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Hyper-learning from hyper-teaching: what might the future hold for learning mathematics from & with electronic screens?

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Summary

Traditionally, mathematics has been learned through learners manipulating the content of their mental screens (mental imagery). When the situation became complicated, devices were found to augment and support the mental screen, holding some aspects invariant while transformations and anticipations were experienced and performed mentally. Now electronic e-screens further augment, but threaten to overwhelm, our mental screens. In this paper different modes of interaction with e-screens are outlined, and attention is drawn to the discipline required to use them effectively. Questions are raised about how e-screens could alter what we mean by learning mathematics, at least for the majority.

Keywords

Learning mathematics, e-screens, interaction

Introduction

Traditional mathematics teaching has emphasised the necessity of learner participation. For example, most authors who see their books as teaching texts include exercises or problems for the reader to tackle, and most lessons involve at least teacher-led worked examples as templates or examples of ways of thinking and carrying out techniques. Again, most textbooks include examples to illustrate definitions and theorems. The advice most widely given to learners is that they must stop and work through examples and proofs for themselves (e. g. Polya 1945, MSOR 2004). It is even more powerful to stop and construct other examples for yourself so as to enrich your example space (Watson & Mason 2005), or to construct your own versions of exercises so as to enrich your sense of the scope of the question-space (Sangwin 2004, 2004a) which a technique will resolve.

At face value then, e-screens (used throughout to refer metonymically to what is presented on electronic screens by software) provide a significantly more powerful resource for teaching mathematics than has been available hitherto, and the enthusiasm of the early years of computers in classrooms bears this out. However, the attractiveness of the novel often obscures what possibilities are afforded in the longer term, what attunements are needed to use them effectively, and what constraints they impose (Gibson 1979), whether positively or negatively.

In this paper, learning is seen as an action in and through which the structure of learner attention is transformed in some way, whether through enriching sensitivity to notice, altering (affective) relationships with some mathematical topics, educating awareness (including sensitivity to notice mathematical thinking, themes, topics, heuristic, techniques, etc. as pertinent in new situations), or increasing proficiency in the use of techniques and methods through integrating behaviour more fully.

Following Bennett (1956-1966), actions are seen as requiring three impulses: initiation, mediation and response. The result of an action can then act as initiator, mediator or response to other impulses in an ongoing chain of actions, the results of which are displayed as conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (Kilpatrick *et al* 2001). The six modes of interaction which arise are used to analyse the potential role of e-screens in supporting the development of mathematical thinking. The final section questions what mathematical experience is actually required by the majority of learners in our fast-paced e-based culture.

Modes of Interaction

Before moving to e-screens particularly, it is worthwhile reviewing the different modes of interaction between teacher or source, learner (Mason 1979, Mason & Johnston-Wilder 2004). These interactions all take place within a milieu (Brousseau 1997) which includes not only the physical context but also the established ways of working (practices within the community) and the pressures and impulses coming from individuals, from colleagueal-peers, and from the wider community within which the learners are embedded. Thus institutional pressures come from government, school, parents and learners themselves in the form of expectations and norms, assumptions and demands, however co- and re-constructed by individuals.

By treating the teacher, learner and content as three independent impulses, there are six different modes of interaction, which can be summarised by the six "ex's":

Expounding	Exploring	Expressing
Explaining	Examining	Exercising

The columns are characterised by the initiative coming from the teacher, the student and the content respectively. The two modes in each column arise from which of the two remaining impulses mediates, with the other responding. Salient features of the important distinctions highlighted by this analysis include the following:

- *Explaining* involves the teacher entering the world of the learner. As soon as the teacher thinks they 'understand the learner's difficulty', the mode is likely to switch to *expounding*, in which the resence of the learner (actual or virtual) brings the teacher into contact with the content in a fresh way.

- *Exploring* involves the learner taking initiative to come into contact with the content, guided or mediated by the teacher (or tasks set by the teacher), while *examining* involves the learner taking the initiative to use the content as mediator to test out their own criteria against the criteria of the expert. Note that this is rare in assessment systems, where learners are called upon to display behaviour expected by the assessor as triggered by assigned tasks.

- *Expressing* involves spontaneous desire in the learner (seen as initiated by the content itself) to articulate their understanding to others, including the teacher, while *exercising* involves spontaneous desire in the learner to master the content, using tasks provided (mediated) by the teacher.

Because each of these are conceived as actions, it is not enough for learners to 'explore', or to 'express themselves' or to 'do exercises'. The ways of working involved and the quality of the interaction between learner, teacher and content within the milieu is critical. This matches Vygotsky's insistence on interdependence between teacher and learner rather than the more usual separation of teaching and learning as independent actions (Valsiner 1988 p162).

The relevance of these modes to learning from e-screens will be developed in the following sections. Suffice it to say for the moment that mathematical software (e-screens) have the potential to contribute substantially to each of these modes by mediating more complex actions than were possible previously.

Using E-Screens

There is neither space nor inclination to attempt a full review of all uses of e-screens for teaching mathematics. Instead I use the six modes to illustrate some of the affordances available, with the corresponding constraints, and the attunements which may be needed in the form of personal discipline, in order to make effective pedagogical use of e-screens.

Expounding

The internet is a wonderful source of factual information, especially in mathematics. There are many websites that offer definitions, theorems, historical backgrounds, and proofs of theorems. What is not immediately obvious is that the apparent factuality of many of these websites obscures the inevitable editorial choices which have had to be made. These choices are sometimes pedagogical, even didactical (depending on the specific topic rather than on a more general pedagogic stance). For example, many webpages seem to be generated from and driven by the mathematician's naïve assumption that if only the definition, theorem or proof can be stated clearly enough, then it will be clear to all (who are competent to read it).

The web provides an ideal outlet for the many frustrated mathematicians and educators who want to express their desire to encompass as much of the totality of their knowledge, insights, approaches, awarenesses as possible. It is a classic example of the expounding mode of interaction, because the presence of an enormous virtual audience brings the author into contact with the content in a fresh an invigorating way, quite different to preparing something for a few students, or even fewer colleagues, or even working away for oneself with no prospect of publishing to an audience at all.

Every attempt at encyclopaedic compilation is subject to the peculiarities and particularities of the editors and authors. Although one of the most famous, Diderot (1751-1776) was only one of many editors, stretching backwards and forwards in time, who wanted to express their awareness to others, to gather together and organise what is known.

The problem with all this organisation is that it is not always easy to detect particular cultural and philosophical biases, which are inescapable even in mathematics. Since one person's organisation is another person's chaos, it is not always useful to someone else. Even though hyper-text and search engines appear to provide access in user-friendly ways, the user is always bound and constrained by the organisation as conceived and expressed by the editor.

What then can people do with the mathematical exposition they find on the web? The key feature is that having, as members of a virtual audience, contributed to the author's exposition, a different mode of interaction is required in order to learn from that exposition. The discipline required in order to make effective use of expounding mode, apart from retrieving factual information, is to turn off the machine and ponder or gaze for a time, to think about what has been read, to explore ramifications, to express what sense has been made.

Explaining

The word ‘explaining’, meaning to ‘make plain’, is often used to refer to elaboration of what has been expounded. Thus there is no clear distinction between expounding clearly and explaining through elaborating: it is all really exposition. I use the term *explaining* more precisely but technically, to refer to the act of a teacher entering the world of the learner. It is some concern about, struggle with or interest in the content which brings the learner into interaction with the teacher. This in direct contrast to expounding in which the learner is drawn into the world of the expositor. Explaining is the fundamental interaction of a tutorial as distinct from a lecture. The role of a tutor is thus to try to experience what it is like for the learner to struggle with the topic. As soon as they think they appreciate the difficulty, they are tempted back into expository mode.

As far as e-screens are concerned, explaining as a mode of interaction is available through e-discussion *fora* in which people can ask questions and get responses. It is interesting to observe the various pedagogic and didactic moves made by people responding to queries. In professional *fora* such as HistoriaMathematica, factual questions get factual responses coupled with greetings from all over the globe; in help-*fora* where learners can ask for help with their homework, different responders give differently structured responses, from full solutions (expounded, because there is no further interaction and it seems to be assumed that the response is clear and so understood) to suggestions and comments which might prompt further thinking by the questioner.

It is in explaining mode the teacher has to cope with what Malara (2005) refers to as (mathematical) *babbling* from learners. The construct is based on the behaviour of very young children learning language who often make utterances that sound like sentences even though they are not using recognisable phonemes. In mathematics lessons some learners develop a practice of responding to a teacher’s request for justification for an assertion or for a description of a method, by producing a sequence of words that have the shape and the connectives expected by the teacher, but which when analysed in detail, prove to be largely gibberish. This is a form of *syntactic babbling*, since attention is directed to the form rather than the meaning, but in an attempt to make it sound meaningful. By contrast, *semantic babbling* describes learners’ attempts to express something but using terms inappropriately or ungrammatically. Teachers marking homework often encounter both of these, and sometimes it may be difficult to distinguish between them.

Exploring

The internet is a rich resource of problems for investigation and exploration. As Freudenthal (1983) suggested, there are numerous phenomena which can be explained and made sense of mathematically, from things that happen in the material world, to things that happen in the mental world of imagery manipulation. Lakoff & Nunes (2000) go so far as to suggest that all mathematical concepts have at their root some bodily experience, and e-screens can be used to invoke or evoke those awarenesses.

The problem arises as soon as the presenter feels compelled to provide some suggestions (note the difference in assumptions and ethos between providing ‘suggestions; and ‘hints’: the former suggests enquiry and possibilities, the latter suggests a specific hidden result to be found). Exploring is a mode of interaction in which the learner takes and retains the initiative, with the teacher acting as mediator by making suggestions.

E-screens can do much more than provide phenomena to make sense of. They can also provide access to ‘computationally expressive and re-constructable software’ (diSessa 2000). This means software in which you can not only express mathematical relationships and properties, but also manipulate these, and you have access to the details of whatever has been provided. One way of analysing software is in terms of the affordances it offers for expressing mathematical relationships, and the constraints it imposes in the way of computational possibilities. For example, although not software, high school algebra affords opportunities to express many but not all types of numerical relationships, and constrains manipulation to what can be carried through by an individual without making errors, usually these days, a few lines or perhaps a page of manipulation.

Examining

In this mode of interaction, the learner seeks to validate their own criteria concerning the depth and breadth of their understanding, against what an expert expects. E-screens afford manifold opportunities to go beyond the short-answer and multiple-guess questions of the past. Instead of or in addition to being asked to solve routine exercises similar to ones they have done in the past, learners can be asked to construct objects meeting specified constraints. In a sense all mathematical tasks of the type ‘to find’ are construction tasks subject to constraints (Watson & Mason 2005), but with e-screen based software, learners can be asked to construct objects which are not simply single answers to specific problems, but rather general objects displaying a range of (computationally verifiable) properties.

Software can do the checking, and can even advise the learner on where their object fails to meet a constraint. Instead of assessment being a snapshot of what a learner can do under pressure at a particular moment, software can track the development of learners’ appreciation of the various dimensions of possible variation in examples of particular concepts or the use of particular techniques. In a sense then, software can track the changes in learners’ ZPD (Vygotsky 1978 p86).

Using software to support as well as prompt object construction, learners extend the example spaces which come to mind in association with various topics, concepts and techniques. Meanwhile the teacher needs to explore the structure of the corresponding question spaces (Sangwin 2004, 2004a). For example, to present a learner with a polynomial that has real roots you can either generate it from the roots, or you can choose the coefficients subject to certain constraints which need to be specified.

Expressing

Expressing as a mode of interaction occurs when the learner finds a desire welling up inside them to express some ideas, perceptions, or awarenesses. In a sense the content itself can be thought of as initiating the interaction, with the learner acting as channel or medium of expression to others, such as the teacher. Mary Boole (Tahta 1972) aptly described this mode when a teacher succumbs to it, as *teacher lust*, that strong desire to tell the learner something that you know. It is entirely natural in social discourse, but not always helpful in a pedagogic context.

As the well known adage has it"you only really learn when you try to teach". Trying to articulate relationships, properties, connections and associated images coherently not only helps to clarify ideas but also contributes to the process of internalisation and condensation (Freudenthal 1978). At the Open University we have found it helpful since 1982 to organise these insights into frameworks for teachers which can then come to mind when planning or when teaching in order to inform pedagogic choices (Mason & Johnston-Wilder 2004). For example,

Manipulating – Getting-a-sense-of – Articulating (MGA)

when internalised, can act as a reminder of the cyclic nature of integration, in which manipulating familiar and confidence-inspiring entities provides experience through which a sense of generality, structure or relationships can emerge, and that as these are articulated and re-articulated they become increasing succinct so as to form entities for further manipulation. Similarly

Doing – Talking – Recording

when internalised, can act as a reminder that the purpose of doing is to have something to talk about, that talking clarifies thinking and often liberates recording, that attempting to record can clarify talking, and that it is not always helpful to rush learners into written records before they have had a chance to rehearse the language connected with a topic or technique. Through attempting to express insights and meanings learners can clarify their own thinking, as well as benefit from the responses of others.

Not only is it often helpful to try to express your thinking in words as if to another person, before trying to talk to others or to make some sort of written record, but it is even helpful to think and express before using mathematical software, whether to do calculations, generate diagrams, or present ideas. Communicating with others often has a component of examining, in that you are looking for affirmation and resonance from others as validation of your own coherence and appropriate use of terms, etc.

The urge to change medium of expression, for example to move from paper to machine in order to try to record in some form after thinking and imagining is often better resisted for a period so that substantial preparation can take place. A prepared mind makes much better use of a tool than an impulsive one.

The internet affords but also constrains, for apart from enabling audio conversations, by and large it requires some sort of recording in pictures, diagrams, words, and to some extent sounds. So it affords opportunities once thoughts can be expressed reasonably coherently. It also affords impetus to reach clarity of articulation in order to participate with others in a global community, rather than simply handing in work to the teacher. However e-screens do not currently facilitate preliminary, tentative and semi-coherent expressions of mathematical ideas, relationships and techniques which often precede clear articulation. Indeed, it is vital to develop a discipline of keeping notes about conjectures and examples, ideas and connections while one is using mathematical software, because it is so easy to get lost in what you are doing, and so difficult to reconstruct from what ever work is saved as a residue.

The most effective use of expressing as a mode of interaction occurs when learners convert it into an expounding mode, treating their audience not as a teacher but as learning from them, because in this mode, attention is drawn to presentational pedagogic issues, rather than simply expressing oneself to the world in general.

Exercising

Exercising also has the content ‘taking initiative’, but here the teacher mediates the contact between learner and content by providing exercises through which the learner can gain competence and proficiency. The point about exercise intended to work on fluency and facility is that the force has to arise within the individual: being told by someone that “you need to exercise” is much less effective than a personal up-welling of desire to rehearse an action. Sometimes this emerges as expressing to others, and sometimes as practice. For example, watching people practice skateboarding reveals just how much concentration and effort people are willing to put in to rehearsing as well as exploring when their own powers are being honed and developed at their own initiative. The use of personal trainers is a reminder that one of the roles of a teacher is to act as personal trainer for learners, providing an element of discipline which individuals find hard to sustain on their own.

A good deal of computer aided instruction and computer aided learning is based around a sequence of tasks which are adjusted and extended according to the learners' performance. The issue is whether the tasks are providing the initiating impulse because of desire to learn, or whether the learners' intentions are to do the minimum in order to surmount some hurdle imposed by outside forces. As a teacher, it is often more effective to engage learners in exploration which affords them the opportunity to construct their own particular cases on which to use some technique in pursuit of some conjecture, thus drawing their attention away from what they are doing rather than, as with routine exercises, directing attention to the very actions which are intended to be internalised.

I am not aware of websites offering sets of exercises on which learners can rehearse their techniques, since this seems to be the domain of commercial software, although people are beginning to put diagnostic tests of one sort and another on the web. The advantage is that these can be undertaken piecemeal, privately, and in your own time.

Exploiting Modes Pedagogically

It has long been recognised that pedagogical tasks are devices to stimulate activity (Christiansen & Walther 1986) to afford the possibility of effective interaction between learner, teacher and content within the socio-cultural and physical milieu, and that even activity and interaction is insufficient by itself to ensure learning. This despite the common learner assumption that by more or less 'doing' the set tasks, the requisite learning will somehow happen, which Brousseau described as the *contrat didactique*. Piaget summarised what was needed as three processes: interiorisation of an action, encapsulation, and thematisation of a schema, summarised asf *reflective abstraction* (Piaget & Garcia 1989). Vygotsky articulated something similar in terms of internalising higher psychological functioning through being in the presence of its manifestations in others, while Gattegno (1988) spoke in terms of *integration through subordination*, and in terms of the *mind teaching the brain*. It is currently popular to advocate peer discussion as a mechanism to achieve similar aims.

The six modes provide a slightly more elaborate structure within which to consider what might be needed in order to prompt an inner transformatory action (learning) through participation in the various modes. Thus expressing (to and for oneself) is a useful action following or even during participating in exposition and exploration, and there are delicate differences when this takes place face-to-face, synchronously but virtually, and asynchronously; exercising on your own is much more difficult (requires more discipline and more energy) than when embedded in a programme of work to be handed in for checking; exploring similarly requires more investment of time and energy and the strength to cope with disappointment and confusion when working at a distance than when working with colleagues.

Directing Attention

The purpose of any mediating educational tool is to direct attention, not only in terms of what is attended to, but also in terms of *how* people attend, that is, the structure of that attention. In Mason (1982) I began my studies of the structure of attention, and in Mason & Davis (1989) we outlined various macro structures concerned with locus, focus and multiplicity: you can direct your attention (locus); you can gaze with a very fuzzy attention or you can focus sharply on some feature; and you can attend to several things at the same time (or in very quick succession). Furthermore although perhaps we like to think that we are in control of our attention, our attention is easily attracted by movement (especially in peripheral vision), by sounds and by other changes in out sensed environment. Learning to control and direct attention is one of the major features of early childhood and again in adolescence.

In Mason (1989) I reported noticing that mathematical abstraction could be described as a delicate shift of attention, and since then I have been trying to describe the various micro structural forms of attention which show up in mathematics lessons. Inspired by insights from Bennett (1956-1966) I found it useful to distinguish five different attentional structures (Mason 1998). These turn out to match closely the 'levels' in geometric thinking delineated by Dina and Pierre van Hiele (see van Hiele 1986, Usiskin 1982).

A major difference between my approach and the van Hiele's is that the structured forms of attention as I perceive them can be fleeting as well as moderately stable, and certainly need not progress sequentially from one to another. It makes no sense to speak of a person being 'at or in one of these structures' corresponding to the activity of classifying learners as being at one of the 'van Hiele levels', because they may be jumping rapidly from one to another in response to stimuli, or they may be stuck in one for some reason. What I have found is that being sensitised to the possibility that even when learners are attending to the same thing they may not be attending in the same way, coupled with being aware of various structural forms of attention in oneself, makes it much easier to make pertinent pedagogic and didactic choices (Mason 1998).

The structural forms can be summarised as follows:

- Gazing: attending to a whole (which may be a part of some other whole)
- Discerning Details: distinguishing some element or part *as* part of some other whole
- Recognising Relationships: relating elements of a whole or elements between wholes in the particular situation

- Perceiving Properties: seeing through the particulars to generalities, being aware of particular relationships as instances of a more general property.

- Reasoning on the Basis of Stated Properties: mathematical reasoning

The question to be considered here is to what extent and in what ways e-screens can prompt learner attention to adopt a useful macro as well as micro structure in order to make efficient pedagogic use of the affordances.

Using E-screens to Direct Attention

Although the various forms can be highly transitory and in no particular order, it is convenient to address each one in turn with respect to how electronic media can contribute to them.

Gazing

Gazing is a peculiar phenomenon because it can sometimes be highly productive and sometimes a form of ‘time out’, ‘wool gathering’ or ‘day dreaming’. For example, you can gaze at a diagram or you can hold something in your mind, much as Poincaré describes prior to getting on a bus (Poincaré 1908). After a while ideas may come to mind, but you can also stare into space and be attending to something else entirely.

E-screens are ideal for producing phenomena which can be gazed at, then analysed. What is needed is the discipline to stop and analyse, perhaps to move away from the machine, in order to make sense not only of the particular, but to consider possible general classes of situations which might give rise to the same phenomenon, and classes of related phenomena. This use of attention to engage in reflection, even reflective abstraction (Piaget & Garcia 1989) is an example of a ‘higher psychological functioning’ (Vygotsky 1986). The frozen metaphor of *reflection* is interesting because mathematically you can only manifest reflection as a physical action by moving into a higher dimension. So too, psychological and social reflection requires drawing back from immediate engagement and moving to a higher plane. It also requires intention on the part of the learner. It involves the (re)direction of human energy and thus requires discipline, whet5her from the individual or guided by a teacher who has an over-view and is able to provide ‘consciousness for two’ (Bruner 1986 p75-76).

Discerning Details

Classic e-screen techniques include the use of flashing certain elements, increasing their intensity or changing their colour or structure, pointing to them in some way, fading down other elements, and building up or deconstructing in animated sequences (see Gutierrez 2007 for elaborate examples).

Often producers like to use voice-over, telling the viewer what they are to look at. It may be however that adding in sound actually reduces the opportunity for the user to exercise choice, so that, instead of strengthening their power to discern details, the effect may be to create greater dependency. Working with silent mathematical animation by viewing, then turning it off and mentally reconstructing as much as possible, then trying to give a description before attempting to account for or justify conjectures, is certainly a powerful way to develop control over attention as well as the power to express what has been imagined (Jaworski 1989).

Recognising Relationships

Juxtaposition is one of the few devices that e-screens can use to draw attention to relationships. These juxtapositions may be in time or in space. Thus two consecutive scenes in a drama are more likely to be connected than two scenes separated in time; the farther apart in time the more nearly similar they need to be to trigger recognition. As Ference Marton points out, if you want a learner to detect variation, then the elements that are being varied need to be sufficiently close in time and space that the learner actually associates them with each other (Marton & Booth 1997).

Structured Variation Grids (Mason 2005) feature the possibility of drawing attention systematically to either or both of two features and in close spatial and temporal juxtaposition, so as to activate learners immense powers of ‘going with the grain’ and experiencing generalisation. The teacher can then prompt and promote generalisation by asking for conjectures about particular cells, initially simultaneously visible, and then ones that are currently out of sight (‘going across the grain’ Watson 2000). The whole process of generalisation, of perceiving properties and reasoning on the basis of those properties can be stimulated. But as with any educational experience, once the activity finishes, once the machine is turned off, the residue that remains may be very particular, and it may be quite general. What is needed is disciplined reflection and gazing.

Simulations are often intended to direct users’ attention to relationships, but as with any other activity, it is essential to prompt learners to draw back from immediate engagement and to reflect in some manner, to become aware beyond ‘theorem in action’ as Vergnaud (1981) described it. Diagrams are usually about relationships, but it is quite difficult to ‘point to’ a relationship. In geometry it is usual to use labels and to articulate relationships in words, but supporting learners in recognising relationships for themselves is not at all easy. The result is that presentations usually fall into expository mode rather than exploratory mode for fear that learners might not alight on what is intended, and might take a liong time over it. So disciplined response to exposition is essential if learners are to internalise and integrate what they have been shown and told.

Perceiving Properties

E-screens are not good at generalisation, because generalisation takes place in the mind of the beholder, not on the screen. Whatever is chosen to instantiate a particular is no longer available to depict or signify a generality encompassing that particular. Therefore properties are referred to in spoken language and the learner is expected to specialise for themselves, instantiating either with recent shared examples, or with examples of their own. The teacher can, as at a blackboard or other display, speak and point at the same time, but no matter how explicit they are, in the end the appreciation of generality and particularity is something the learner has to do for themselves. The art of the teacher lies in constructing situations in which learners will spontaneously use their own powers, rather than depending on the teacher to try to do that work for them.

Problems and Issues

Thinking in terms of learner attention, where it is directed and with what macro and micro structure raises numerous questions about the efficacy of hyper-teaching as an aid to hyper-learning. It seems that age-old insights into the actions necessary for learning to take place still apply: the learner needs to engage in reflection, rehearsal and re-construction. Before asking whether the facility provided by e-screens might alter what we mean by learning mathematics' it is appropriate to raise one or two other problems and issues occasioned by ubiquitous use of e-screens.

Click-and-Go

Television producers have learned to manipulate camera shots so that the shot changes just before the producer thinks audience attention is likely to wander. So observing the structure of television programmes tells us a great deal about the expectations of producers. Their aim is to keep audience attention glued to the screen. It was thought at first that films would not transfer well, because in a darkened cinema with a big screen there is no temptation to look away, get up and get a drink, or turn and talk to a neighbour. In the privacy of your own home however, social conventions are different, and people talk over what is on the screen, get up and move about and so on. To counter this audience independence, advertisers and e-screen designers seek ways of keeping attention on the screen. The result is the co-emergence of the quick-change: as attention spans decrease, things have to happen more rapidly on a screen, and as things happen more rapidly, attention spans decrease. The result is a habit of 'click and go': as soon as what is on the screen appears to be problematic or complicated, or to require effort, there is a strong temptation to change screen and see if there is something more stimulating, more engaging, less complicated elsewhere.

Of course it is not a simple matter, as anyone knows who has, for instance, invited children to display one of their hobbies at a 'bring and show' at school. It is curious how children with apparently short attention spans in school sometimes display hobbies which require extended focused attention. A major factor may be the difference between having your attention controlled and directed by someone else, and having your attention directed by something of interest to you. This would suggest that lesson pacing is a critical matter, and that maintaining a fast pace in a lesson may not be maximally effective. Periods of individual reflection, indeed of gazing, may contribute to sustained attention at other times.

Explaining and Expounding

A concomitant feature of click-and-go is the desire to get help as soon as a learner is stuck, amplified by the teacher's desire to help the learner and to get them unstuck (*teacher lust* referred to earlier). Teaching is often more effective if learners are invited and provoked into actions in which they are the ones who make use of their own powers and who have the desire to express, and through expressing and articulating, to clarify things for themselves. Thus learner and teacher connive, through their mutual desires, the one to expound, the other to be told, to increase learner dependency on the teacher.

So if something is unclear on a screen, what is the learner to do? Pause and gaze? Or click-and-go in case the next screen is more helpful or more stimulating? This is where the discipline enacted by the presence of a teacher, or perhaps, sometimes, the quality of a task which includes the use of the e-screen is vital. At the very least, learners will need to be enculturated into ways of working with e-screens which involve pausing, gazing, going away and thinking, trying to articulate what sense has been made, make conjectures, confer with colleagues, and so on.

There is a Geometry Forum on the web which is predominantly learners seeking help with their homework. When homework was based on textbooks, some teachers would choose the exercises which did have the answers in the back of the book, and others would choose ones that did not. The impulse when stuck to give up and 'get a hint' is very strong. It is true that sometimes, when stuck, seeing the answer, especially if it is fairly complete and so informative, can hep a learner get unstuck. However, it requires personal discipline to learn from whatever help is received, rather than simply rushing on to the next task.

Of course what matters is not the getting of the particular answer, but whether an answer could be obtained to a similar question in the future. Some learners seem convinced that what matters is the immediate answer, and that somehow, through their assumptions about the *didactic contract* (Brousseau 1998) obtaining the asnwe will be sufficient to produce the learning. Again what is required is enculturation into effective ways of seeking and making subsequent use of help.

Easy to Ask Very Difficult Questions

A problem with computer algebra systems is that it is all too easy to ask the machine to undertake a task which is either too difficult (involves solving high degree polynomials) or form which the result is pages and pages of output which are too difficult to make sense of. For example, trying to use coordinate geometry to find the four equations of the mutual tangents of two non-intersecting circles requires the use of considerable algebraic awareness in order to succeed. Having access to computer algebra systems tempts the user into undertaking more general problems than they would try by themselves on paper, but often results in pages of symbols, or nothing at all if the solution is mathematically too sophisticated.

Overheads

The more sophisticated software packages become, the longer it takes people to become proficient users. It seems that relatively few people know how to use styles in a word processor, yet a considerable amount of the power afforded by such software comes from the use of styles to reformat. Furthermore, for a tool to be an effective mediator between user and content or problem, it is important that the tool does not divert attention to itself. Yet a user trying to work on a challenging problem with sophisticated software may find themselves, as I often do, having to find out what possibilities the tool offers of which I am not yet aware.

Disciplined Use of ICT

The six modes of interaction together with the different ways in which attention can be structured lead to one major conclusion: learners require discipline in order to disengage with activity and draw back to reflect on or make sense of what they have been doing. Where is this discipline likely to come from? As a higher psychological process it can only come from being in the presence of others who manifest it, and teachers who support learners by acting as ‘consciousness for two’ in developing and maintaining conditions in which that discipline is encouraged and found to be effective. So in as sense it all comes down to a question I asked in 1985: “What do you do when you turn off the machine?” (Mason 1985) together with the self-discipline to delay using a machine until it is really appropriate, and to have the discipline to keep records of conjectures and examples and other thoughts while using the e-screen. Making an explicit conjecture before clicking amplifies potential surprise when expectation is broken, and condenses understanding when it is confirmed.

Mathematical Experience in an E-based Culture

These observations about hyper-teaching and hyper-learning could lead to a rethink about what learning and doing mathematics is desirable for a majority of the population. What sort of proficiency is necessary when there are machines that can outperform most humans in mathematics up to and into tertiary level mathematucs?

Proficiency

What does it mean, and what is it sensible to expect in the way of mathematical proficiency in an era dominated by every more sophisticated and complex e-support for mathematical representations, reasoning, and computation? For example, how does mathematical proficiency as delineated by Kilpatrick *et al* (2001): conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition.

People who are not going to be working as mathematicians do not need to know all sorts of definitions and theorems. Those that they do need they can follow up if and when it is appropriate. What people need is a sense of mathematics as a domain (or collection of domains) of enquiry. What sorts of questions can mathematicians help answer? What does it mean to reason mathematically, to justify a conjecture, to generalise an assertion? They need to experience the *necessity* of something proved, and the impossibility of something disproved. They need to appreciate that some things must happen, some things may happen, and some things cannot happen. For example, the sum of four consecutive whole numbers must be divisible by 2 but cannot be divisible by 4; the medians of a triangle must intersect in a single point, but the lines formed by joining each vertex to the point one third of the way along the opposite edge in a counter-clockwise direction cannot intersect in a single point.

Whereas in the past we have cajoled and threatened learners to rehearse techniques on sets of exercises in order, we hope, that they might gain proficiency in techniques, perhaps what is needed in an e-culture is rather different.

License to Practise

In Mason, Hewitt & Brown (1993) we proposed the idea that competence and proficiency in mathematics could be recast using the metaphor of a driving license. What learners need most is to be confident and proficient users of mathematical support, rather than creators of mathematics. Thus learners could expect to display behaviour which justifies them in receiving a ‘license to use unaided’ various specific mathematical tools such as dynamic geometry, spreadsheet, CAD and CAS systems. Each system would have a number of levels corresponding to the different complexities and sophistication of the particular software. For example, in dynamic geometry, the construction of figures with given constraints using the provided tools, and the construction and use of macros, would constitute different levels of expertise. At each level learners would have to demonstrate that they can make sense of someone else’s (appropriate) construction, that they can initiate and carry out constructions for testing and refuting conjectures following someone else’s description, and that they can initiate constructions and make generalisations of their own on the basis of some provided inspiration. These three formats correspond to the three impulses comprising an action: initiating, mediating, and responding, which generate the six modes of interaction.

As well as mathematical tools themselves, there could be a 'license to use unaided' the web as a source of conjectures as to meanings of terms, mathematical and mathematically related facts and figures, and so on. This would involve taking a critical stance, checking conjectures where possible, displaying data in appropriate forms (using mathematical tools) and so on.

Beyond a 'license to use unaided' various mathematical tools, mathematical proficiency could then be concentrated on disposition to notice opportunities for mathematical thinking and reasoning, and a disposition to think first then reach for a tool rather than reach for a tool and then, maybe, think about what went wrong. It could encompass awareness and use of pervasive mathematical themes and heuristics, and a propensity to pose as well as solve problems, to generalise as well as specialise. It could emphasise that formulating a conjecture is an important step, but justifying conjectures is where the real mathematical work lies, and this involves more than 'it works in lots of cases on the screen'.

Conjecturing and Reasoning

It is important that all learners experience contexts in which they want to justify conjectures through mathematical reasoning, so that they become aware of the process of articulating properties which are assumed to hold, and then reasoning on the basis of those announced properties. One of the difficulties in teaching mathematical reasoning heretofore has been that it is either tedious to go through all the early deductions from a simple set of axioms such as those of Euclid or Peano, and too complicated to write down all the assumed properties needed to start with more interesting results. Consequently at every level learners are left wondering what they may assume and what they have to justify. By treating mathematical reasoning as a top-down rather than bottom-up activity (Leron 1983) learners can decide when to stop further elaboration of justification and declare their assumed properties.

The key feature of teaching reasoning, whether through deducing bottom-up or through refining top-down, is the awareness of reasoning on the basis of assumed properties. This is a sophisticated awareness, indeed, as I have argued elsewhere (Mason 2001, Mason & Johnston-Wilder 2004a), a sophisticated structure of attention. What blocks learner access to thinking in this manner is not the absence of powers of reasoning and logical deduction, but where their attention is placed and how it is structured. Learners who are gazing at a whole (whether all or part of a diagram, problem statement, or algebraic expression) are not in a position to make sense of distinctions being talked about by someone else; learners trying to discern distinctions being referred to are not in a position to recognise relationships between those distinguished elements; learners seeking relationships amongst particular elements may not be in a position to perceive those relationships as particular cases of more general properties; perceiving properties as applying to particular relationships amongst discerned elements is not sufficient for reasoning on the basis of those properties.

Counter arguments

The position outlined regarding a possible future mathematics curriculum is certainly not universally accepted. Indeed I also have some conflicting doubts which I articulate here.

You need to know a lot to use a powerful tool

The principal counter argument against a fundamental reformulation of mathematical proficiency in an e-screen culture is the observation that to make cogent use of CAS, you actually need to know quite a bit of algebra, and be able to manipulate algebra yourself. For example computer algebra systems are not yet able to isolate subsets of symbols so as to highlight patterns which a competent algebraicist would notice and exploit, and it requires considerable user fluency with algebra to achieve this. For a tool to be useful as a mediator between user and content it is essential that the user's attention is not diverted to the functioning of the tool itself.

You need to integrate current functioning in order to be able to use it effectively to progress further

If you don't know your number bonds to ten, you will find it difficult to add and subtract even moderate sized numbers; if you don't have facility in adding and subtracting even moderate sized numbers, you are going to have difficulty engaging in productive exploring and expressing, or keeping up with expounding. So you have t be able to do things yourself in order to make effective use of a more powerful tool which can extend the scope of your competence. This does not justify a simplistic move 'back to basics' which requires everyone to master arithmetic before they use a calculator, but it does suggest that teachers have a responsibility to support learners in developing the necessary discipline in using e-screen tools so that they internalise not only appropriate productive dispositions, but also conceptual understanding, procedural fluency, strategic competence and adaptive reasoning (Kilpatrick *et al.* 2001).

Conclusion

My principal conclusion is that teachers are more necessary perhaps than ever before, because the forms of mental and emotional not to say behavioural discipline required to learn from and with the use of e-screens in mathematics require the presence of the awareness of a relative expert. This may seem odd coming from someone who has spent his career in a distance-teaching university, and who has seen many thousands of students successfully learn (some) mathematics using text, video and audio, and the occasional tutorial at best. However I am not arguing that face-to-face teaching is necessary, only that learners be in the presence of awareness which transcends their own. While it is perfectly possible for a Diderot to teach himself geometry by starting in the middle of Euclid, tracking back to theorems on which reasoning depends until he gets to the axioms, then proceeding to build up the deductive structure under Euclid's guidance, this is rare, and rarer with the richness of sense impressions available through e-screens.

What I am arguing for is a careful analysis of the ways of working which support learners in using e-screens effectively for their purposes. Integrating the six modes of interaction with the structure of attention seems to be a useful way of undertaking at least some of that analysis. In the process, it is vital to continue to question what it means to be proficient as a mathematics learner in an e-culture.

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