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Skolem Number of Subgraphs on the Triangular Lattice

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Abstract

A Skolem sequence can be thought of as a labelled path where any two vertices with the same label are that distance apart. This concept has naturally been generalized to graph labelling. This brings rise to the question; "what is the smallest set of consecutive positive integers we can use to properly Skolem label a graph?" This is known as the Skolem number of the graph. In this paper we give the Skolem number for three natural vertex induced subgraphs of the triangular lattice graph.

Keywords: Skolem graph labelling, Skolem number, triangular grid graph

1 Introduction

Skolem defined and worked with specific distance preserving integer sequences while working on problems involving Steiner triple systems.

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Definition 1.1. *A Skolem sequence, is a sequence* $S = (s_1, s_2, \ldots s_{2n})$ *consisting of each integer in* $\{1, 2, \ldots, n\}$ *twice so that whenever* $s_i = s_j = k$, $|i - j| = k$.

Example 1.2. The sequence $\{1, 1, 3, 4, 2, 3, 2, 4\}$ is a Skolem sequence of order 4.

Theorem 1.3 ([7]). *There exists a Skolem sequence whenever* $n \equiv 0$ *or* $1 \pmod 4$, *and otherwise none exists.*

One can think of a Skolem sequence as a labelling on a path P_{2n} with labelling set $\{1, 2, \ldots, n\}$, which we will refer to as $[n]$, so that whenever a pair of vertices $v, u \in V(P_{2n})$ are labelled i, $d(u, v) = i$. This naturally gave rise to the concept of Skolem labellings which was introduced in 1992 by Mendelsohn and Shalaby [6]. In subsequent years, further results have appeared about Skolem labellings of ladders and grids [2, 5], cycles [3], and Dutch windmills [4] to name a few. All the aforementioned papers assume the definition first posed by Mendelsohn and Shalaby that a *d-Skolem labelled graph* is a triple (G, L, d), where

(a) d is a natural number,

(b) $G = (V, E)$ is an undirected graph,

(c) $L : V \to \{d, d + 1, \ldots, d + n - 1\}$,

(d) $L(v) = L(w) = d + i$ exactly once for $i = 0, 1, \ldots, n - 1$ and $d(v, w) = d + i$.

(e) if $G' = (V, E')$ for $E' \subset E$, then (G', L, d) violates (c)

Note that if a triple (G, L, d) satisfies parts (a), (b), (c) and (d), but doesn't satisfy part (e), it is said to be a *weak Skolem labelled graph*.

In the results on Skolem sequences and Skolem labellings some have used the idea of a *hook*, which is a vertex $v \in V$ such that $L(v) = 0$. It has even been extended so that more than one vertex may be a hook, in which case hook vertices may be any distance apart from one another.

Another abstraction of a Skolem sequence to a graph labelling was introduced in [1], which allows for more than two vertices to have the same label, provided they are mutually the label's distance apart. To emphasize the slight difference, yet recognize their similarities, we will use $\mathcal{L}$ for the bijection in this abstraction of a Skolem sequence on a graph.

Definition 1.4. Given a graph $G = (V, E)$, we say $\mathcal{L} : V \to [n]$ is a *proper Skolem labelling* of G if whenever $\mathcal{L}(v) = \mathcal{L}(w) = i$ for $v \neq w$, then $d(v, w) = i$. Where $\mathcal{L}$ is surjective, the proper Skolem labelling is said to be of order n.

If $|\mathcal{L}^{-1}(i)| = 1$, then we call the label i or the vertex $\mathcal{L}^{-1}(i)$ a *hook label* or *hook vertex* respectively. Trivially every graph $G = (V, E)$ contains a proper Skolem labelling of order $|V|$ using any bijection between V and $[|V|]$. Since it is *never* the case that $\mathcal{L}(v) = \mathcal{L}(w)$ for $v \neq w$, this style of proper Skolem labelling contains only hooks. The bijective assignment makes a vacuously proper Skolem labeling, hence the desire to find the smallest order proper Skolem labelling.

Definition 1.5. The *Skolem number* of a graph G is the smallest positive integer s such that there exists a proper Skolem labelling of G of order s.

Carrigan and Asplund [1] provided a way to proper Skolem label C_n with no hooks when $n \equiv 3 \pmod 6$, extending the results of [3] to obtain the Skolem number of C_n.

Theorem 1.6 ([1]). *The Skolem number of C_n is*

- $\frac{n-3}{2} + 1$ *when* $n \equiv 3, 9, 15, 21 \pmod{24}$
- $\lceil \frac{n}{2} \rceil$ *when* $n \equiv 0, 1, 2, 8, 10, 11, 16, 17, 18, 19 \pmod{24}$
- $\lceil \frac{n}{2} \rceil + 1$ *when* $n \equiv 4, 5, 6, 7, 12, 13, 14, 20, 22, 23 \pmod{24}$

Asplund and Carrigan also extended the results of [6], [2], and [5] to obtain the Skolem number of $P_a \square P_b$.

Theorem 1.7 ([1]). *The Skolem number of $P_a \square P_b$, where $a \leq b$ is:*

- $ab - a - 2b + 4 - \lfloor \frac{a-1}{2} \rfloor$ *if* $b = 2a - 1$,
- $(a-1)(b-a) + 1 - 2(a-1) - \lfloor \frac{a-1}{2} \rfloor$ *if* $b > 2a - 1$ *and* $(a, b) \neq (3, 6)$,
- *and* $(a-1)(b-a) + 1 - \lfloor \frac{a}{2} \rfloor - \lfloor \frac{1}{3}(a + b - 2) \rfloor$ *if* $b < 2a - 1$, *(except for a few exceptions, where the Skolem numbers was found and exhaustively proven.)*

Recognizing $P_a \square P_b$ is a vertex induced subgraph of the rectangular lattice graph, we wish to explore subgraphs of the triangular grid graph. Let us define the triangular lattice graph as $T = (\mathbb{Z}^2, E)$ (Shown in Figure 1) on the vertex set of integer ordered pairs (a, b). Two vertices $i = (i_a, i_b)$ and $j = (j_a, j_b)$ of T are adjacent if and only if one of the following is true

1. $|i_a - j_a| + |i_b - j_b| = 1$
2. $i_a - j_a = 1$ and $i_b - j_b = -1$
3. $i_a - j_a = -1$ and $i_b - j_b = 1$

This leads us to define two natural vertex induced subgraphs of the triangular lattice graph

Definition 1.8. *A triangular grid graph of order n, denoted as T_n, is the subgraph of T induced by the vertices (a, b) such that $a, b \in \mathbb{N}$ and $a + b \leq n$. (T_3 is shown in Figure 2)*

Definition 1.9. *A parallelogram grid graph, denoted as $P_{(a,b)}$, is the subgraph of T induced by the vertices $\{(i, j) \mid i, j \in \mathbb{N}$ for $0 \leq i < a$ and $0 \leq j < b\}$. ($P_{(4,4)}$ is shown in Figure 3)*

The hexagonal lattice graph H is an infinite subgraph of T. Assuming a proper three coloring of T, we will call the subgraph induced by the vertices of two of the color classes the Hexagonal lattice graph H. More precisely, let $R = \{(x, y) \mid x \equiv y \pmod 3\}$ and define $H = T - R$. This is achieved by removing the red vertices in Figure 4 and their incident edges from the triangular lattice graph.

Just as before, we will define a finite subgraph of H.

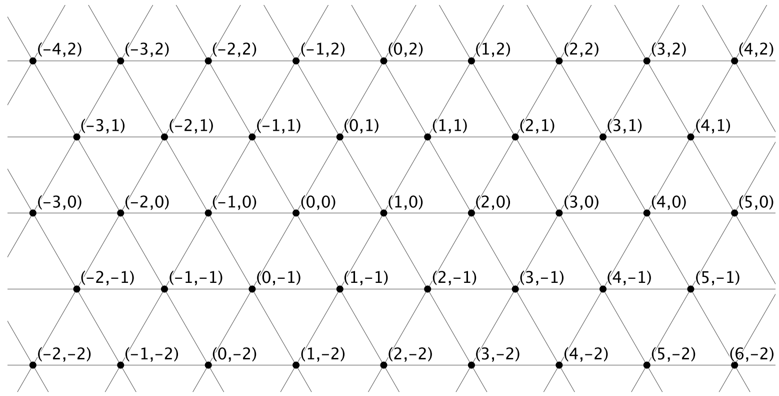

Figure 1: Triangular lattice graph

Definition 1.10. *Consider a proper three coloring of T_{3n}, with colors r, g, and b, so that vertex $(0,0)$ is colored r. Define the set of vertices colored r to be R_n. A hexagonal grid graph of order n is $H_n = T_{3n} - R_n$, namely the subgraph of T_{3n} induced by the vertices $(x,y) \in T_{3n}$ such that $x \not\equiv y \pmod 3$.*

2 Skolem Number of Triangular Grid Graph

While we have defined T_n as a graph induced by a subset of the vertices of the triangular lattice graph as we indicated, it is also common to define T_n on 3-tuple as shown in Figure 6.

Definition 2.1. *Let $T_n = (V,E)$ be a graph on the vertex set $V = (a,b,c)$, where a, b, and c are non-negative integers which sum to n and two vertices are adjacent if and only if the sum of their absolute differences is 2. (Otherwise stated as: (i_a, i_b, i_c) is adjacent to (j_a, j_b, j_c) $\iff |i_a - j_a| + |i_b - j_b| + |i_c - j_c| = 2$).*

Defining T_n in this fashion on vertices as 3-tuple helps us in defining the distance between two vertices. More specifically the definition provides us with the following observations.

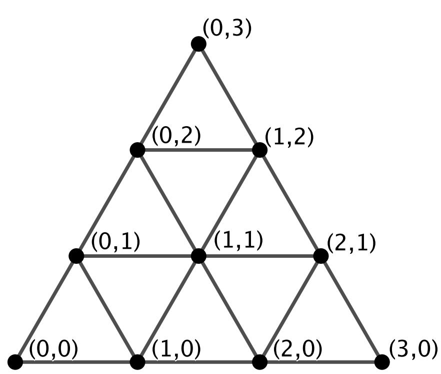

Figure 2: T_3 - Triangular grid graph

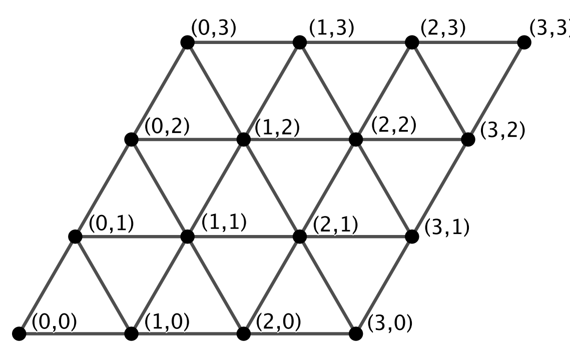

Figure 3: $P_{(4,4)}$ - Parallelogram grid graph

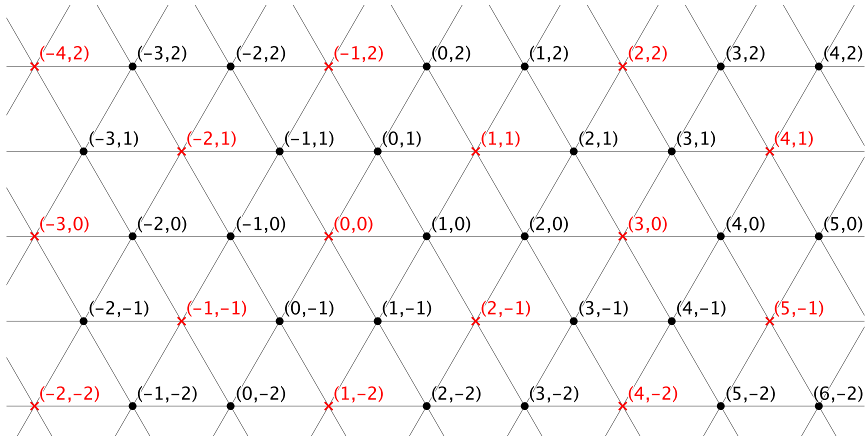

Figure 4: Creating the hexagonal lattice graph

Observation 2.2. If two vertices are adjacent, they have exactly one coordinate that is equal and the difference of the other two coordinates is one.

Observation 2.3. The distance between vertices $A = (a_x, a_y, a_z)$ and $B = (b_x, b_y, b_z)$ in a Triangular Grid Graph is $d(A, B) = \max(|a_x - b_x|, |a_y - b_y|, |a_z - b_z|)$.

Observation 2.4. For any positive integer d, there are at most 3 distinct vertices in T that are mutually distance d apart.

With these observations, we can proceed to find the Skolem Number of T_n.

Theorem 2.5. *The Skolem number of T_n is $\frac{n(n-1)}{2} + 1$ for all $n > 0$ such that $n \neq 3$, and the Skolem number of T_3 is 5.*

Proof. First we consider the case of $n = 3$. A proper Skolem labelling of T_3, with hook labels excluded, is provided in Figure 7.

Now we want to show that it is impossible for T_3 to have three distinct vertices labelled each of 1, 2, and 3.

By contradiction, assume that T_3 can be labelled where the integers of $\{1, 2, 3\}$ each appear three times. Since there is only one set of vertices in T_3 such that all vertices in the set are mutually distance 3 apart let $\mathcal{L}(3, 0, 0) = \mathcal{L}(0, 3, 0) = \mathcal{L}(0, 0, 3) = 3$. The subgraph induced by removing these vertices is symmetric on three axis, and $(1, 1, 1)$ is distance 1 from all other vertices on this subgraph, hence it shall not be labelled 2. Without loss of generality, define $\mathcal{L}(2, 0, 1) = \mathcal{L}(0, 1, 2) = \mathcal{L}(1, 2, 0) = 2$. Since $(1, 1, 1)$ is distance 1 from all the remaining vertices but all other remaining vertices are mutually distance 2 apart we have reached a contradiction that each label may appear three times.

Therefore we have shown the Skolem labelling in Figure 7 is optimal, hence the Skolem number of T_3 is 5.

We will proceed by way of induction for all $n \neq 3$. So Skolem labellings for T_1, T_2, and T_6 are provided in Figures 8, 9, and 10 respectively. Here we see that all integers less than n appear three times in the labelling, thus by Observation 2.4 we have the Skolem number

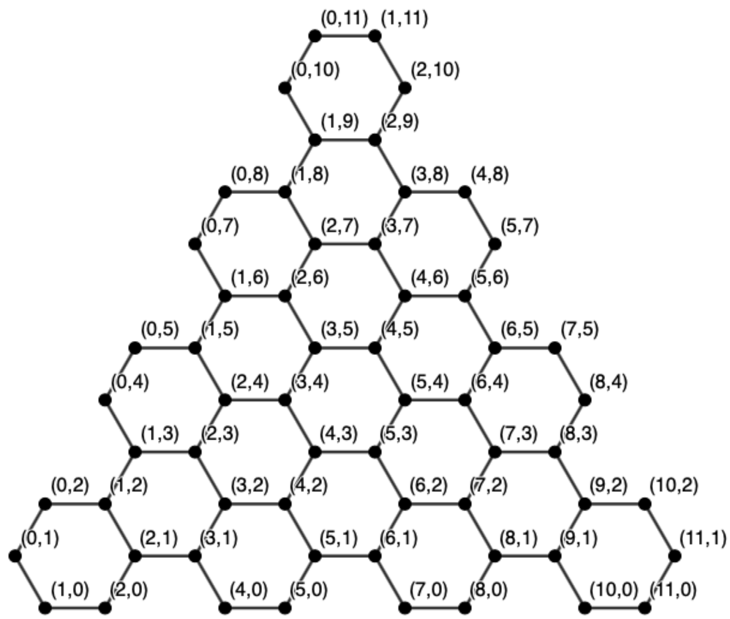

Figure 5: H_4 - Hexagonal grid graph

equal to the $(n+1)^{\text{st}}$ triangular number minus $2n$, or $\frac{(n+1)(n+2)}{2} - 2n = \frac{n(n-1)}{2} + 1$, for these triangular grid graphs.

For all $n > 3$, T_{n-3} is the vertex induced subgraph of T_n with all vertices in T_n that have no zero coordinates, using the 3-tuple definition. Let vertices $V \in T_n$ and $U \in T_{n-3}$ be (v_x, v_y, v_z) and (u_x, u_y, u_z) respectively. For all U such that $v_x, v_y, v_z \neq 0$ Define the mapping to be $f : V \to U = (v_x - 1, v_y - 1, v_z - 1)$. For induction we assume $\mathcal{L}'$ is proper Skolem Labelling of T_{n-3} of order $\frac{(n-3)(n-4)}{2}+1$. Now for all $n \neq 1$, 2, 3, or 6, we construct a labelling, $\mathcal{L}$, on T_n. For all vertices, v of T_n with no zero-coordinates, set $\mathcal{L}(v) = \mathcal{L}'(f(v))$ whenever $f(v)$ is not a hook vertex in the labelling $\mathcal{L}'$. Then define $\mathcal{L}(n,0,0) = \mathcal{L}(0,n,0) = \mathcal{L}(0,0,n) = n$, $\mathcal{L}(n-1,1,0) = \mathcal{L}(0,n-1,1) = \mathcal{L}(1,0,n-1) = n-1$, and $\mathcal{L}(n-2,2,0) = \mathcal{L}(0,n-2,2) = \mathcal{L}(2,0,n-2) = n-2$. Lastly, let all remaining unlabelled vertices be a hook vertex with a

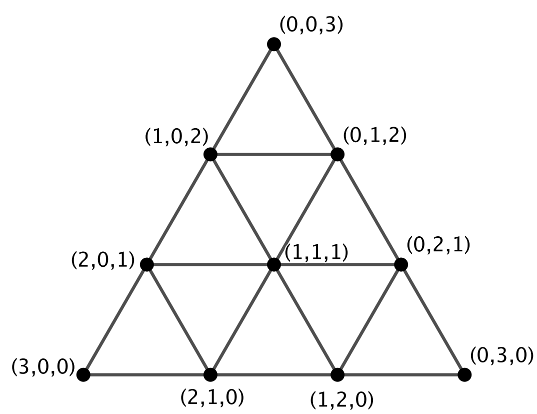

Figure 6: T_3 graph

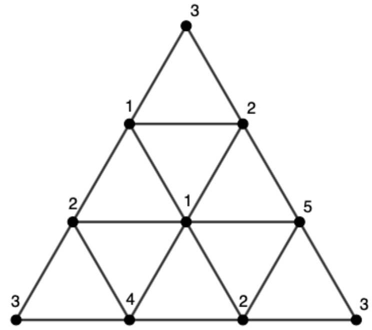

Figure 7: Skolem labelling T_3

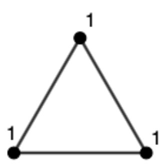

Figure 8: T_1

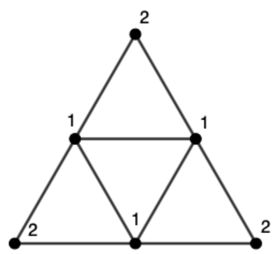

Figure 9: T_2

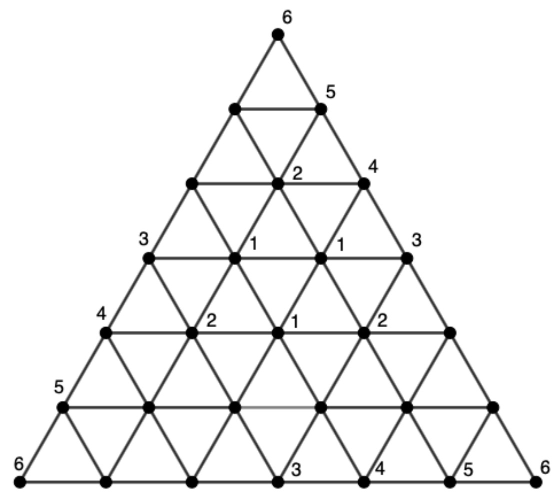

Figure 10: T_6

label between n and $\frac{n(n-1)}{2}+2$. $\mathcal{L}$ is a proper Skolem labelling of T_n that uses all positive integers less than $n+1$ exactly three times. Furthermore, since the diameter of T_n is n, $\mathcal{L}$ is a proper Skolem labelling of smallest order. Since there are $\frac{(n+1)(n+2)}{2}$ vertices in T_n and n labels appear three times, the Skolem number is $\frac{(n+1)(n+2)}{2} - 2n = \frac{n(n-1)}{2} + 1$. □

3 Skolem Number of Parallelogram Grid Graphs

Just like the results in [1], we will make use of the following two "Skolem type" labellings of a path.

Definition 3.1. *If k is even, there exists a proper Skolem labelling of P_k with the odd integers of the set $[k]$, called the odd Skolem labelled path as shown in Figure* 11*. If k is odd, there exists a proper Skolem labelling of P_{k+1} with the even integers of $[k]$ and one hook, called the even Skolem labelled path as shown in Figure* 12*.*

For the computation of the Skolem number, we will use the definition of the vertices as ordered pairs.

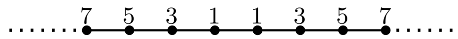

Figure 11: Odd Skolem labelled path

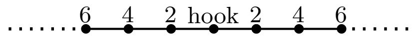

Figure 12: Even Skolem labelled path

Example 3.2. Let us begin by showing the Skolem number of the parallelogram graph $P_{(a,b)}$ when $a = 2$ is b. The lower bound argument, that no proper Skolem labelling of order less than b exists, can easily be extracted from the proof of Theorem 3.3. Thus we proceed by way of induction to show the existence of a proper Skolem labelling of order b. The proper Skolem labelling of $P_{(2,2)}$ (Figure 13) and $P_{(2,3)}$ (Figure 14) will serve as base cases.

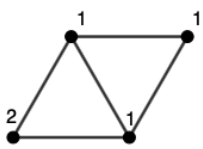

Figure 13: Skolem labelled $P_{(2,2)}$

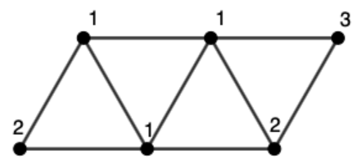

Figure 14: Skolem labelled $P_{(2,3)}$

Now assuming a proper Skolem labeling of order $b-2$ on $P_{(2,b-2)}$ exists. Here we will prescribe this labelling on the vertices of $P_{(2,b)}$ of degree 4, that is the vertices x, y where $x \neq 0$ or $b-1$. Then we will label vertices $(0,0)$ and $(b-1,1)$ with b and vertices $(0,1)$ and $(b-1,0)$ with $b-1$. Thus we have shown a proper Skolem labelling of $P_{(2,b)}$ of order b.

Theorem 3.3. *The Skolem number of $P_{(a,b)}$ is $ab-2a-b+4$ for all $1 < a \leq b$.*

Proof. Firstly, note that $P_{(a,b)}$ is a vertex induced subgraph of T_{a+b-2} and Observation 2.4 holds, hence no label may appear on more than three distinct vertices. Furthermore, the diameter of $P_{(a,b)}$ is $a+b-2$, so any label greater than $a+b-2$ must be a hook. However, there is only one pair of vertices possible to label $a+b-2$ but one of those vertices would be required for selecting a pair of vertices to label $a+b-3$. So we note that labels $a+b-2$ and $a+b-3$ cannot both exist twice, hence we can assume label $a+b-2$ is also a hook. Finally, since T_n for $n \geq a$ is not a subgraph of $P_{(a,b)}$, we can conclude no label greater than $a-1$ appears more than twice. This establishes the lower bound for the Skolem number to be $ab-(a+b-2)+1-(a-1) = ab-2a-b+4$.

Example 3.2 shows a labelling that satisfies this bound when $a = 2$. Now we aim to provide the necessary base cases and an inductive step to establish the existence of a Skolem labelling of $P_{(a,b)}$ of order $ab-2a-b+4$ when $a > 2$.

Base Cases: Figure 15 and Figure 16 illustrates a proper Skolem labelling for $P_{(a,a)}$ and $P_{(a,a+1)}$ respectively, except for when $a = 4$. Begin by labelling $(0,0)$, $(0,a-1)$ and $(a-1,0)$ with the integer $a-1$. Then label the T_{a-2} subgraph induced by the vertices (x,y) where $x \in \{1,2,\ldots,a-1\}$ and $x+y \leq a$ with the non hook labels of the Skolem labelling given in Theorem 2.5. (Since T_3 cannot be properly Skolem Labelled using three of each non-hook label we cannot use this construction for $a = 5$.)

Finally consider two paths on the boundary of $P_{(a,b)}$. Let U be the path of length $a+b-1$ formed by the vertices (x, y) where $x = 0$ or $y = a-1$ excluding the vertex $(0, 0)$ and L be the path of length $a+b-2$, formed by the vertices (x, y) where $y = 0$ or $x = b-1$ excluding $(0, 0)$ and $(b-1, a-1)$. When $a = b$, U is an odd length and L is an even length, hence each has a proper Skolem label of an even and odd path respectively and then the parity of each path is reversed when $b = a+1$. The vertices of $P_{(a,a)}$ and $P_{(a,a+1)}$ on the paths U and L that already contain a label are assigned either a hook, 1 or 2 in the respective labelling of these paths. Therefore we will use the assignments from the odd and even Skolem paths for any integer $x \geq a$ for labelling the vertices of U and L respectively.

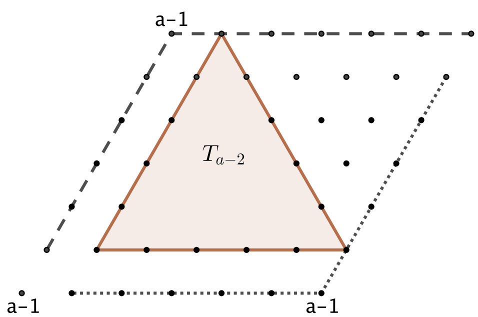

Figure 15: Skolem labelled $P_{(a,a)}$

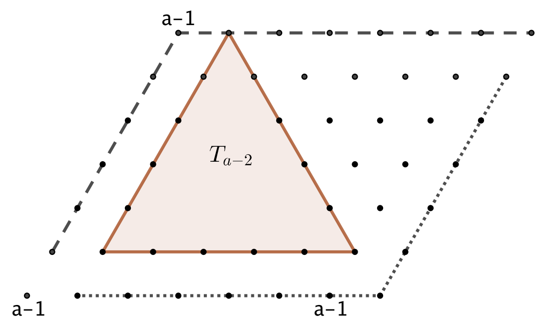

Figure 16: Skolem labelled $P_{(a,a+1)}$

Since Theorem 2.5 showed that it is not possible to properly Skolem label T_3 so that each integer $n \leq 3$ exists three times, we have provided a properly Skolem label of $P_{(5,5)}$ (Figure 17) and $P_{(5,6)}$ (Figure 18) that satisfy the given lower bound, where the hook labels are omitted.

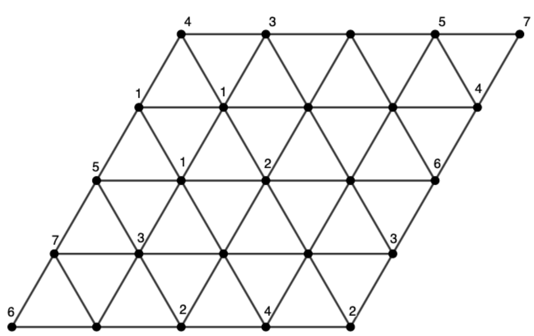

Figure 17: Skolem labelled $P_{(5,5)}$

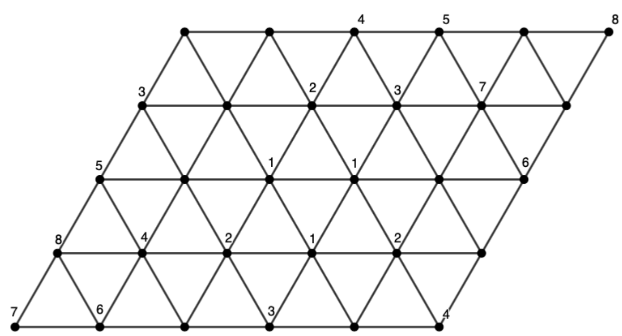

Figure 18: Skolem labelled $P_{(5,6)}$

Thus we have shown for all $P_{(a,b)}$, where $b = a$ or $a+1$ the Skolem number is $a \times b - 2a - b + 4$.

Inductive Step: Now we assume a proper Skolem labelling $\mathcal{L}'$ of $P_{(a,b-2)}$ of order $a(b-2) - 2a - (b-2) + 4$. We will use all the non-hook labels of $\mathcal{L}'$ to label the vertices of the $P_{(a,b-2)}$

subgraph induced by the vertices $(x, y) \in P_{(a,b)}$ such that $x \neq 0$ or $b-1$. Then label vertices $(0, 0)$ and $(b-1, a-3)$ with $a+b-4$ and vertices $(0, 1)$ and $(a-1, b-1)$ with $a+b-3$. Finally label each remaining unlabelled vertex with a hook using the consecutive integers $a+b-2$ to $ab-2a-b+4$. This process is illustrated in Figure 19.

Figure 19: Induction step

□

4 Skolem Number of Hexagonal Grid Graphs

One might first assume we can apply the results on T_{3n} to find the Skolem number of H_n, however the removal of vertices or edges from a graph drastically affects the Skolem number of it's subgraph. For instance in T_{3n} vertices $(0, 1)$ and $(2, 1)$ are distance 2 apart, but are distance 3 apart in H_n, while other pairs of vertices are the same distance apart in both graphs. Hence determining the Skolem number for H_n will require a new construction. When $n = 1$, we note $H_1 = C_6$, for which the Skolem number was given in Theorem 1.6, therefore we will assume $n > 1$.

Let us begin by making a few observations about H_n.

Observation 4.1. Just as in Observation 2.4, for any positive integer d, there are at most 3 distinct vertices in H that are mutually distance d apart.

Observation 4.2. Taking the 2-tuple coordinates of T_{3n} that sum to a multiple of 3, we see that there are exactly $1 + 3\binom{n+1}{2}$ vertices in R_n. Therefore H_n has $3n(n+1)$ vertices.

Observation 4.3. H_n is bipartite, so there is no set of three vertices mutually an odd distance apart.

Theorem 4.4. *The Skolem number of H_n, for $n \geq 2$, is $3n^2 - 3n + 3$.*

Proof. Observation 4.1 provides that no label may appear on more than three distinct vertices. Also, since H_n is bipartite (Observation 4.3), no odd integer can appear more than twice. Then, because the diameter of H_n is $4n-1$, any label greater than $4n-1$ must be a hook. However, there is exactly three pairs of vertices that are distance $4n-1$ apart and there is exactly two sets of three vertices that are mutually distant $4n-2$ apart. Unfortunately, each of the sets of three vertices that are mutually distance $4n-2$ apart contains a vertex from each of the pairs of vertices that are distance $4n-1$ apart, so these labels cannot both appear their maximum number of times in H_n. Hence we can assume label $4n-1$ is also a hook. This reasoning and Observation 4.2 establish the lower bound for the Skolem number to be $3n(n+1)-(4n-2)-(\frac{4n-2}{2}) = 3n^2-3n+3$.

Now, by way of induction, we will show a construction for proper Skolem labelling H_n so that all odd integers less than $4n-1$ appear twice and all even integers less than $4n-1$ appear thrice.

Base Case: Figure 20 shows a proper Skolem labeling of H_2 of order 9.

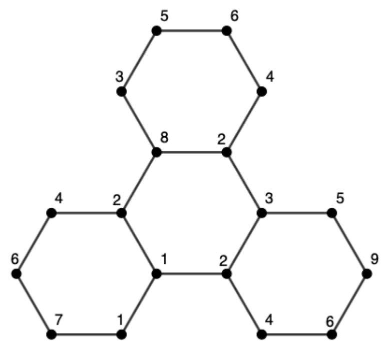

Figure 20: Skolem labelled H_2

Inductive Step: Assume there exists a proper Skolem labelling, $\mathcal{L}'$ of order $3(n-1)^2-3(n-1)+3$ on H_{n-1}. Label the $H_{(n-1)}$ subgraph induced by the vertices $(x, y) \in H_n$ such that $x \neq 0$ and $y \neq 0$ and $x+y \neq 3n$ with the non hook labels of $\mathcal{L}'$ in the same fashion as Theorem 2.5. Label vertices $(0,1)$, $(0,3n-1)$, and $(3n-1,0)$ with $4n-2$ and vertices $(0,2)$, $(2,3n-2)$, and $(0,3n-2)$ with $4n-4$ to use all the even integers less than $4n-1$ thrice. Then label vertices $(0,3n-1)$ and $(3n-2,2)$ with $4n-3$ and vertices $(0,3n-2)$ and $(3n-3,2)$ with $4n-5$. While $(3n-3,2)$ belongs to the H_{n-1} subgraph already labelled, our construction leaves vertex $(3n-1,1)$ as a hook vertex in H_n, thus $(3n-3,2)$ has yet to be labelled. Therefore we have used all odd integers less than $4n-1$ twice. Lastly, assign the remaining vertices as hooks on the integers $4n-1$ to $3n(n+1)$ and we have a proper Skolem labelling of order $3n^2-3n+3$ on H_n. □

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