$. Assume its derivative exists and denote it $\rho(t)$. This is the time rate of change of the expected number of failures and is called the rate of occurrence of failures (ROCOF).

Also, assume that when a failure occurs, the repair begins immediately and the times to repair $D_1, D_2, \ldots$ follow a repair probability distribution $G(t) = \Pr\{D \leq t\}$, probability density function $g(t) = dG(t)/dt$ and repair rate function ρ_D as per Equation 3.41.

5.2.1 Failure Process

Remember the Weibull-Power Law Process and its special cases, presented in Section 3.3. Given those special cases the models are said to be nested. Those results are important as it shows that the WPLP is a generalised version of very common repairable system models, which makes it a good candidate for a general purpose modelling tool.

Therefore, the proposed model assumes that the failure process is governed by a Weibull-Power Law process, WPLP(α, a, b), as per definition 3.6, where α, a and b are the renewal, shape and time-trend parameters, respectively. Given observations $\{X_i : i = 1, 2, \ldots, N\}$ (failure truncation scheme), the parameters may be estimated

Symbol	Description
	Failure Model
T_i	The ith failure time in the failure time scale
X_i	The ith inter-failure time
N_F	Number of failure events
T_{N_F}	The last failure time
$N(t)$	Number of failures at time t
$E[N(t)]$	Expected number of failures at time t
$\rho(t)$	Rate of occurrence of failures (ROCOF)
$\lambda(t)$	Trend function
F	Renewal function
	Downtime Model
D_i	The ith downtime
$G_D(t)$	Repair probability distribution function
$g_D(t)$	Repair probability density function
$\rho_D(t)$	Repair rate function
	Simulation Model
$t^{(G)}$	Global time scale
$T_i^{(G)}$	The ith event time in the global time scale
$T_E^{(G)}$	Minimum simulation clock total time
$N_{E,j}$	Total number of events of type j where $j = 0, 1$ for failure and downtime events, respectively
N_t	Number of divisions in the time mesh
M	Number of replications
K	Number of random samples
N_S	Random sample size
$Y(t)$	Binary state characterisation variable
$R_i(t) = \Pr\{X_i > t\}$	Reliability function for the ith inter-failure time
$A(t)$	Point availability at time t
$\overline{A}(t)$	Mean availability in the time interval $[0, t]$

Table 5.1: Summary of the relevant variables and parameters.

using the maximum-likelihood method. The general log-likelihood equation is given in Equation 3.58, under failure truncation the equation simplifies to

$$\tilde{\ell} = n(\ln\varphi + \ln b + \ln\alpha) + \sum_{i=1}^{n}[(b-1)\ln T_i + \\ + (\alpha-1)\ln(T_i^b - T_{i-1}^b) - \varphi[T_i^b - T_{i-1}^b]^\alpha] \quad (5.1)$$

Thus, the parameter estimates for the WPLP model are calculate by the direct maximisation of Equation 5.1.

5.2.2 Expected Number of Failures

From definition 3.5 and from the discussion in [14] the expected number of failures at time t for a general trend-renewal process can be defined. From the definition of a TRP$(F, \lambda(\cdot))$, $\Lambda(T_1)$, $\Lambda(T_2)$, ... is a renewal process on a time axis different than t, let it be $s > 0$. Let $N^*(s)$ be the counting process on that time scale. Therefore,

$$N(t) = N^*(\Lambda(t)) \quad (5.2)$$

and $N^*(s)$ is a renewal process. Let $\mu^*(s) = E[N^*(s)]$ be the renewal function for the renewal process $N^*(s)$ on the s-scale. Thus, from equation 5.2,

$$E[N(t)] = E[N^*(\Lambda(t))] = \mu^*(\Lambda(t)) \quad (5.3)$$

Therefore, in order to find the renewal function for the TRP, calculate $\mu^*(s)$ on the s-scale, using 3.15, and then use equation 5.3 to go back to the t-scale. Any suitable numerical method may be used, in this specific situation the RS-Method was used. The method was described in Section 3.2.3.

5.2.3 Rate of Occurrence of Failures

Since usually one is interested in the ROCOF, defined in equation 3.2, for the case of a trend-renewal process its definition may be applied to equation 5.3 to then get,

$$\begin{aligned} \frac{d}{dt}E[N(t)] &= \frac{d}{dt}\mu^*(\Lambda(t)) \\ &= \frac{d}{ds}\mu^*(s)\frac{d}{dt}\Lambda(t) \\ \rho(t) &= \lambda(t)\rho^*(\Lambda(t)) \end{aligned} \tag{5.4}$$

To calculate the necessary $\rho^*(\cdot)$ the standard finite difference method is used, as described in Section 3.2.3. For the other models, namely the HPP and NHPP, the ROCOF is analytically defined and was described in Section 3.2.1 and 3.2.2, respectively.

5.2.4 Failure Prediction

In [46], the authors explore general problems in the point and interval prediction in a trend-renewal process (TRP). In general, they consider 'possible ideas and methods for constructing the predicted next failure time and the prediction interval for the next failure time' [46]. Although they develop methods for a general TRP, for the purpose of this thesis the interest is in the predictor for the Weibull-Power Law Process (WPLP). The point predictor for the next time to failure in a general WPLP, T_{N_F+1}, given the maximum-likelihood estimates $\widehat{\alpha}$, $\widehat{a}$, $\widehat{b}$ in the failure truncation scheme, is [46]

$$\widehat{T}_{N_F+1} = \left(\frac{1}{\widehat{a}} + T_{N_F}^{\widehat{b}}\right)^{1/\widehat{b}} \tag{5.5}$$

Now, a interval predictor is also in order. Such predictor is not the confidence interval for the point predictor $\widehat{T}_{N_F+1}$ and should not be interpreted as such. Following [46], the estimated lower and upper bounds $\widehat{T_L}$ and $\widehat{T_U}$, respectively, of the prediction interval for

the next failure time T_{N_F+1} in the WPLP(α, a, b) are given by

$$\widehat{T}_L = \left[T_{N_F}^{\widehat{b}} + \left(\frac{1}{\widehat{\varphi}} \ln \frac{1}{1-\varepsilon_1} \right)^{1/\widehat{\alpha}} \right]^{1/\widehat{b}} \tag{5.6}$$

$$\widehat{T}_U = \left[T_{N_F}^{\widehat{b}} + \left(\frac{1}{\widehat{\varphi}} \ln \frac{1}{\varepsilon_2} \right)^{1/\widehat{\alpha}} \right]^{1/\widehat{b}} \tag{5.7}$$

where $\widehat{\varphi} = [a\Gamma(1+1/\widehat{\alpha})]^{\widehat{\alpha}}$, as per Equation 3.37, and ε_1 and ε_2 determine the percentage of the confidence interval. For a 95% confidence interval, $\varepsilon_1 = \varepsilon_2 = 0.025$.

5.2.5 Repair Process

As already mentioned, three of the most common model for downtimes are the exponential, normal and log-normal distribution presented in Section 3.4. The estimation process for the model parameters is done by the maximum-likelihood method as described in Section 3.5.

5.3 Model Selection

Consider a complete data set of inter-failure and repair times, that is $\{X_i : i = 1, 2, \ldots, N_F;\}$ and $\{D_j : j = 1, 2, \ldots, N_D\}$, respectively. For the failure process, consider the model selection square for the general WPLP(α, a, b) [14], shown in Figure 5.3. In the case of the downtime model, consider the model selection triangle, in Figure 5.4. Both model selection diagrams represent all the possible fitted models for the data set, and illustrate the parameters of each along with the value of the estimated log-likelihood function, $\ell(\cdot)$.

From the estimated values of the log-likelihood functions along with the knowledge of the amount of estimated parameters involved, the Akaike information criterion (AIC), explored in Section 3.6, may be applied to select the best fitted model among the model set.

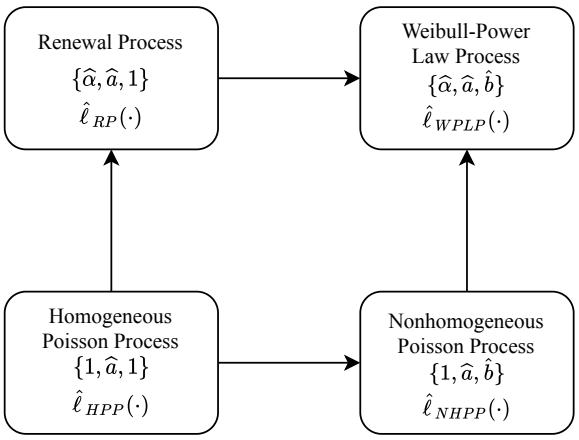

Figure 5.3: Failure Models

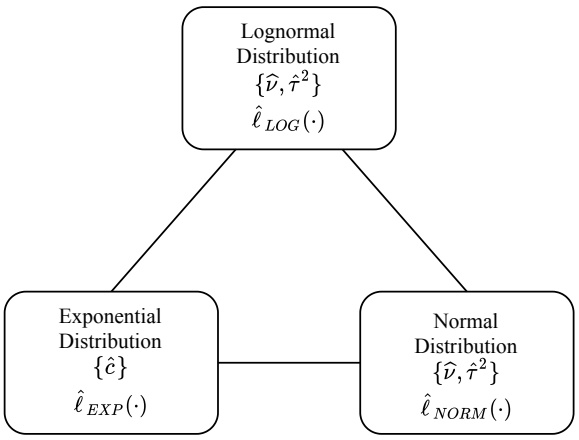

Figure 5.4: Downtime Models

5.4 Simulation Model

Let $\{T_{i,m}^{(G)} : i = 1, 2, \ldots, N_E; m = 1, 2, \ldots, M\}$ represent the output of a simulation run in the time interval $[0, T_E]$, such that $T_{i,m}^{(G)}$ is the ith event time in the mth replication of the process. Thus, each replication is a terminating simulating with stopping criterion such that the minimum simulation clock equals T_E and the minimum number of events is N_E. However, as making predictions about the next inter-failure times X_{N_F+1} is the real interest, the minimum number of failure events is such that $N_{E,1} = N_F + 1$, and $N_E = (N_F + 1) + N_{E,0}$, where $N_{E,j}$ is the number of events of type j, with $j = 0, 1$ representing failure and downtime events, respectively, given the two-state characterisation. Therefore every even i is a failure event, while the odds are end-of-repair events. Across replications, all systems start in a up-state, such that $Y_m(t = 0) = 1 \; \forall \; m$.

Using simulation, deriving performance measures of the system which are not available by analytical means is the goal. The common framework is to design the simulation model with the Central Limit Theorem (CLT), presented in Definition 4.2, in mind.

Consider the calculation of a general performance measure, $f(t \mid \text{data})$, at some point t in time given the available data. A general scheme is to fix the time t, at some interest value t^*, and calculate it across M replications, such that $\{f^{(m)}(t^*) : m = 1, 2, \ldots, M\}$. Now, split the simulated data into K smaller samples of size $N_S = M/K$ and calculate the

mean for each sample. According to the CLT, this value resulting sample should converge to a normal distribution, with a mean equal to the average of the resulting sample that will converge to the expected value of the original sample of $f^{(m)}(t^*)$'s, regardless of its underlying distribution. The same holds true for the sample standard deviation when divided it by the factor $\sqrt{K}$.

5.4.1 Reliability

For a general stochastic point process, the inter-arrival times are neither independent nor identically distributed. Therefore, finding a well defined formula for the reliability of a system is a difficult task. In the case of the trend-renewal process, combining the definitions of reliability and the TRP, given in Definitions 2.15 and 3.5, respectively, one may write the reliability function as

$$\Pr\{\Lambda(T_i) - \Lambda(T_{i-1}) > t\} = R(t) = 1 - F(t) \tag{5.8}$$

If the cumulative trend function, $\Lambda(\cdot)$, is linear, an analytical solution for the equation may be found. In general, however, this is not the case, and defining the reliability function becomes complicated. To come around this issue, an estimator for the reliability of the system is used, based on simulation. A classical non-parametric approach to estimating the reliability function is using the Kaplan-Meier estimator.

Remember that, according to [35], for terminating simulations, it is assumed that the random variables across replications are comparable. That is, the $X_i^{(m)}$, for $m = 1, 2, \ldots, M$, are i.i.d. random variables. Therefore, one may use the Kaplan-Meier estimator to give an idea of the behaviour of the reliability function for each of the X_i in the original process. The Kaplan-Meier estimator was discussed in Section 3.2.3.

In this specific case, the Kaplan-Meier estimator is used only to illustrate the reliability of a system modelled by a Weibull-Power Law Process (WPLP), since all the other three possible models, namely the HPP, the NHPP and the RP, have their reliability functions well defined. Table 5.2 summarises the definition of the different reliability functions for

each of the models. In the case of the HPP, NHPP and RP, their reliability function have been derived in Sections 3.2.1, 3.2.2, and 3.2.3, respectively.

Models	Reliability Function
Homogeneous Poisson Process	$R_{HPP}(t) = \exp\{-at\}$
Power Law Process	$R_{NHPP}(t) = \exp\{-at^b\}$
Weibull Renewal Process	$R_{RP}(t) = \exp\{-[a\Gamma(1+1/\alpha)]^\alpha t^\alpha\}$
Weibull-Power Law Process	Kaplan-Meier Estimator

Table 5.2: Summary of the reliability functions

Thus, with the simulated data resulting from a WPLP model with M replications and $N_F + 1$ failures and as $X_i^{(m)}$, for a fixed i and $m = 1, 2, \ldots, M$, are i.i.d. random variables the Kaplan-Meier estimators for each i inter-failure time may be calculated. Plotting several of them in the same figure, gives an idea of the reliability as a function of the failure index i.

5.4.2 Mean Availability

An important reliability measure is the *mean availability*, given in Definition 2.20. When under the two-state characterisation, the state variable $Y(t)$ may be used to calculate the mean availability in the time interval $[t, t + \Delta t]$. This is done by using the formula for the calculation of the time average of the sample path, as per Equation 4.14, combined with the general formula for piecewise output processes, given in Equation 4.15. Here, the general formula for the calculation of the average availability between time 0 and t for a single run of a simulation is derived.

First, consider the case where one is interested in the average availability in the time interval $[0, t_1]$, and t_1 is located such that $Y(t_1) = 1$, as illustrated by Figure 5.5.

For this first case the area of the process up until time t_1 may be written as

$$\begin{aligned}\text{Area}_1(0, t_1) = (T_1 - T_0)Y(T_0) + (T_2 - T_1)Y(T_1) + (T_3 - T_2)Y(T_2) + \\ + (T_4 - T_3)Y(T_3) + (t_1 - T_4)Y(T_4) \quad (5.9)\end{aligned}$$

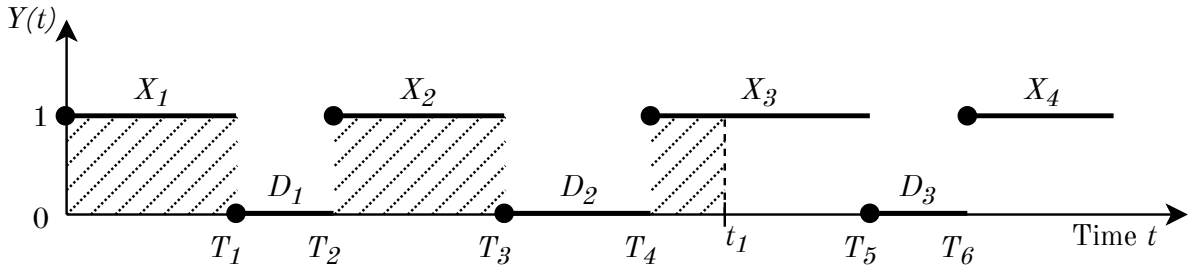

Figure 5.5: Mean Availability from $[0, t_1]$ with $Y(t_1) = 1$.

Dividing Equation 5.9 by the time interval of interest, the mean availability between 0 and t_1 is calculated:

$$\overline{A}(0, t_1) = \frac{1}{t_1 - 0}\text{Area}_1(0, t_1) = \frac{1}{t_1}\sum_{i=1}^{4}(T_i - T_{i-1})Y(T_{i-1}) + (t_1 - T_4)Y(T_4) \tag{5.10}$$

For the second case, let the time of interest be t_2 such that $Y(t_2) = 0$, as shown in Figure 5.6. Thus, the area is:

$$\text{Area}_2(0, t_2) = (T_1 - T_0)Y(T_0) + (T_2 - T_1)Y(T_1) + \\ + (T_3 - T_2)Y(T_2) + (t_2 - T_3)Y(T_3) \tag{5.11}$$

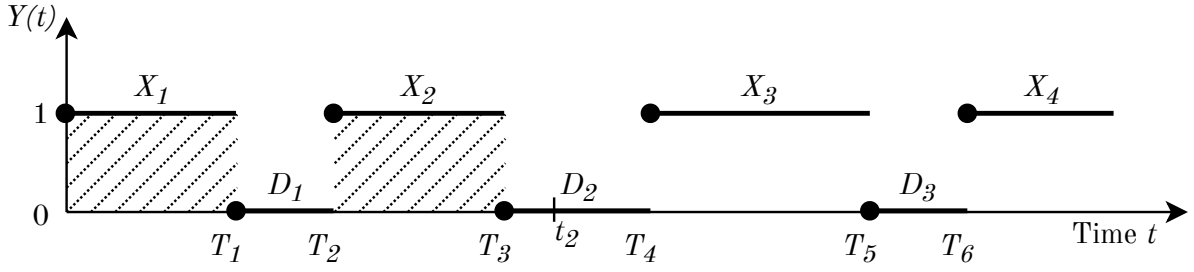

Figure 5.6: Mean Availability from $[0, t_2]$ with $Y(t_2) = 0$.

And the mean availability in the interval $[0, t_2]$ becomes:

$$\overline{A}(0, t_2) = \frac{1}{t_2 - 0}\text{Area}_2(0, t_2) = \frac{1}{t_2}\sum_{i=1}^{3}(T_i - T_{i-1})Y(T_{i-1}) + (t_2 - T_3)Y(T_3) \tag{5.12}$$

Comparing Equations 5.10 and 5.12, they look a lot like Equation 4.15, as one would expect. However, as the goal is a formula that works for any interval $[0, t]$, some care must be taken, since Equation 4.15 only works when in a time interval $[0, T_i]$, $i = 1, 2 \ldots$, where T_i is some event. The natural way of writing a general equation, combining what Equations 5.10 and 5.12, then is

$$\overline{A}(0,t) = \overline{A}(t) = \frac{1}{t}\sum_{i=1}^{r}(T_i - T_{i-1})Y(T_i) + (t - T_r)Y(T_r) \text{ where} \tag{5.13}$$
$$r = \text{argmin}_j\{T_j \leq t < T_{j-1} : j = 1, 2, \ldots, n\}$$

With Equation 5.13, the mean availability up to some time t for a single sample-path/simulation run ma be calculated. This is, of course, not useful given the stochastic nature of the process, which means that high variability in the mean availability between replications is to be expected. As before, making use the scheme of dividing the M replications into K random samples of size N_S. Calculate the mean for each of the K random samples with a fixed time t, such as:

$$\overline{A}_t^{(K)} = \frac{1}{N_S}\sum_{n=1}^{N_S}\overline{A}(t) \tag{5.14}$$

This results in the random variables $\overline{A}_t^{(1)}, \overline{A}_t^{(2)}, \ldots, \overline{A}_t^{(K)}$ which are, according to the CLT, normally distributed. Therefore, the expected value of the random variable will simply be the average across K, that is

$$\overline{A}_t^{(M)} = \frac{1}{K}\sum_{k=1}^{K}\overline{A}_t^{(k)} \tag{5.15}$$

With variance calculated using the sample variance formula as per Equation 4.18.

$$S^2 = \frac{1}{K-1}\sum_{k=1}^{K}(\overline{A}_t^{(k)} - \overline{A}_t^{(M)})^2 \tag{5.16}$$

Lastly, the standard deviation may be calculated by taking the square root of the

variance. Then, Equation 4.12 may be used to calculate an approximate $100(1-\varepsilon)$ percent $(0 < \varepsilon < 1)$ confidence interval for the mean at each time t, that is

$$\overline{A}_t^{(M)} \pm t_{K-1,1-\varepsilon/2}\sqrt{\frac{S^2}{K}} \tag{5.17}$$

and $t_{K-1,1-\varepsilon/2}$ is the upper $1-\varepsilon/2$ critical point for the t distribution with $K-1$ degrees of freedom [35].

5.4.3 Point Availability

Another measure of performance of the system is the *point availability*, formalised as per Definition 2.19. With this measure one is able to see how the availability of the system is changing with time, adding another indicator to characterise whether the system is stable, improving or degrading with time.

Using the idea that a probability may be regarded as a proportion, point availability may be seen as the proportion of systems that are functioning at time t across the M replications. Now, using the general framework for applying the CLT, one has M replications divided into K random samples of size N. Thus, the point availability estimator, for a fixed time t, may be written as

$$\widehat{A}(t) = \frac{1}{K}\sum_{k=1}^{K}\widehat{A}^{(k)}(t) = \frac{1}{K}\sum_{k=1}^{K}\frac{1}{N_S}\sum_{n=1}^{N_S}Y_n^{(k)}(t) = \frac{1}{KN_S}\sum_{k=1}^{K}\sum_{n=1}^{N_S}Y_n^{(k)}(t) \tag{5.18}$$

$$\therefore\ \widehat{A}(t) = \frac{1}{M}\sum_{k=1}^{K}\sum_{n=1}^{N_S}Y_n^{(k)}(t) \tag{5.19}$$

where $Y_n^{(k)}(t)$ is the value of the state variable at time t for the nth samples of the k random sample. As for the mean availability estimator, the variance will be given by,

$$S^2 = \frac{1}{K-1}\sum_{k=1}^{K}(\widehat{A}^{(k)}(t) - \widehat{A}_t)^2 \tag{5.20}$$

and the standard deviation is given by the square root of the variance. Then, an approximate $100(1-\varepsilon)$ percent $(0 < \varepsilon < 1)$ confidence interval for the mean at each time t may be written,

$$\widehat{A}(t) \pm t_{K-1,1-\varepsilon/2}\sqrt{\frac{S^2}{K}} \tag{5.21}$$

and $t_{K-1,1-\varepsilon/2}$ is the upper $1-\varepsilon/2$ critical point for the t distribution with $K-1$ degrees of freedom [35].

5.5 Numerical Model Validation

Consider the special case of the FRP, when the inter-failure times are independent and identically distributed exponential random variables, that is a WPLP$(1, a, 1)$, and the downtimes are exponentially distributed with mean c. For this case, the point availability of the system is given as [4]:

$$A(t) = \frac{c}{a+c} + \frac{a}{a+c}\exp\{-(a+c)t\} \tag{5.22}$$

and the mean availability in $[0, t]$ is

$$\begin{aligned}\overline{A}(0,t) = \overline{A}(t) = \frac{1}{t}\int_0^t A(u)du &= \frac{1}{t}\int_0^t \frac{c}{a+c} + \frac{a}{a+c}\exp\{-(a+c)u\} \\ &= \frac{1}{t}\left\{\frac{c}{a+c}t + \frac{a}{(a+c)^2}[1-\exp(-(a+c)t)]\right\}\end{aligned}$$

$$\therefore \overline{A}(t) = \frac{c}{a+c} + \frac{1}{t}\frac{a}{(a+c)^2}(1-\exp\{-(a+c)t\}) \tag{5.23}$$

Now, let both remaining parameters a and c be equal to 1, for the sake of simplicity. As before, let M be the total replications divided into K random samples of size N_S. The goal is to compare the approximations of the point and mean availability, given in Equations 4.11 and 5.19, respectively, against its analytical counterparts, presented above.

Additionally, it is necessary to verify the approximations accuracy with respect to the number of replications. This is done by varying the values of K, such that $K = 100$, 1000,

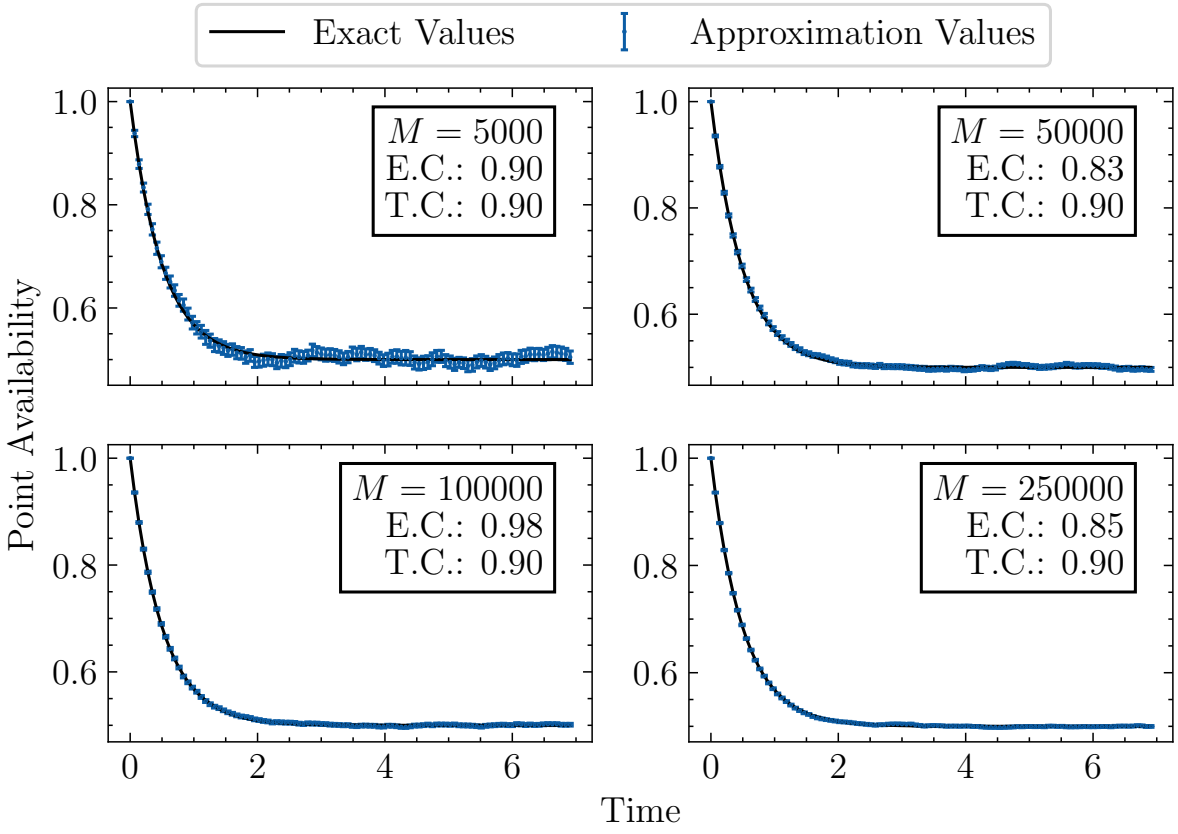

Figure 5.7: Point availability for different numbers of total replications

2000,5000 with a fixed $N_S = 50$. Thus, the total number of replications will be $M = 5000$, 50000, 100000, 250000. The total time of the simulations is given by the formula

$$t_{\text{sim}} = -\frac{1}{a+c}\ln\varepsilon \tag{5.24}$$

where ε is any small number, such as $\varepsilon = 1\text{e}{-6}$. For replicability, set the random number generator seed available in Numpy [44] to 0.

For comparison, the results are shown in Figures 5.7 and 5.8. In this context E.C. is the empirical coverage, which measures the proportion of the exact values of the point/mean availability that fall within the confidence intervals, and T.C. is the theoretical coverage, which is simply the confidence level, in this case being 90%.

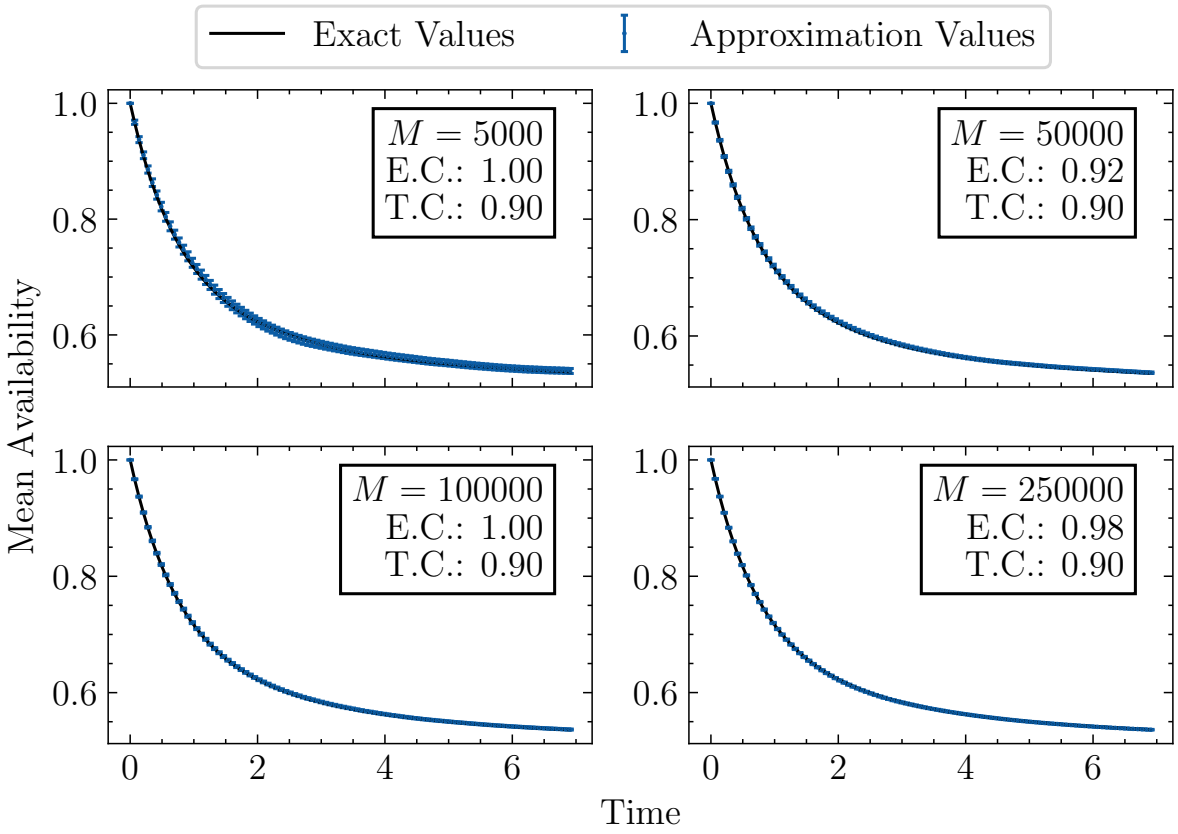

Figure 5.8: Mean availability for different numbers of total replications

Maximum Relative Error		
K	Point Availability	Mean Availability
100	0.032220	0.006970
1000	0.011843	0.003567
2000	0.006294	0.001451
5000	0.006602	0.000883

Table 5.3: Maximum relative error for the point and mean availability estimators

The point availability estimator seems to perform slightly worse that the mean availability one. This is likely to be cause by the binomial nature of the point availability estimator, while the mean availability is more normally characterised. However, both estimators appear to be reasonable. This is clear from Table 5.3, where the maximum relative errors for the point and mean availability estimators are 3.2% and 0.7% respectively.

Chapter 6

Application to real data

6.1 General Concepts

In this chapter some numerical results are presented illustrating the proposed model taking into account some real time between failures and downtimes data, namely the data set available in [47], given in Tables 6.1 and 6.2, respectively. These data contain 123 time between failure and 124 downtimes from an enrobing machine in a food industry located in the South region of Brazil [48].

For clarity, first it is explored, step-by-step, the process the implemented tool goes through. After each chunk of the model logic, some code snippets are presented to help visualise the background calculations that occur. Lastly, the implemented tool interface is shown, along with the software flow a general user would have to go through.

6.2 Modelling failure times

The algorithm has 4 different failure models to choose from, namely: the Weibull-Power Law TRP (WPLP); the Nonhomogeneous Poisson Process (NHPP) with power-law trend function; the Homogeneous Poisson Process (HPP); and the Weibull Renewal Process (WRP). These models are said to be nested, since the WPLP is a general model that encapsulates all the others. The general WPLP has 3 parameters α, a and b, named

126.0	100.5	60.85	144.85	119.2	165.5	201.0	81.5	245.0	142.0
187.0	56.0	122.5	11.0	122.2	228.5	731.0	123.0	104.0	209.3
165.25	247.7	146.8	251.8	82.0	62.3	103.85	206.65	166.0	288.5
81.5	5.0	136.5	125.0	141.0	311.75	102.5	82.85	7.5	36.5
12.5	208.5	104.5	61.0	354.75	396.75	671.25	166.0	61.5	205.5
333.5	565.25	124.0	144.5	964.5	291.25	56.74	209.2	375.5	293.5
40.5	138.5	754.75	58.0	82.0	78.5	6.5	100.0	482.0	239.0
40.2	8.5	226.75	313.0	480.0	59.0	40.0	121.0	414.0	206.5
123.0	103.5	271.0	102.5	290.25	334.0	333.25	61.0	310.5	102.5
543.25	8.75	374.0	483.0	206.0	187.25	418.0	499.25	58.25	248.15
351.0	356.25	145.5	1026.5	523.5	59.0	313.0	185.25	124.5	101.5
604.75	460.0	267.5							

Table 6.1: Chronologically ordered (left to right) inter-failure times [47]

4.5	3.33	2.93	3.9	3.62	4.52	4.23	4.2	4.29	4.06	4.33	4.26	4.07
3.92	3.83	3.84	3.79	3.63	3.48	3.44	3.48	3.63	3.48	3.42	3.31	3.23
3.23	3.19	3.27	3.24	3.23	3.41	3.35	3.43	3.42	3.4	3.34	3.31	3.31
3.3	3.26	3.22	3.16	3.13	3.17	3.15	3.1	3.08	3.05	3.08	3.06	3.04
3.02	3.01	2.98	2.98	3.04	3.0	2.99	2.95	2.92	3.01	2.99	3.02	3.0
3.04	3.01	3.02	2.99	3.14	3.12	3.11	3.07	3.05	3.05	3.06	3.05	3.08
3.11	3.12	3.12	3.11	3.17	3.15	3.14	3.14	3.12	3.14	3.12	3.13	3.12
3.11	3.09	3.09	3.1	3.08	3.08	3.07	3.08	3.07	3.06	3.06	3.05	3.04
3.05	3.03	3.03	3.02	3.03	3.04	3.07	3.05	3.03	3.03	3.02	3.03	3.02
3.02	3.01	3.01	3.02	3.02	3.04	3.03						

Table 6.2: Chronologically ordered (left to right) downtimes [47]

renewal, shape and time trend parameters, respectively.

For fitting the four models, the **statsmodels** [45] module is used, which 'provides classes and functions for the estimation of many different statistical models, as well as for conducting statistical tests, and statistical data exploration' [45]. The main interest is with the **GenericLikelihoodModel**, which allows the fitting of any likelihood function via maximum likelihood given that at least the loglikelihood function be specified. Consider one wants to fit a general WPLP under failure truncation. To do so, simply write the loglikelihood function of the process (equation 3.58) and then write a sub-class inheriting the methods from **GenericLikelihoodModel**. That way, all the previously implemented routines are accessible, like loglikelihood optimisation for parameter estimation, calculation of the covariance matrix, which allows for the construction of Wald-type confidence intervals and the AIC values for the model selection phase. In listing 6.1, the code for fitting the WPLP is shown.

```
1  import numpy as np
2  from statsmodels.base.model import GenericLikelihoodModel
3
4  def loglike_WPLP(T, b, alpha, phi):
5      n = len(T) - 1
6      ret = n * (np.log(phi) + np.log(b) + np.log(alpha)) + (b - 1) * np.
       log(T[1]) + (alpha - 1) * np.log( T[1] ** b) - phi * T[1] ** (b *
       alpha) + sum([(b - 1) * np.log(T[i]) + (alpha - 1) * np.log(T[i] ** b
        - T[i-1] ** b) - phi * (T[i] ** b - T[i-1] ** b) ** alpha for i in
       range(2, n+1)])
7      return ret
8
9  class WPLP_Model(GenericLikelihoodModel):
10     def __init__(self, endog, exog=None, **kwds):
11         if exog is None:
12             exog = np.zeros_like(endog)
13
14         super(WPLP_Model, self).__init__(endog, exog, **kwds)
15
```

```
16     def nloglikeobs(self, params):
17         b, alpha, phi = np.exp(params[0]), np.exp(params[1]), np.exp(
    params[2])
18         return -loglike_WPLP(self.endog, b=b, alpha=alpha, phi=phi)
19 
20     def fit(self, start_params=None, maxiter=10000, maxfun=5000, **kwds)
    :
21         if start_params is None:
22             start_params = np.array([-.5, -.5, -.5])
23         else:
24             start_params = start_params
25 
26         return super(WPLP_Model, self).fit(start_params=start_params,
    maxiter=maxiter, maxfun=maxfun, **kwds)
```

Listing 6.1: Code snippet for statistical inference under the WPLP

Writing all the classes inheriting from **`GenericLikelihoodModel`** for each of the models: WPLP, NHPP, HPP and WRP, and fitting them to the failure data in Table 6.1, results in Table 6.3. Looking at the WPLP, there is evidence of a renewal-type behaviour in the failure process, given that $\widehat{\alpha} \neq 1$, along with some time dependence, indicated by $\widehat{b} \neq 1$. However, one must be careful in drawing conclusions purely from the point estimates of the parameters. A closer look in the confidence intervals reveals a behaviour that could be explained by a NHPP, since the lower CI is 1.081, or a HPP, since the upper CI is 0.978. This is where model selection is important, to give a statistical reason for choosing a particular model.

	$\widehat{\alpha}$ [95% CI]	$\widehat{a}$ [95% CI]	$\widehat{b}$ [95% CI]
WPLP	1.240 [1.081, 1.423]	0.021 [0.002, 0.118]	0.848 [0.735, 0.978]
NHPP	1.000 [1.000, 1.000]	0.019 [0.004, 0.091]	0.859 [0.719, 1.026]
HPP	1.000 [1.000, 1.000]	0.005 [0.004, 0.005]	1.000 [1.000, 1.000]
WRP	1.201 [1.048, 1.377]	0.005 [0.001, 0.019]	1.000 [1.000, 1.000]

Table 6.3: Fitted failure models parameters

In Table 6.4, the AIC values are calculated for each model along with the other AIC

statistics, Δ_i and w_i. The results points towards the WPLP as the best model with a 91.1% probability. There is also some evidence that the WRP could be a potential model, by the rule of thumb $4 \leq \Delta_i \leq 7$. However, plotting the times to failure of the system, against the observed number of failures, that is (T_i, i), there is a slight curvature in the data, which points towards a time trend of some sort, as seen in Figure 6.1. Therefore, the final model by the Akaike information criterion is the Weibull-Power Law process, WPLP$(1.240, 0.021, 0.848)$.

	WPLP	NHPP	HPP	WRP
AIC	1549.39	1557.78	1560.74	1554.46
Δ_i	0	8.396	11.354	5.078
w_i	0.911	0.014	0.003	0.072

Table 6.4: Fitted failure models selection

As means of a final visual inspection, in Figure 6.1, the comparison between the fitted model and the failure data are shown. The dashed line represents the expected number of failures under the WPLP, calculated as explored in Section 5.2.1, with a number of subdivisions in the time mesh of the RS-Method equal to $n = 2000$, and total solution time equal to 10.9 seconds. The fitting seems pretty reasonable, with a clear time trend behaviour of the data.

The time derivative of the expected number of failure is the ROCOF, by definition. This is a very important performance measure, and is plotted in Figure 6.2. The system is clearly in a decreasing ROCOF region, which means that the system is improving with time. Now, the resulting WPLP have a Weibull renewal distribution with shape parameter 1.240 which implies a increasing failure rate. This means that the conditional intensity function will jump downwards at each failure, which may be an indicative of efficient maintenance, since the interventions will happen at high values of renewal failure rate, on average. Further, the general aging of the system, governed by an estimated time trend of the form $\widehat{\lambda}(t) = 0.021 \cdot 0.848 \cdot t^{0.848-1} = 0.018 \cdot t^{-.152}$ which decreases with t, introduces a decreased failure frequency.

The final metrics extracted from the failure model are the next time to failure. There

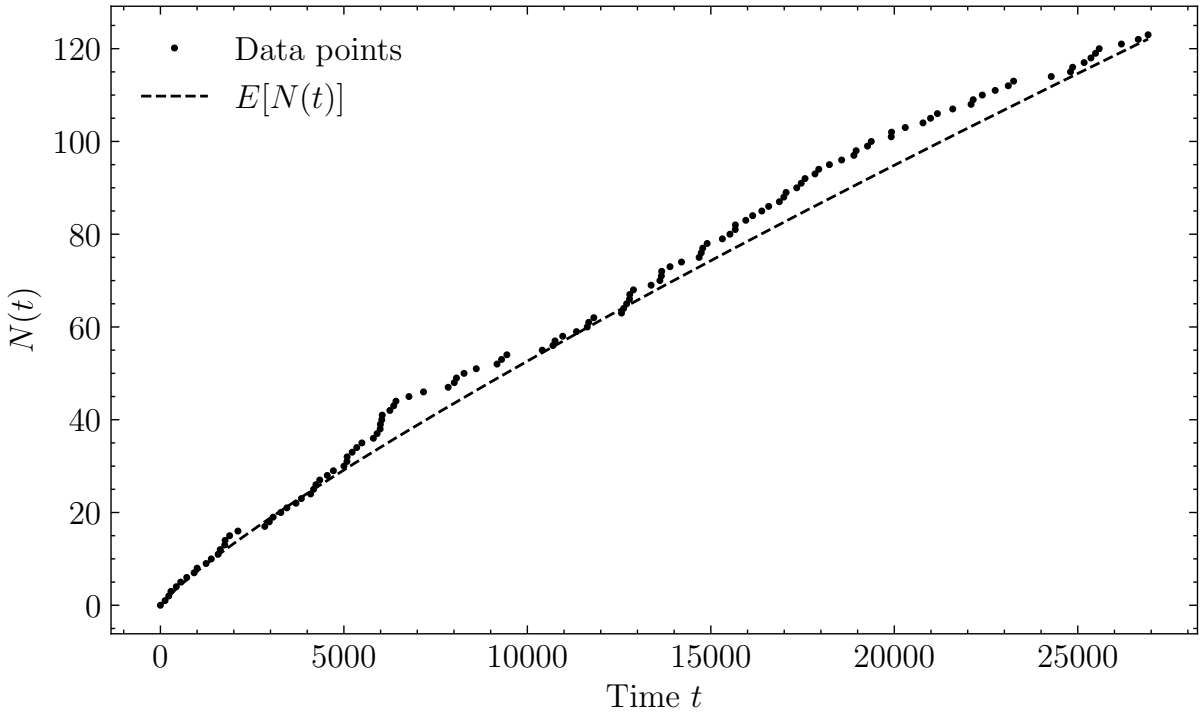

Figure 6.1: Expected number of failures for the WPLP$(1.240, 0.021, 0.848)$

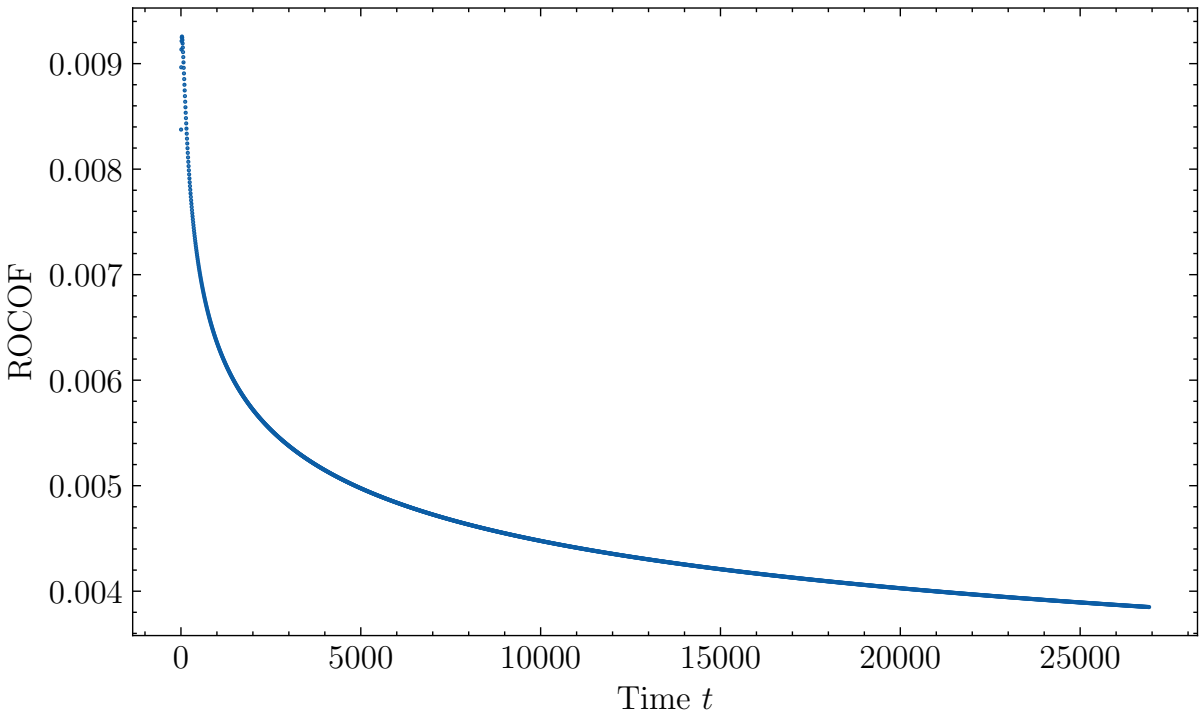

Figure 6.2: Rate of Occurrence of Failures for the WPLP$(1.240, 0.021, 0.848)$

are two estimators, analytically derived: a point predictor and a interval predictor, given in Equations 5.5 and 5.6, respectively. For the fitted WPLP$(1.240, 0.021, 0.848)$ model:

$$\widehat{T}_{N_F+1} = 27176.70 \text{ hours}$$
$$\widehat{X}_{N_F+1} = 260.21 \text{ hours}$$

and, with 90% confidence:

$$[\widehat{T}_L, \widehat{T}_U] = [26941.89, 27592.81] \text{ hours}$$
$$[\widehat{X}_L, \widehat{X}_U] = [25.40, 676.32] \text{ hours}$$

The implementation is very straightforward and is presented in Listing 6.2.

```
import math
import numpy as np

def point_predictor(last_failure, failure_model):
    alpha, a, b = failure_model[1]
    ret = (1 / a + last_failure ** b) ** (1/b)
    ret2 = ret - last_failure
    return ret, ret2

def interval_predictor(last_failure, failure_model, epsilon_1=.05,
   epsilon_2=.05):
    alpha, a, b = failure_model[1]
    phi = (a * math.gamma(1+1/alpha)) ** alpha
    T_L = (last_failure ** b + (1/phi * np.log(1 / (1-epsilon_1))) **
   (1/alpha)) ** (1/b)
    T_U = (last_failure ** b + (1/phi * np.log(1 / (  epsilon_2))) **
   (1/alpha)) ** (1/b)
    return [T_L, T_U], [T_L-last_failure, T_U-last_failure]
```

Listing 6.2: Code snippet for the prediction module

6.3 Modelling of downtimes

The implemented tool works with 3 different possible downtime models, namely the: 2-parameter exponential distribution; the normal distribution; and the 3-parameter lognormal distribution. Those are commonly used models under the assumption of independent and identically distributed data. For statistical inference, the **SciPy** module [30] is used and, specifically, its **scipy.stats** sub-module, which contains a large number of probability distributions and statistical functions. Say a variable **data** stores some sample observations. Given the **scipy.stats** module, a general **Python** function may be written to get both the parameter estimates for each model and the AIC values, as displayed in Listing 6.3.

```
1  import scipy.stats as stats
2
3  def downtime_models(data):
4      params_expon = stats.expon.fit(data)
5      params_lognorm = stats.lognorm.fit(data)
6      params_norm = stats.norm.fit(data)
7      expon_aic = 2*len(params_expon) - 2*(stats.expon.logpdf(data,*
       params_expon).sum())
8      lognorm_aic = 2*len(params_lognorm) - 2*(stats.lognorm.logpdf(data,*
       params_lognorm).sum())
9      norm_aic = 2*len(params_norm) - 2*(stats.norm.logpdf(data,*
       params_norm).sum())
10     model = list()
11     model.append(["expon", params_expon, expon_aic])
12     model.append(["lognorm", params_lognorm, lognorm_aic])
13     model.append(["norm", params_norm, norm_aic])
14     return model
```

Listing 6.3: Code snippet for fitting downtime models

In the present case, **data** stores the downtime data in Table 6.2. The estimated parameters for each model are given in Table 6.5. The data is clearly skewed to right, since the location parameters are positive across all models, justifying the necessity of the

location parameter in a general purpose tool. The same conclusion may be drawn from the histogram plot of the data, which is imply a frequency plot, presented in Figure 6.3.

	Exponential	Lognormal	Normal
Shape	–	0.851	–
Location	2.920	2.905	3.247
Scale	0.327	0.232	0.354

Table 6.5: Fitted downtime models parameters

As before, the resulting AIC values and its related statistics for model selection are used, as displayed in Tabel 6.6. In this case, there is no doubt that the lognormal model is the best one, with probability tending to 1. Thus, the final downtime model is the 3-parameter lognormal.

	Exponential	Lognormal	Normal
AIC	-25.262	-44.894	98.430
Δ_i	19.632	0	143.324
w_i	$5e-05$	0.999	$8e-32$

Table 6.6: Fitted downtime models selection

A comparison between the fitted model and the observed downtime data is shown in Figure 6.3. The lognormal model seems to capture the skewed behaviour of the data, as well as its general frequency distribution shape.

6.4 Simulation model

Consider the proposed Failure-Repair Process (FRP), which is a simple next event simulation model for a single repairable system with a single failure mode. The sample size N_S is fixed in 50, while the random samples K vary as $100, 500, 1000, 2000$. By doing so, it is expected to verify how the non-analytical performance measures vary as a function of the total replications $M = N_S K$.

In order to simulate the resultant FRP, both the final failure and downtime models parameters are needed. With those the random variates may be generated, which allows

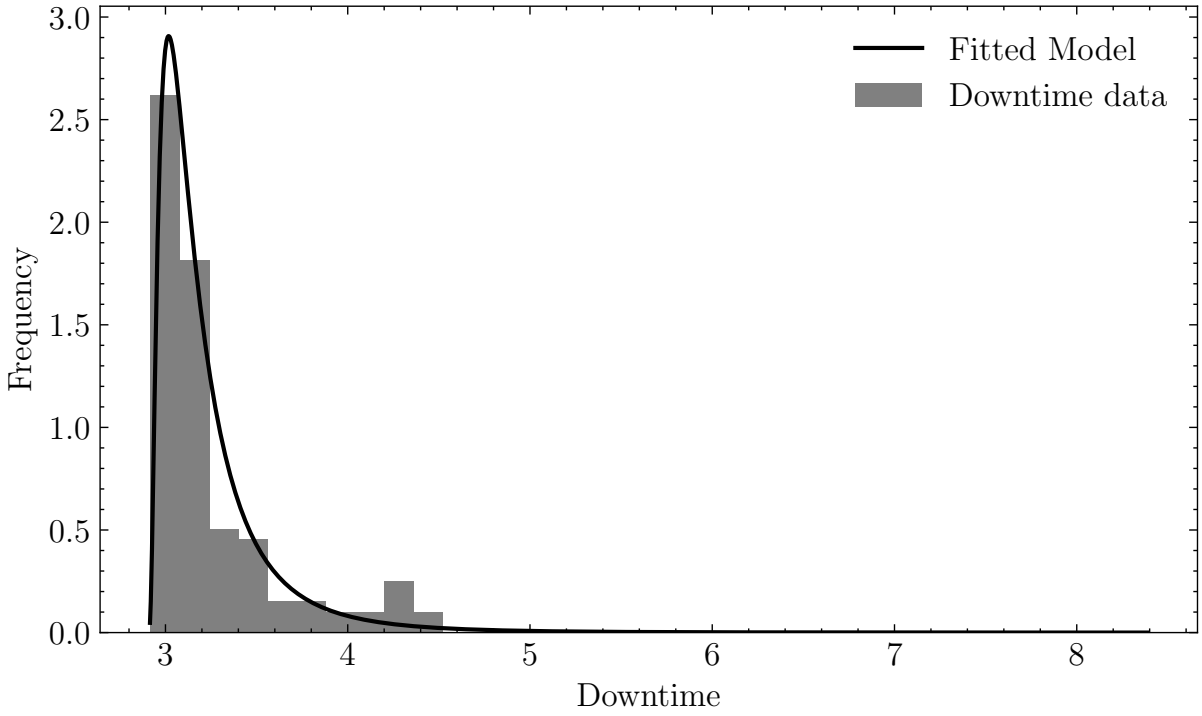

Figure 6.3: Distribution of the $D \sim \text{lognorm}(0.851, 2.905, 0.232)$

for the creation of different simulation runs for the system. Generation of the failure times is done by using the formula presented in Equation 4.2, while the downtimes are generated using the **numpy** and **scipy** modules. The code for the model simulation is presented in Listing 6.4.

```
import scipy.stats as stats
import numpy as np
import math

def next_ttf(previous_ttf, failure_model):
    alpha, a, b = final_failure_model[1]
    U = np.random.uniform()
    tmpA = previous_ttf ** b
    tmpB = 1 / (a * math.gamma(1+1/alpha))
    tmpC = np.log(1/(1-U)) ** (1/alpha)
    tmpD = (tmpA + tmpB * tmpC) ** (1/b)
    return tmpD
```

```
13
14
15 def next_ttr(downtime_model, CRN=None):
16     dist = final_downtime_model[0]
17     params = final_downtime_model[1]
18     if dist == "expon":
19         return stats.expon.rvs(*params, random_state=CRN)
20     elif dist == "lognorm":
21         return stats.lognorm.rvs(*params, random_state=CRN)
22     elif dist == "norm":
23         return stats.norm.rvs(*params, random_state=CRN)
24
25 def model_sim(failure_model, downtime_model, time_horizon, n_f):
26     t, d, t_G, s, x = list(), list(), list(), list(), list()
27     t.append(0)
28     d.append(0)
29     s.append(1)
30     t_G.append(0)
31     x.append(0)
32     i = 1
33     while (max(t_G) < time_horizon) or (len(t) <= n_f+1):
34         if s[-1] == 1:
35             t.append(next_ttf(t[-1], failure_model))
36             tmp_x = t[-1] - t[-2]
37             x.append(tmp_x)
38             t_G.append(tmp_x + t_G[-1])
39             s.append(0)
40         elif s[-1] == 0:
41             d.append(next_ttr(downtime_model))
42             t_G.append(d[-1] + t_G[-1])
43             s.append(1)
44         i = i + 1
45     return t_G, s, x, d, t
```

Listing 6.4: Code snippet for model simulation

Now, some other parameters are needed in order to run the simulation, namely: minimum number of failures, **n_f**, and maximum simulation time, **time_horizon**. Those are taken from the original sample path, which is the converted failure and downtime data, taking place in a global time scale. Such process is done by the code presented in Listing 6.5.

```
def failure_repair_process(X=list(), D=list()):
    """
    Description: This function converts the time-between-failures (TBF)
   and the repair times to a single global time scale.
    Input  | X: list containing the TBF of the system.
           | D: list containing the downtime times between failures of
   the system.
    Output | T: failure and repair times in global time
           | S: state of the system at each event epoch
    """
    S, T = list(), list()
    N = max([len(X), len(D)])
    if X[0] == 0:
        X = X[1:]
    else:
        pass
    n = 0
    T.append(0)
    S.append(1)
    while n < N:
        for i in range(0, 2):
            if S[-1] == 1:
                S.append(S[-1] - 1)
                try:
                    T.append(T[-1] + X[n])
                except IndexError:
                    pass
            else:
                S.append(S[-1] + 1)
```

```
                try:
                    T.append(T[-1] + D[n])
                except IndexError:
                    pass
        n = n + 1
    return T, S
```

Listing 6.5: Code snippet for converting the original data to a single time scale

With the original sample path is very simple to simulated the M simulation runs of interest. In Listing 6.6, the code for that is presented.

```
import numpy as np

complete_sample = np.array([model_sim(final_failure_model,
final_downtime_model, t_sim, n_f) for i in range(0, M)])
random_sample   = np.split(complete_sample, K)
```

Listing 6.6: Code snippet for random samples simulation of the FRP

With the FRP sample paths, the first metric shown is the proposed approximate reliability using the Kaplan-Meier estimate. The main assumption for the calculation of this metric is that each time between failures (TBF), X_i, in the system is independent and identically distributed across replications. The Kaplan-Meier estimates are easily calculated with the help of the **lifelines** module [49]. Listing 6.7 display the code for the calculation of the KM estimates for the nth TBF, based on the TBF data. The **alpha** parameter dictates the confidence interval for the estimate.

```
from lifelines import KaplanMeierFitter

def reliability(n, TBF, alpha=.1):
    kmf = KaplanMeierFitter()
    if str(n)[-1] == '1':
        label = f'{n}st TBF'
    elif str(n)[-1] == '2':
        label = f'{n}nd TBF'
    elif str(n)[-1] == '3':
```

```
        label = f'{n}rd TBF'
    elif n == len(TBF) - 1:
        label = r'$N_F+1$' + ' TBF'
    else:
        label = f'{n}th TBF'
    kmf.fit(TBF[n], label=label, alpha=alpha)
    return kmf
```

Listing 6.7: Code snippet for the Kaplan-Meier estimates of the X_n TBF

The resultant reliability plot for the data is shown in Figure 6.4. It agrees with past conclusions, as it shows that the system is improving with time, as the reliability had a big increase from X_1 to X_{42}. However, the reliability improvement is slowing down, as seen when comparing the next time between failure of system $N_F + 1$ and 83^{rd}, showing that the reliability functions does not differ greatly.

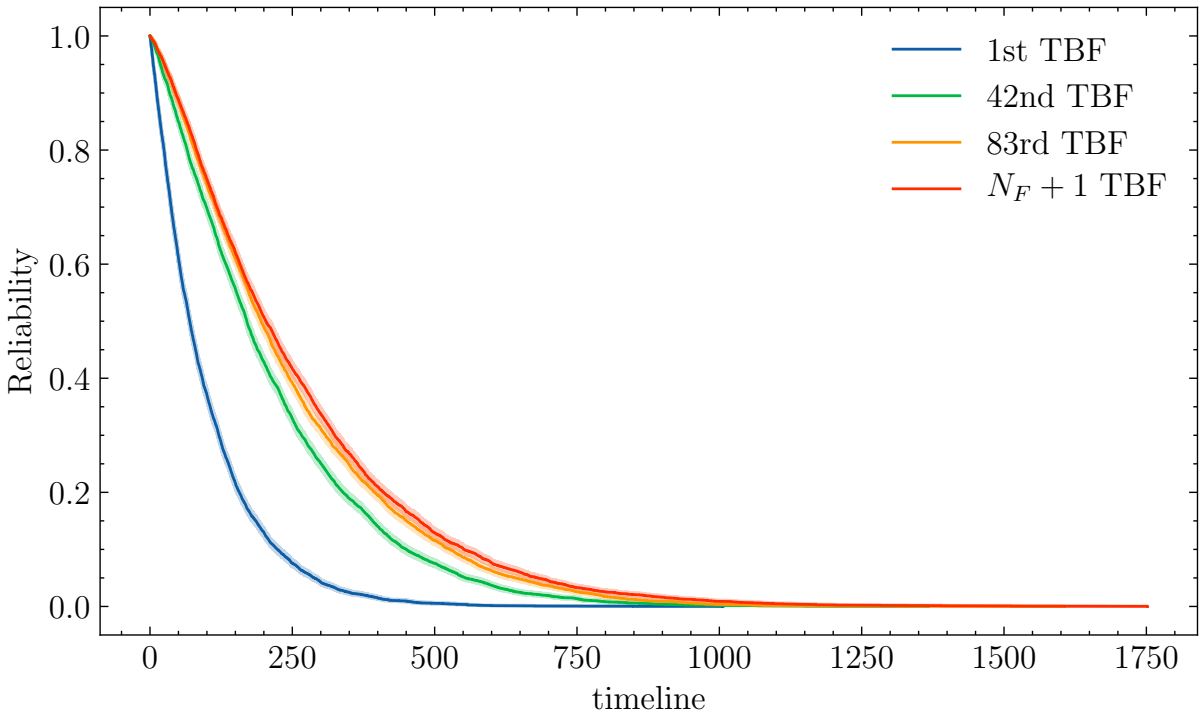

Figure 6.4: Kaplan-Meier Estimates for the X_n's of the WPLP$(1.240, 0.021, 0.848)$

A more specific measure of reliability of repairable system is the availability [4]. Both the point and mean availability estimators were described in Sections 5.4.3 and 5.4.2. For

the considered data of the enrobing machine, the plots of both estimators are presented in Figures 6.5 and 6.6. For visualisation, the Savitzky-Golay Filter is applied, which smooths the data and gives the user a sense of direction among the data noisiness. As their accuracy is dependent in the number of random samples considered, the previously discussed values for $K = 100, 500, 1000, 2000$ are considered, with fixed $N_S = 50$.

The point availability behaves somewhat poorly for $K = 100$, but seems to stabilise fairly soon at $K = 500$, where a clear idea of the behaviour of the availability of the system is present, with a minimum value of 0.976, and a last value of 0.987. By increasing the number of K to 1000 and 2000, there is no significant gain, but a lot of drawbacks when it comes to elapsed solution times, as shown in Table 6.7. For $K = 2000$, the minimum value continues 0.976 while the maximum one becomes 0.988, a negligible difference.

For the mean availability estimator, there is a very well behaved curve since $K = 100$, with a minimum value of 0.973 and maximum of 0.986. While for $K = 2000$, the minimum and maximum values continue the same. For completeness sake, one could argue that the Savitzky-Golay Filter does not serve any good here. However, as a guarantee measure that the data is interpretable in any circumstance, it is always plotted.

K	Time elapsed $[s]$
100	152.48
500	769.05
1000	1590.56
2000	3135.43

Table 6.7: Solution times for each value of K

All the previously discussed performance indicators are shown in Figure 6.7.

6.5 General-purpose tool

In this section, the flow of the tool is discussed, as experienced by a general user. The tool consists of 3 windows, namely: **`main`**, **`configurations`** and **`results`**. The **`main`**

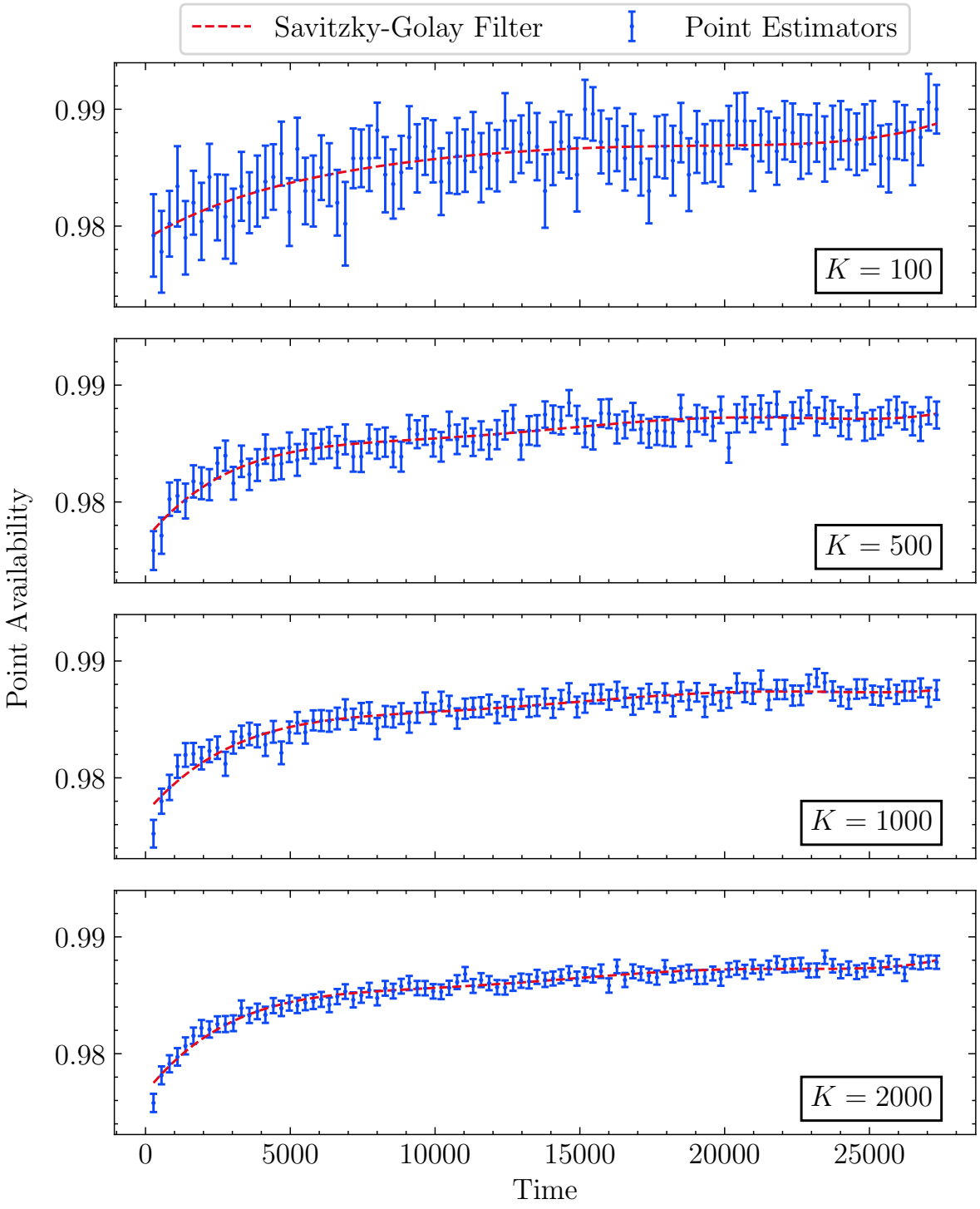

Figure 6.5: Point availability for the WPLP$(1.240, 0.021, 0.848)$ with varying random samples

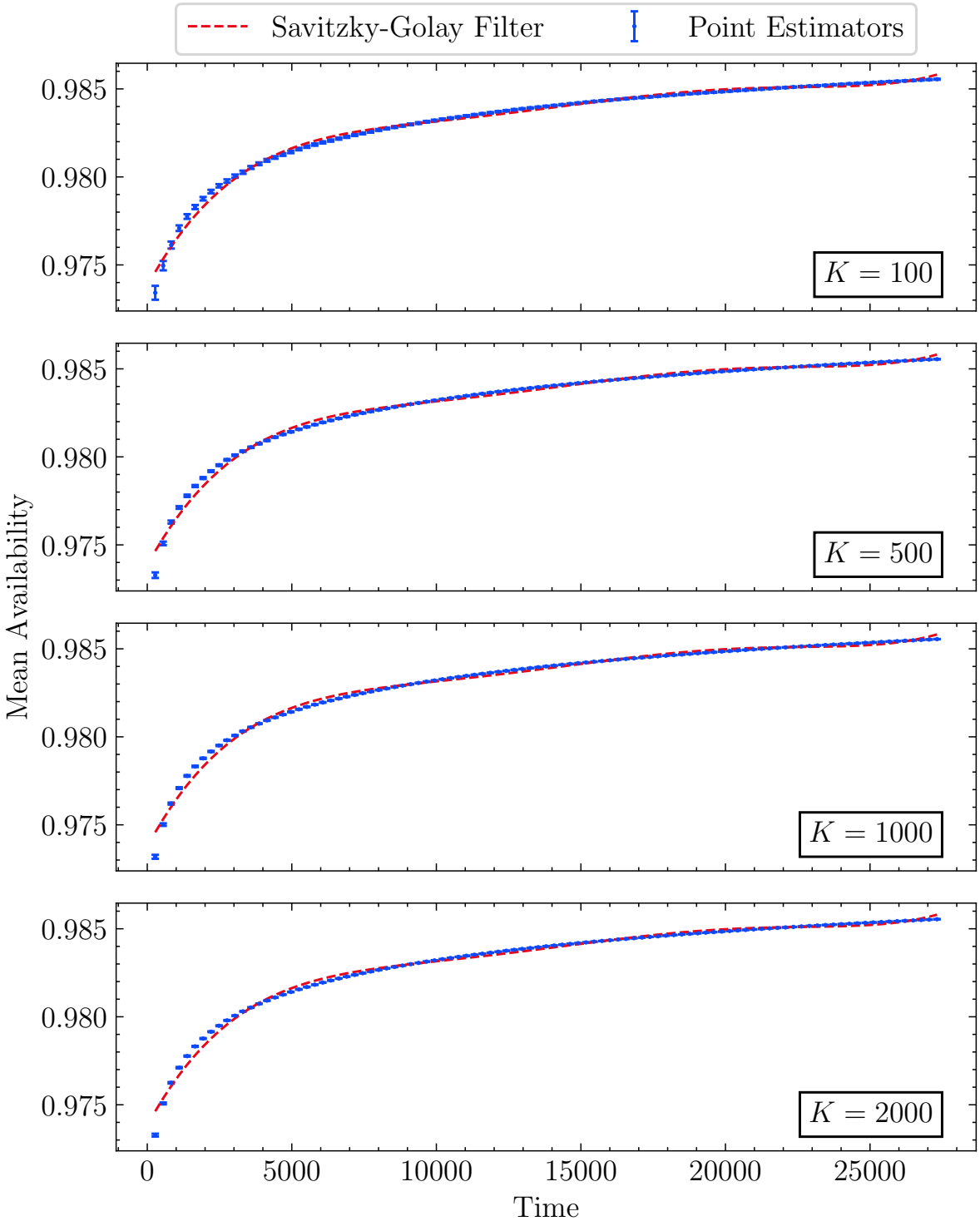

Figure 6.6: Mean availability for the WPLP$(1.240, 0.021, 0.848)$ with varying random samples

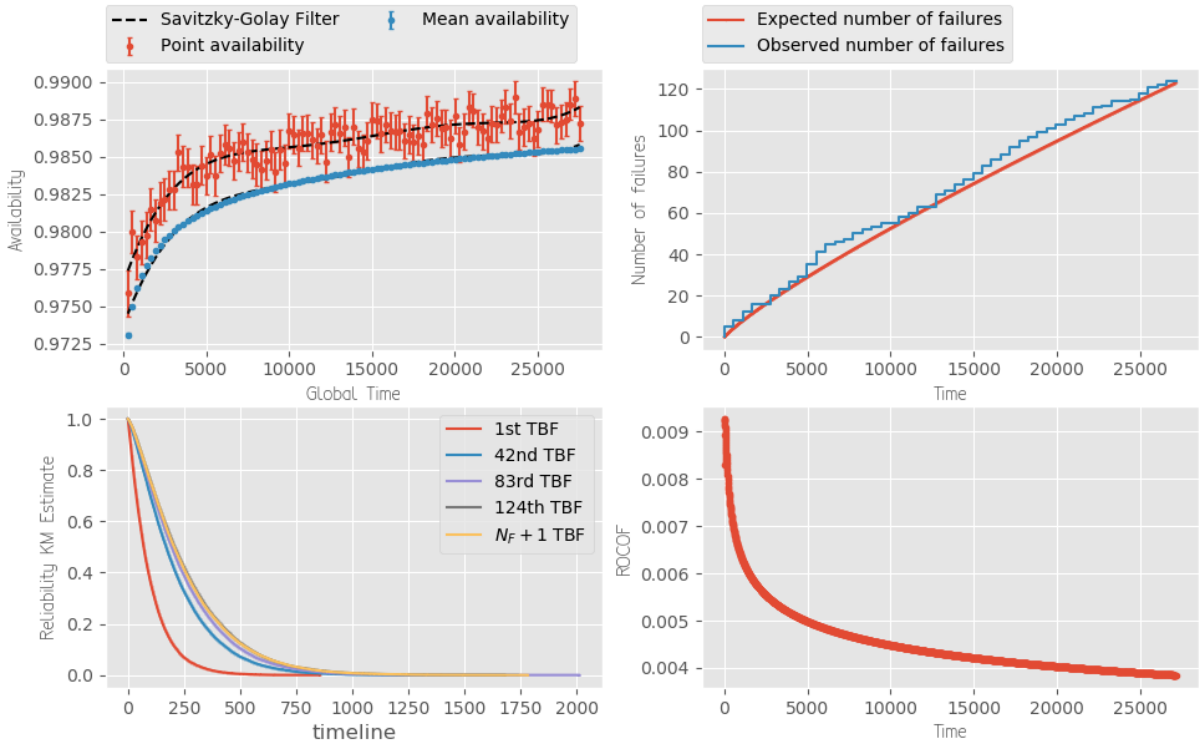

Figure 6.7: Results for the WPLP$(1.240, 0.021, 0.848)$

window allocates the access to all others, besides also importing the failure and downtime data. The **configurations** one is responsible for setting some global parameters pertinent for the model. Lastly, the final output is displayed in **results**.

The **main** window, shown in Figure 6.8, consists of 3 buttons, where the first one, from left to right, gives the user access to the **configurations** environment, displayed in Figure 6.9, and, very importantly, allows the user to import the data. After setting up the software parameters in **configurations** and importing the data, the user then may begin the analysis with the **Run Analysis** button. After the calculations are done, the **Show Results** button will become enabled and the user is able to display **results** environment.

In the **configurations** window, presented in Figure 6.9, the user may choose between a few states of model selection, that is, the user may continue to use the **Default** method, that utilises the Akaike information criterion for model selection, or have a predetermined failure and/or downtime model. Lastly, a few parameters are available for controlling the

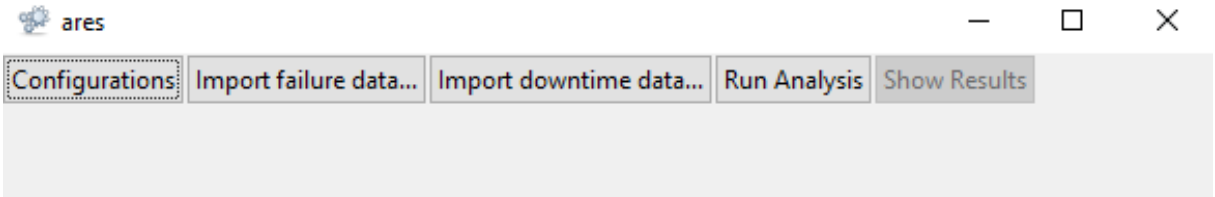

Figure 6.8: Main window

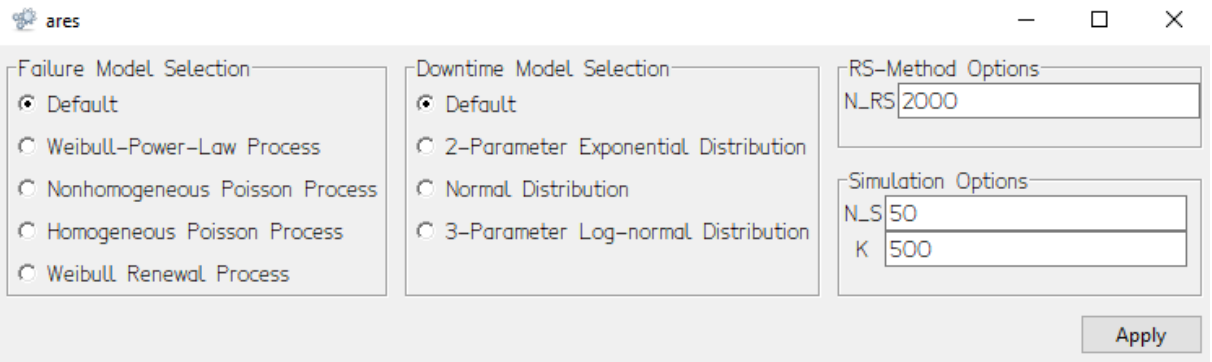

Figure 6.9: Configurations window

accuracy of both the RS-Method and the simulation model.

Lastly, **`results`** window, shown in Figure 6.10, displays all the Key Performance Indicators that are available in the tool: failure model parameters, failure prediction, point and mean availability, expected number of failures, rate of occurrence of failures and the Kaplan-Meier estimates for the reliability. The user is also able to save the plotted performance indicator, in the left hand side corner, by clicking the floppy disk symbol.

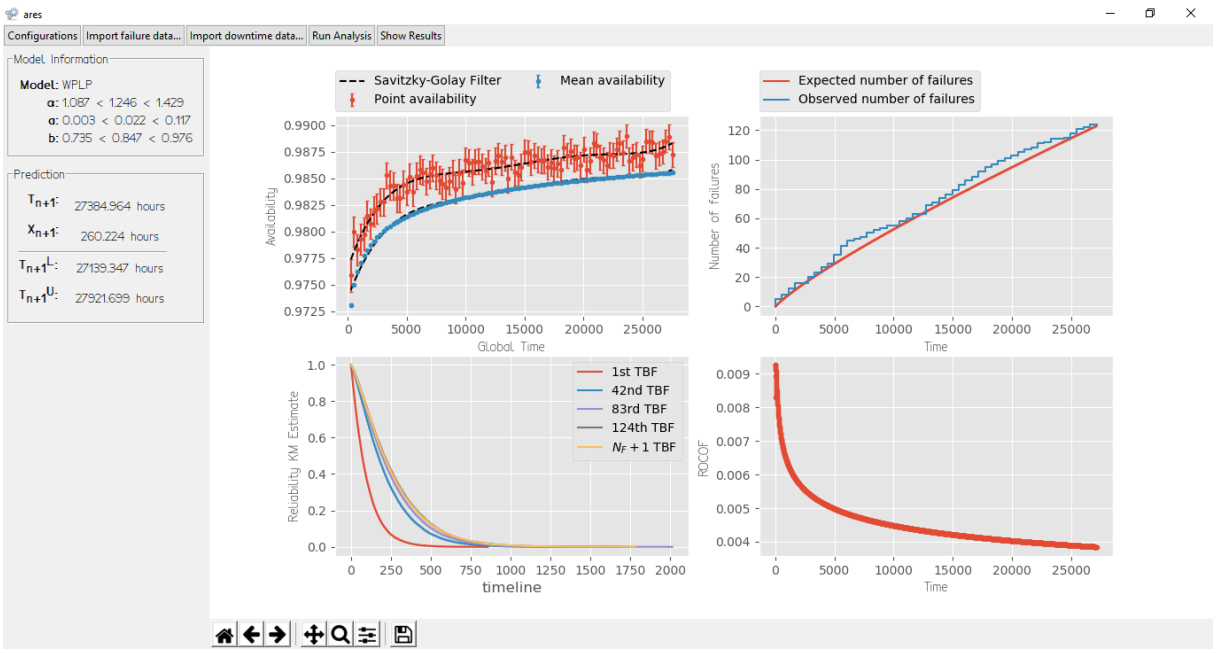

Figure 6.10: Results window

Chapter 7

Conclusions and Future Work

The purpose of this thesis was to implement a general-purpose tool for modelling repairable systems, that works under a automated model selection framework and describes the system in terms of several performance indicators.

First, a thorough research into failure and downtime models was conducted. Along with methods for estimating model parameters and confidence intervals.

Then, the main techniques used in simulation of general stochastic processes were described. This allowed us to calculate the availability of the system, which is not possible under the purely failure modelling, commonly used.

With the mathematical models for the failure and repair processes together with a model selection framework, we proposed the final algorithm. The use of the Akaike information criterion (AIC) is what allows the tool to be completely automated, and glues the whole algorithm together. Although the software also works without automated selection, we believe that its main appeal is for users that do not know a lot about repairable system modelling.

Finally, we analysed a real application case using the proposed algorithm. All the different failure models were fitted, and the most suitable model was selected based on the AIC. The same is true for the downtime models. In the end, the models was a Weibull-Power Law TRP with a 3-parameter lognormal distribution. The analysed system showed a general improving behaviour and its parameters could be a indicative of efficient

maintenance. Also, the availability estimators were compared for different values of total replications, and 25000 seemed to be a good enough number. In general, the system presented a very high availability, with some upward trend, which could be an indicative of posterior improvement.

Since this is a general tool, two main points may be improved upon: adding new imperfect repair processes, other than the WPLP; and using different model selection criterion. When it comes to efficiency, one could improve both the point availability and the reliability estimators proposed here, by more robust and general ones.

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