An Archimedean Proposition Presented by the Brothers Banū Mūsā and Recovered in the *Kitāb al-Istikmāl* (eleventh century)

Lluís Pascual

Abstract: Archimedes developed a geometrical method to obtain an approximation to the value of π , that appears in the encyclopaedic work of the prince al-Mu'taman ibn Hūd in the eleventh century. This article gives the edition, translation and transcription of this Archimedean proposition in his work, together with some comments about how other medieval authors dealt with the same proposition.

Keywords: al-Mu'taman, Archimedes, *al-Istikmāl*, Banū Mūsā, Gerard of Cremona, al-Ṭūsī, the value of π, geometrical.

Appearance of the proposition in Saragossa during the eleventh century

There was a great burst of scientific activity in Saragossa during the eleventh century, especially in the fields of mathematics and philosophy. In mathematics, the most notable work produced in eleventh-century Saragossa was the « *Kitāb al-Istikmāl* », or « The book of perfection ». This encyclopaedic work was written by the prince al-Mu'taman ibn Hūd, who reigned from 1081 until the date of his death in 1085. His reign was very short, but during the very long reign of his father (al-Muqtadir ibn Hūd, king from 1046 to 1081) crown prince al-Mu'taman devoted himself to the study of mathematics, on subjects originated by Euclid and by others, and started the composition of the « *Kitāb al-Istikmāl* ».

As Hogendijk has noted (Hogendijk, 1991) and (Hogendijk, 1995), al-Mu'taman sometimes transcribes the propositions as if he had copied them from their Euclidian origin, but more often he takes the initiative himself: he simplifies demonstrations, he merges symmetrical propositions into a single one, and he often changes the flow of the demonstration. In this way he shows that he was not a mere copyist, but a researcher who introduced innovations.

After ten other propositions (some of them new, others related to the contents of book XIII of Euclid's *Elements*) al-Mu'taman rather unexpectedly terminates sub-species 3.2 of « *al-Istikmāl »* with proposition 11 in chapter 3.2.2 ([1]). Being a good mathematician, he is unlikely to have put this proposition here purely on a whim; the decision probably responds to the internal logic of the general layout that the author has adopted in the *Kitāb al-Istikmāl.*

The aim of proposition 11 is to provide a geometrical approximation to the measurement of π, probably taken from proposition 6 of the « *Book of measurement of plane and spherical figures* » by the brothers Banū Mūsā. This book was the object of a translation into Latin by Gerard of Cremona in the twelfth century, and also of a rewriting in the thirteenth century by Naṣīr al-Dīn al-Ṭūsī (m. 1274), founder of the Marāga observatory in Persia.

This book was a key text in the geometry of the Middle Ages, and traces of its contents can be found in works by Arabic and European authors, such as Thābit ibn Qurra, Ibn Haytham (d. 1039), Leonardo Fibonacci of Pisa (d. 1250), Jordanus de Nemore (d. 1260), and Roger Bacon (d. 1294). (Casulleras, 2007, pages 92 - 94).

History of the proposition

The history of this geometrical method in the classical and medieval times can be divided in several well-known steps. However, between the end of the ninth century and the middle of the eleventh century, other authors in Islamic countries may have published a text on this subject. (Rashed, 1993).

Archimedes

The geometrical method that permits to obtain an approximation to the value of π has its origin in proposition 3 of the work « *The measurement of the circle* », in Greek «*Κύκλου μέτρησις*», [*Kuklou metrēsis*], by Archimedes of Syracuse (c.287 BC - 212 BC).

[1] The logical classification of knowledge established by Aristotle had reached al-Andalus through the writings of al-Fārābī (Forcada, 2006). As a result the prince distributes the parts of the mathematical science according to the disciplines defined by the philosophers, and divides its contents according to Porphyry's predicables: genus, species, sub-species, and chapter. (Hogendijk, 1991) and (Forcada 2011, pages 226 -227). Thus 3.2.2 means (for genus 1) species 3, sub-species 2, chapter 2.

The text is to be found in Greek and in French in (Mugler, 1970, pages 140 – 143), and in English in (Heath, 1897, pages 93 - 98).

Archimedes' text is quite succinct, because he only gives the results of his calculations. Fuller comments were added later by Eutocius of Ascalon (c.480 – c.540), a Greek mathematician, who revived the works of Apollonius and Archimedes. The proposition is quoted in the works of Ptolemy (c.90 – c.168), Simplicius (c.490 – c.560) and Heron of Alexandria (c.10 – 70).

Having seen that, if the diameter of a circle has a length of one, its perimeter escapes an exact measurement, Archimedes draws regular polygons in the interior and in the exterior of the circle, and he increases the number of their sides from 6 (he begins with a regular hexagon) up until 96.

The procedure leads him to give a value between $3+\frac{10}{71}$ and $3+\frac{10}{70}$ for the ratio between perimeter and diameter.

Banū Mūsā

At the beginning of the ninth century, at the time of the large-scale translation movement in Abbasid Baghdad which produced Arabic versions of Greek and Sanskrit texts, this method of geometrical approximation to the value of π reappears in proposition 6 of the book « *Kitāb fī maʿrifat misāḥat al-ashkāl al-basīṭa wa al-kuriyya* », « *Book to ascertain the measurement of plane and spherical figures* », by the brothers Banū Mūsā (Muḥammad, Aḥmad and al-Ḥasan, sons of Mūsā ibn Shākir).

Sadly the Arabic version of this book has not survived, but the contents of the proposition can be deduced; its demonstration in later works, in Arabic and in Latin, bear witness to the book's wide dissemination among the mathematicians of the Middle Ages.

Al-Mu'taman

A new version of the method appears in Saragossa at the end of the eleventh century, in proposition 11 of the second part of sub-species 3.2 of the encyclopaedic work « *Kitāb al-Istikmāl » (Book of perfection)* by the prince of Saragossa al-Mu'taman ibn Hūd.

In the present article, al-Mu'taman's text of Archimedes' proposition is edited, translated and analysed.

Gerard of Cremona

A Latin translation by Gerard of Cremona (c.1114 – 1187) appears in the twelfth century (Clagett, 1964, pages 264 - 279).

The author translates the work of the brothers Banū Mūsā under the title « *Liber trium fratrum de geometria et Verba filiorum Moysi filii Sekir, id est Maumeti, Hameti et Hasen* ».

Al-Ṭūsī

A new version of the proposition is presented by Naṣīr al-Dīn al-Ṭūsī (1201 – 1274) in the mid-thirteenth century, either in 1255, or in 1260.

It is a rewriting or a new redaction of Banū Mūsā's book. Al-Ṭūsī retakes the original text, simplifies the steps of the demonstration that he regards as unnecessary, and eliminates all the introductory sentences that were retained in the Latin translation by Gerard of Cremona; however, he does not alter the mathematical text (Rashed, 1996, pages 74 - 83).

Proposition 11 by al-Mu'taman ibn Hūd

The current article contains the analysis of proposition 11 in chapter 2 of the sub-species 3.2 in « *al-Istikmāl* », which describes Archimedes' method for the evaluation of π.

Transcription in standard Arabic

The proposition has been edited from the contents of two manuscripts:

Manuscript of Leiden, Bibliotheek der Rijksuniversiteit, mss Or123a (L) Fols. 45 v – 49 r.

Manuscript of Copenhagen, Kongelige Biblioteket Kobenhavn, mss Or82 (K), Fols. 60 r – 61 r.

Rules and conventions adopted in the edition of the Arabic text

The edition of the Arabic text respects the texts of the manuscripts, but the two present several differences:

Names of the points and segments

(L) gives the points a name identified by a letter: A (ا) , B (ب).

(K) uses for a point the name of the Arabic letter: alif (ألف) , bā' (باء).

This edition will use the naming in (L).

Reconstruction of diacritical signs omitted in the consonants

Diacritics in the consonants are very often omitted by the copyist in (L), but they are generally present in (K). This edition restores them without notes. The hamza is written according to contemporary rules.

The word qaws (قوس)

The word qaws (arc) may be feminine or masculine. It is feminine in (K), but masculine in (L); the relative pronouns and the verbs that follow are consequently in agreement. This edition of the text takes it to be feminine (as is more customary in Arabic geometry), without notes.

// ل : 45 ظ// ك : 60 ظ/ [المبرهنة 11 (يا)]

كل خطّ يُحيط بدائرة، فإنّه زائد على ثلاثة أضعاف قطرها بأقلّ من سُبع القطر [1]، وبأكثر من عشرة أجزاء من واحد وسبعين جزءًا من القطر.

[مثاله :] فنخطّ دائرة **ا ط ب** [2]، وقطرها **ا ب** ، ومركزها نقطة **جـ** ، ونخرج خطّ **جـ ز** يُحيط مع خطّ **ب جـ** بثُلث زاوية قائمة، ونقيم على نقطة **ب** من خطّ **ب جـ** خطًّا على زاوية قائمة، وهو خطّ **ب ز** . فبيّن أن القوس التي توتّر زاوية **ب جـ ز** نصف سُدس محيط دائرة **ا ط ب** ، وأن خطّ **ب ز** نصف ضلع المسدّس المحيط بدائرة **ا ط ب** . ونقسم زاوية **ب جـ ز** بنصفين بخطّ **جـ هـ** ، ونقسم زاوية **ب جـ هـ** بنصفين بخطّ **جـ و** ، ونقسم زاوية **ب جـ و** بنصفين بخطّ **جـ د** ، ونقسم زاوية **ب جـ د** بنصفين بخطّ **جـ ح** .

فبيّن أن القوس التي توتّر زاوية **ب جـ ح** هي جزء من اثنين وتسعين جزءًا ومائة جزء من الخطّ المحيط بدائرة **ا ط ب** ، وأن خطّ **ب ح** نصف ضلع ذي الستة وتسعين ضلعًا المحيط بدائرة **ا ط ب** .

وإذا كان هذا هكذا، فإنا نصير خطّ **جـ ز** ثلاثمائة وستّة لسهولة استعمال هذا العدد فيما يحسب. فإذا كان خطّ **جـ ز** ثلاثمائة وستّة كان مربّعه ثلاثة وتسعين ألفًا [3] وستّمائة وستّة وثلاثين، وكان خطّ **ب ز** مائة وثلاثة **// ل : 46 و/** وخمسين، لأن زاوية **ب جـ ز** ثُلث زاوية قائمة، وزاوية **جـ ب ز** قائمة. وكان مربّع خطّ **ب ز** ثلاثة وعشرين ألفًا وأربعمائة وتسعة، ومربّع خطّ **جـ ب** سبعون ألفًا ومائتان وسبعة وعشرون. فخطّ **جـ ب** أكثر من مائتين وخمسة وستّين. ولكن نسبة خطّي **ب جـ جـ ز** مجموعين إلى **ب ز** كنسبة **ب جـ** إلى **ب هـ** من أجل أن خطّ **جـ هـ** يقسم زاوية **ب جـ ز** بنصفين، وخطّا **ب جـ جـ ز** مجموعان أكثر من خمسمائة وواحد وسبعين، وخطّ **ب ز** مائة وثلاثة وخمسون. فنسبة **جـ ب** إلى **ب هـ** أعظم من نسبة خمسمائة وواحد وسبعين إلى مائة وثلاثة وخمسين [4]. فيكون خطّ **جـ ب** أكثر من **// ك : 61 و/** خمسمائة وواحد وسبعين. إذا كان **ب هـ** مائة وثلاثة وخمسين، ومربّع **جـ ب** أكثر من ثلاثمائة ألف وستّة وعشرين ألفًا وواحد وأربعين، ومربّع **ب هـ** [5] ثلاثة وعشرين

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1 - القطر في (ك)، قطر في (ل)

2 - ا ط ب في (ل)، ا ط في (ك)

3 - ألفًا : ألف في (ك)

4 - العدد الصحيح في حافة الصفحة (ك)

5 - خط ب هـ : به (ل)

ألفًا وأربعمائة وتسعة. فمربّع **جـ هـ** أكثر من ثلاثمائة ألف وتسعة وأربعين ألفًا وأربعمائة وخمسين. فخطّ **جـ هـ** أكثر من خمسمائة وواحد وتسعين وثُمن واحد.

وعلى المثال الذي وصفنا يتبيّن أن نسبة خطّ **جـ ب** إلى **ب و** أعظم من نسبة ألف ومائة واثنين وستّين وثُمن واحد إلى مائة وثلاثة وخمسين. فإذا كان **ب و** مائة وثلاثة وخمسين، **// ل : 46 ظ/** كان **جـ ب** أكثر من ألف ومائة واثنين وستّين وثُمن، ومربّع **جـ ب** أكثر من ألف ألف وثلاثمائة ألف وخمسين ألفًا وخمسمائة وأربعة [6] وثلاثين، ومربّع **ب و** ثلاثة وعشرين ألفًا وأربعمائة وتسعة، ومربّع **جـ و** أكثر من ألف ألف وثلاثمائة ألف وثلاثة وسبعين ألفًا وتسعمائة وثلاثة وأربعين. فخطّ **جـ و** أكثر من ألف ومائة واثنين وسبعين وثُمن.

وعلى هذا المثال الذي وصفنا يتبيّن أن نسبة **جـ ب** إلى **د ب** أعظم من نسبة ألفين وثلاثمائة وأربعة وثلاثين ورُبع إلى مائة وثلاثة وخمسين. فإذا كان خطّ **ب د** مائة وثلاثة وخمسين، كان **جـ ب** أكثر من ألفين وثلاثمائة وأربعة وثلاثين ورُبع، ومربّع **جـ ب** [7] أكثر من خمسة آلاف ألف وأربعمائة ألف وثمانية وأربعين ألفًا وسبعمائة وثلاثة وعشرين، ومربّع **ب د** ثلاثة وعشرون ألفًا وأربعمائة وتسعة، ومربّع **جـ د** أكثر من خمسة آلاف ألف وأربعمائة ألف واثنين وسبعين ألفًا واثنين وثلاثين. فخطّ **جـ د** أكثر من ألفين وثلاثمائة وتسعة وثلاثين ورُبع.

وعلى هذا المثال الذي وصفنا يتبيّن أن نسبة **جـ ب** إلى **ب ح** أعظم من نسبة أربعة آلاف وستّمائة وثلاثة وسبعين ونصف واحد إلى مائة وثلاثة وخمسين. فإذا كان خطّ **ح ب** مائة وثلاثة وخمسين **// ل : 47 و/** كان خطّ **جـ ب** أكثر من أربعة آلاف وستّمائة وثلاثة وسبعين ونصف. وهذا هو قدر ضلع ذي ستّة وتسعين ضلعًا عند [8] القطر. فقدر القطر عند جماعة أضلاع ذي ستّة وتسعين ضلعًا الذي يُحيط [9] بالدائرة أعظم من قدر أربعة آلاف وستّمائة وثلاثة وسبعين ونصف إلى أربعة عشر ألفًا وستّمائة وثمانية وثمانين. فقد تبيّن أن قدر جماعة أضلاع ذي ستّة وتسعين ضلعًا عند القطر أقلّ من ثلاثة وسُبع من الواحد.

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6 - العدد الصحيح في حافة الصفحة (ك)

7 - جـ ب : جـ د، في (ل) و (ك)

8 - كلمات بين "كان خطّ" وبين "ضلعًا عند" مطموسة عند صفحة 47و في (ل)

9 - "الذي" ناقصة في (ل) و (ك)، "يحيط" مطموسة في (ل)

ثم نخرج في دائرة **اط ب** وتر المسدّس، وهو **ط ب** ، ونخرج **اط** ، ونقسم زاوية **ط ا ب** بنصفين بخطّ **اع ت** ، ونخرج وتر **ت ب** ، ونقسم زاوية **ت ا ب** بنصفين بخطّ **اك** ، ونخرج وتر **ك ب** ، ونقسم زاوية **ك ا ب** [10] بنصفين بخطّ **ال** ، ونخرج [11] وتر **ل ب** ، ونقسم زاوية **ل ا ب** بنصفين بخطّ **ام** ، ونخرج وتر **م ب** .

فتبيّن أن وتر **م ب** هو ضلع ذي ستّة وتسعين ضلعًا التي تُحيط به الدائرة. ثم نجعل خطّ **ا ب** ألفًا وخمسمائة وستّين لسهولة استعمال هذا العدد فيما نريد. فيكون وتر **ط ب** سبعمائة وثمانين، ويكون مربّع **ا ب** ألفي ألف وأربعمائة ألف وثلاثة وثلاثين ألفًا وستّمائة، ومربّع **ط ب** ستّمائة ألف وثمانية آلاف وأربعمائة. ويكون مربّع **اط** // **ل : 47 ظ/** ألف ألف وثمانمائة ألف وخمسة وعشرين ألفًا ومائتين. فخطّ **ط ا** أقلّ من ألف وثلاثمائة وواحد وخمسين. ولكن نسبة خطّي **ط ا** ، **ا ب** مجموعين إلى **ط ب** كنسبة **اط** إلى **ط ع** . ونسبة **اط** إلى **ط ع** كنسبة **ات** إلى **ت ب** ، وخطّا **ط ا** **ا ب** مجموعان [12] أقلّ من ألفين وتسعمائة وأحد عشر. فنسبة **ات** إلى **ت ب** أقلّ من نسبة ألفين وتسعمائة وأحد عشر إلى سبعمائة وثمانين. وإذا كان **ت ب** سبعمائة وثمانين، كان **ات** أقلّ من ألفين وتسعمائة وأحد عشر، ومربّع **ات** أقلّ من ثمانية آلاف ألف وأربعمائة ألف وثلاثة وسبعين ألفًا وتسعمائة واحد وعشرين، ومربّع **ت ب** ستّمائة ألف وثمانية آلاف وأربعمائة، ومربّع **ا ب** أقلّ من تسعة آلاف ألف واثنين وثمانين ألفًا وأحد وعشرين. فخطّ **ا ب** // **ك : 61 و/** أقلّ من ثلاثة آلاف وثلاثة عشر وثلاثة أرباع.

وعلى هذا المثال الذي وصفنا يتبيّن أن نسبة **اك** إلى **ك ب** أقلّ من نسبة خمسة آلاف وتسعمائة وأربعة وعشرين وثلاثة أرباع واحد إلى سبعمائة وثمانين. فإذا [13] كان خطّ **ك ب** سبعمائة وثمانين، كان خطّ **اك** أقلّ من خمسة آلاف وتسعمائة وأربعة وعشرين وثلاثة أرباع. وقدر خمسة آلاف وتسعمائة وأربعة وعشرين وثلاثة أرباع عند سبعمائة // **ل : 48 و/** وثمانين كقدر ألف وثمانمائة وثلاثة وعشرين عند مائتين وأربعين. فإذا كان خطّ **ك ب** مائتين وأربعين، كان **اك** أقلّ من ألف وثمانمائة وثلاثة وعشرين، ومربّع **اك** أقلّ من ثلاثة آلاف ألف وثلاثمائة ألف وثلاثة وعشرين ألفًا وثلاثمائة وتسعة وعشرين، ومربّع **ك ب** سبعة وخمسون ألفًا وستّمائة، ومربّع **ا ب** أقلّ من ثلاثة آلاف ألف وثلاثمائة ألف وثمانين ألفًا وتسعمائة وتسعة وعشرين. فخطّ **ا ب** أقلّ من

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[10] - ك ا ب : ك ا

[11] - ونخرج : ونقسم

[12] - كلمات بين "إلى طب" وبين "طا اب مجموعان" ناقصة في (ك)

[13] - فإذا في (ل) ، فإن في (ك)

ألف وثمانمائة وثمانية وثلاثين وتسعة أجزاء من أحد عشر.

وعلى هذا [14] المثال الذي وصفنا يتبيّن أن نسبة **ا ل** إلى **ل ب** أقلّ من نسبة ثلاثة آلاف وستّمائة واحد وستّين وتسعة أجزاء من أحد عشر عند مائتين وأربعين، التي هي قدر ألف وسبعة عند ستّة وستّين. فإذا كان خطّ **ل ب** ستّة وستّين، كان خطّ **ا ل** أقلّ من ألف وسبعة، ومربّع **ا ل** أقلّ من ألف ألف وأربعة عشر ألفًا وتسعة وأربعين. ومربّع **ل ب** أربعة آلاف وثلاثمائة وستّة وخمسون، ومربّع **ا ب** أقلّ من ألف ألف وثمانية عشر ألفًا وأربعمائة وخمسة. فخطّ **ا ب** أقلّ من ألف وتسعة وسُدس.

وعلى المثال الذي وصفنا يتبيّن أن قدر **ا م** عند **م ب** أقلّ من قدر ألفين وستّة عشر **// ل : 48 ظ/** وسُدس عند ستّة وستّين. فإذا كان **م ب** ستّة وستّين، كان **ا م** أقلّ من ألفين وستّة عشر وسُدس، ومربّع **ا م** [15] أقلّ من أربعة آلاف ألف وأربعة وستين ألفًا وتسعمائة وثمانية وعشرين، ومربّع **ا ب** أقلّ من أربعة آلاف وتسعة وستّين ألفًا ومائتين وثمانين. فخطّ **ا ب** أقلّ من ألفين وسبعة عشر وربُع. ولكن خطّ **م ب** بهذا المقدار ستّة وستّون، وخطّ **م ب** هو ضلع ذي ستّة وتسعين ضلعًا، الذي تُحيط به الدائرة. فنسبة القطر إلى أضلاع ذي ستّة وتسعين ضلعًا، الذي تُحيط به الدائرة، أقلّ من نسبة ألفين وسبعة عشر إلى ستّة آلاف وثلاثمائة وستّة وثلاثين.

فقد تبيّن أن نسبة جميع أضلاع ذي ستّة وتسعين ضلعًا، الذي تُحيط به الدائرة، إلى القطر، أعظم من نسبة ثلاثة وعشرة أجزاء من واحد وسبعين، إلى واحد، والخطّ المحيط بالدائرة أطول من جماعة أضلاع ذي ستّة وتسعين ضلعًا الذي تُحيط به الدائرة، وأقصر من جماعة أضلاع ذي ستّة وتسعين ضلعًا الذي تُحيط به الدائرة.

فقد صحّ [16] ممّا وصفنا أن نسبة الخطّ المحيط بالدائرة إلى قطرها أعظم من نسبة ثلاثة وعشرة أجزاء من واحد وسبعين جزءًا، وأقلّ من [17] نسبة ثلاثة وعشرة أجزاء من سبعين.

وذلك ما أردنا أن نبيّن.

// ل : 49 و/

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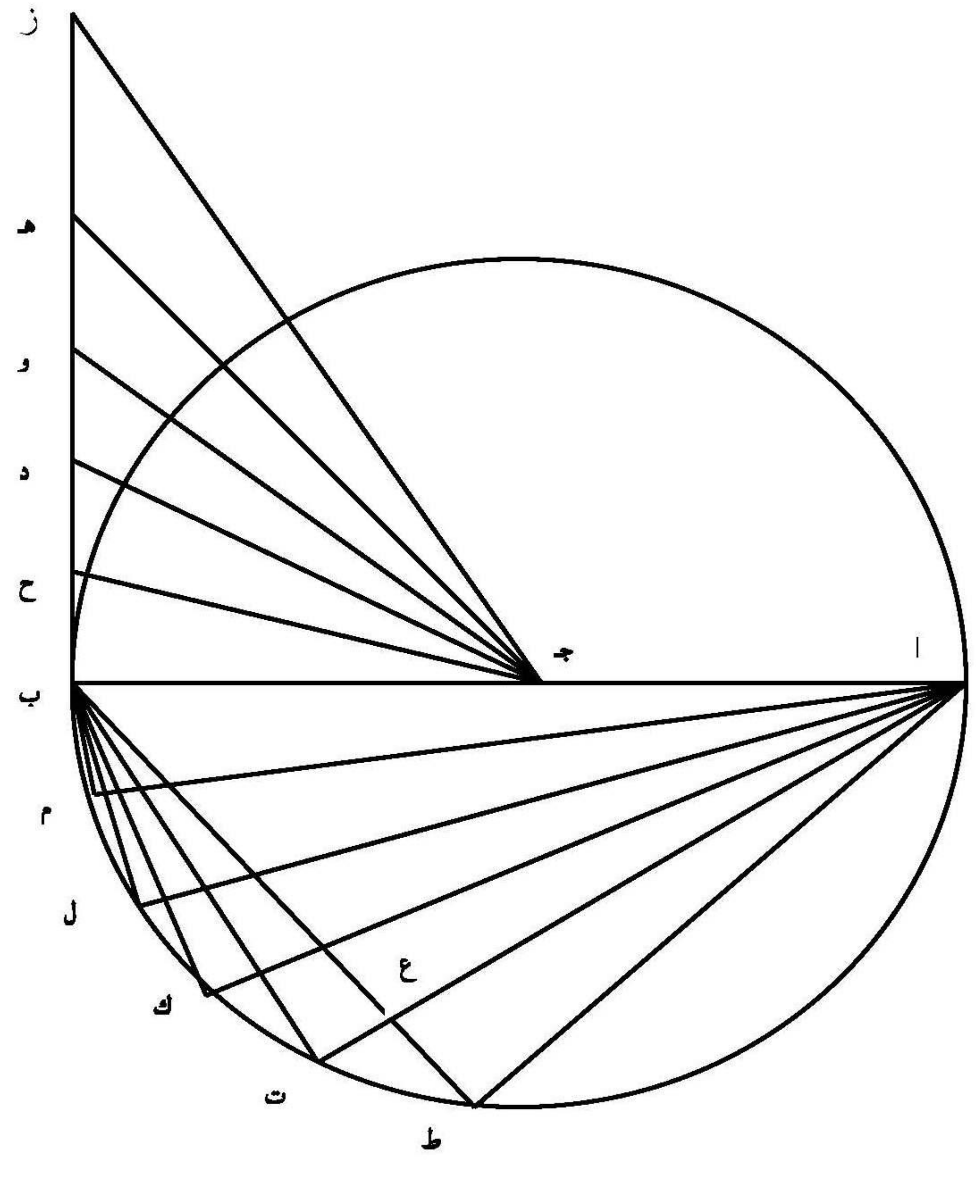
[14] - كلمة ناقصة في (ك)

[15] - ا م في (ل) ، ا جـ في (ك)

[16] - صحّ في (ل) ، تبيّن في (ك)

[17] - كلمات "جزءا وأقل من" مطموسة في (ك)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English translation

Concerning the geometrical letters: In the figures of the proposition, the transcription of the Arabic to Latin letters has been done following the convention published by Kennedy (Kennedy, 1983, page 745).

Al-Mu'taman uses a supplementary letter, namely tā' (ت), which is quite uncommon in geometrical figures in the Arabic language. It has been transcribed here by letter V.

[Lines 1 – 3 of the Arabic text]

Proposition 11

The length of the circumference exceeds three times the diameter by [a length] less than 1/7 of the diameter, and more than 10/71 of the diameter.

[Lines 5 – 11 of the Arabic text]

[Example:] We draw circle $\overline{ATB}$, and its diameter $\overline{AB}$, and its centre is at point $\overline{G}$, and we draw line $\overline{GZ}$ that encloses with line $\overline{BG}$ a third of a square angle, and we draw at point $\overline{B}$ a line perpendicular to line $\overline{BG}$, which is $\overline{BZ}$. It is clear that the arc intercepting angle $\overline{BGZ}$ is a half of a sixth of the perimeter of circle $\overline{ATB}$, and that line $\overline{BZ}$ is half the side of the hexagon circumscribed around circle $\overline{ATB}$. And we divide angle $\overline{BGZ}$ in two halves by means of line $\overline{GE}$, and we divide angle $\overline{BGE}$ in two halves by means of line $\overline{GW}$, and we divide angle $\overline{BGW}$ in two halves by means of line $\overline{GD}$, and we divide angle $\overline{BGD}$ in two halves by means of line $\overline{GH}$.

[Lines 12 – 14 of the Arabic text]

It is clear that the arc intercepting angle $\overline{BGH}$ is equal to one part of the 192 parts of the perimeter of circle $\overline{ATB}$, and that line $\overline{BH}$ is equal to half one side of the 96 sides [of a polygon] circumscribed around circle $\overline{ATB}$.

[Lines 15 – 28 of the Arabic text]

And, if it is so, then we give to line $\overline{GZ}$ [a length of] 306 to make the use of this number easier wherever it fits. Then if line $\overline{GZ}$ [measures] 306, its square is 93.636, and line $\overline{BZ}$ [measures] 153, because angle $\overline{BGZ}$ is a third of a square angle, and angle $\overline{GBZ}$ is square. And the square of line $\overline{BZ}$ is 23.409, and the square of line $\overline{GB}$ is 70.227. Thus line $\overline{GB}$ is longer than 265. But the ratio of the sum of the two lines $\overline{BG}$ and $\overline{GZ}$ to $\overline{BZ}$ is equal to the ratio of $\overline{GB}$ to $\overline{BE}$, so that line $\overline{GE}$ divides angle $\overline{BGZ}$ in two halves, and the sum of the two lines $\overline{BG}$

and $\overline{GZ}$ is greater than 571, and line $\overline{BZ}$ [measures] 153. Then the ratio of $\overline{GB}$ to $\overline{BE}$ is greater than the ratio of 571 to 153. Thus line $\overline{GB}$ is longer than 571. If $\overline{BE}$ [measures] 153, and the square of $\overline{GB}$ is greater than 326.041, and the square of $\overline{BE}$ is 23.409, then the square of $\overline{GE}$ is greater than 349.450. Thus line $\overline{GE}$ is longer than 591 plus one eighth.

[Lines 29 – 35 of the Arabic text]

And if we follow the example given, it is clear that the ratio of line $\overline{GB}$ to $\overline{BW}$ is greater than the ratio of 1.162 plus one eighth to 153. Then if $\overline{BW}$ [measures] 153, $\overline{GB}$ is longer than 1.162 plus one eighth, and the square of $\overline{GB}$ is greater than 1.350.534, and the square of $\overline{BW}$ is 23.409, and the square of $\overline{GW}$ is greater than 1.373.943. Then line $\overline{GW}$ is longer than 1.172 plus one eighth.

[Lines 36 – 42 of the Arabic text]

And if we follow the example given, it is clear that the ratio of $\overline{GB}$ to $\overline{BD}$ is greater than the ratio of 2.334 plus one fourth to 153. Thus if line $\overline{BD}$ [measures] 153, $\overline{GB}$ is longer than 2.334 plus one fourth, and the square of $\overline{GB}$ is greater than 5.448.723, and the square of $\overline{BD}$ is 23.409, and the square of $\overline{GD}$ is greater than 5.472.032. Thus line $\overline{GD}$ is longer than 2.339 plus one fourth.

[Lines 43 – 50 of the Arabic text]

And if we follow the example given, it is clear that the ratio of $\overline{GB}$ to $\overline{BH}$ is greater than the ratio of 4.673 plus one half to 153. Thus if line $\overline{HB}$ [measures] 153, line $\overline{GB}$ is longer than 4.673 plus one half. And that is the ratio of the side of a [polygon] of 96 sides to the diameter. Thus the ratio of the diameter to the sum of the sides of a [polygon] of 96 sides circumscribed around the circle is greater than the ratio of 4.673 plus one half to 14.688. Then it is clear that the ratio of the sum of the 96 sides [of the polygon] to the diameter is smaller than 3 plus one seventh of a unit.

[Lines 51 – 54 of the Arabic text]

Then we draw within circle $\overline{ATB}$ a cord of a sixth [of the circle], and it is $\overline{TB}$, and we draw $\overline{AT}$, and we cut angle $\overline{TAB}$ in two halves with line $\overline{AOV}$, and we draw the cord $\overline{VB}$, and we cut angle $\overline{VAB}$ in two halves with line $\overline{AK}$, and we draw the cord $\overline{KB}$, and we cut angle $\overline{KAB}$ in two halves with line $\overline{AL}$, and we draw the cord $\overline{LB}$, and we cut angle $\overline{LAB}$ in two halves with line $\overline{AM}$, and we draw the cord $\overline{MB}$.

[Lines 55 – 68 of the Arabic text]

Then it is clear that cord $\overline{MB}$ is the side of a [polygon] of 96 sides inscribed in the circle. Then we give to line $\overline{AB}$ [a length of] 1.560 to make easier the use of this number wherever it fits. Then the cord $\overline{TB}$ measures 780, and the square of $\overline{AB}$ is 2.433.600, and the square of $\overline{TB}$ measures 680.400. And the square of $\overline{AT}$ measures 1.825.200. Then line $\overline{TA}$ is shorter than 1.351. But the ratio of the sum of the two lines $\overline{TA}$ and $\overline{AB}$ to $\overline{TB}$ is equal to the ratio of $\overline{AT}$ to $\overline{TO}$. And the ratio of $\overline{AT}$ to $\overline{TO}$ is equal to the ratio of $\overline{AV}$ to $\overline{VB}$, and the sum of the two lines $\overline{TA}$ and $\overline{AB}$ is shorter than 2.911. Thus the ratio of $\overline{AT}$ to $\overline{TO}$ is smaller than the ratio of 2.911 to 780. And if $\overline{VB}$ measures 780, $\overline{AV}$ is shorter than 2.911, and the square of $\overline{AV}$ is smaller than 8.473.921, and the square of $\overline{VB}$ measures 608.400, and the square of $\overline{AB}$ is smaller than 9.082.021. Thus line $\overline{AB}$ is shorter than 3.013 and three fourths of a unit.

[Lines 69 – 78 of the Arabic text]

And if we follow the example given, it is clear that the ratio of $\overline{AK}$ to $\overline{KB}$ is smaller than the ratio of 5.924 and three fourths of a unit to 780. Then if line $\overline{KB}$ measures 780, line $\overline{AK}$ is shorter than 5.924 and three fourths of a unit. And the ratio of 5.924 and three fourths of a unit to 780 is equal to the ratio of 1.823 to 240. Then if line $\overline{KB}$ measures 240, $\overline{AK}$ is shorter than 1.823, and the square of $\overline{AK}$ is smaller than 3.323.329, and the square of $\overline{KB}$ is 57.600, and the square of $\overline{AB}$ is smaller than 3.380.929. Thus line $\overline{AB}$ is shorter than 1.838 and 9/11 of a unit.

[Lines 79 – 84 of the Arabic text]

And if we follow the example given, it is clear that the ratio of $\overline{AL}$ to $\overline{LB}$ is smaller than the ratio of 3.661 and 9/11 of a unit to 240, which is equal to the ratio of 1.007 to 66. Then if line $\overline{LB}$ measures 66, line $\overline{AL}$ is shorter than 1.007, and the square of $\overline{AL}$ is smaller than 1.014.049. And the square of $\overline{LB}$ measures 4.356, and the square of $\overline{AB}$ is smaller than 1.018.405. Thus line $\overline{AB}$ is shorter than 1.009 and one sixth of a unit.

[Lines 85 – 93 of the Arabic text]

And if we follow the example given, it is clear that the ratio of $\overline{AM}$ to $\overline{MB}$ is smaller than the ratio of 2.016 and one sixth of a unit to 66. Then if $\overline{MB}$

measures 66, $\overline{AM}$ is shorter than 2.016 and one sixth of a unit, and the square of $\overline{AM}$ is smaller than 4.064.928, and the square of $\overline{AB}$ is smaller than 4.069.280. Then line $\overline{AB}$ is shorter than 2.017 and one fourth of a unit. But line $\overline{MB}$ has a length of 66, and line $\overline{MB}$ is the side of a [polygon] of 96 sides inscribed in the circle. Thus the ratio of the diameter to the sum of the sides of a [polygon] of 96 sides inscribed in the circle is smaller than the ratio of 2.017 to 6.336.

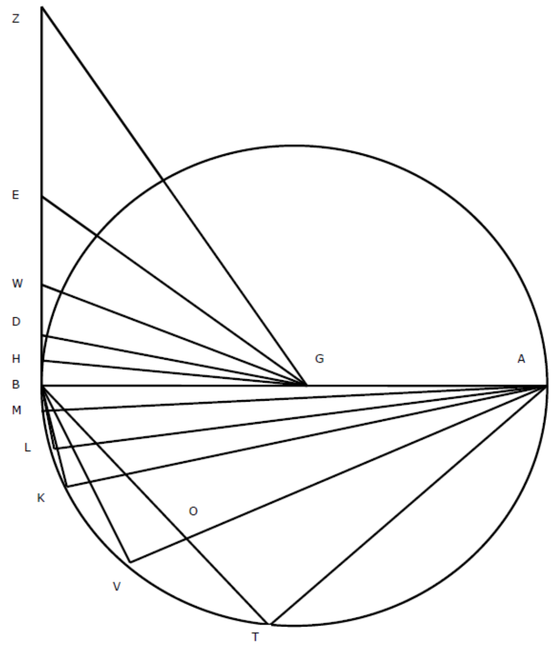
[Lines 94 – 97 of the Arabic text]

Il is thus clear that the ratio of the sum of the sides of a [polygon] of 96 sides inscribed in the circle to the diameter [of the circle] is greater than the ratio of 3 plus 10/71 to the unit. And the perimeter of the circle is longer than the sum of the sides of a [polygon] of 96 sides inscribed in the circle, and it is shorter than the sum of the sides of a [polygon] of 96 sides, circumscribed around the circle.

[Lines 98 – 101 of the Arabic text]

Thus it is certain, according to what we have described, that the ratio of the perimeter of the circle to the diameter [of the circle] is greater than the ratio of 3 plus 10/71 [to the unit], and smaller than the ratio of 3 plus 10/70 [to the unit].

And this is what we wanted to demonstrate.



Mathematical transcription

[Lines 1 – 3 of the Arabic text]

Proposition 11

The length of the circumference exceeds three times the diameter by a length less than 1/7 of the diameter, and more than 10/71 of the diameter.

[Lines 5 – 11 of the Arabic text]

Proof:

Circle $\overline{ATB}$ has a diameter $\overline{AB}$, and its centre is at point $\overline{G}$,
$\overline{GZ}$ is drawn to define with $\overline{BG}$ an angle of 30º,
and at point $\overline{B}$ of $\overline{BG}$ the perpendicular $\overline{BZ}$ is drawn.

The arc intercepting $\angle\ \overline{BGZ}$ = 1/2 * 1/6 * the perimeter of circle $\overline{ATB}$,
and $\overline{BZ}$ = 1/2 * the side of the hexagon circumscribed around circle $\overline{ATB}$.

And $\overline{GE}$ divides $\angle\ \overline{BGZ}$ in two halves,
and $\overline{GW}$ divides $\angle\ \overline{BGE}$ in two halves,
and $\overline{GD}$ divides $\angle\ \overline{BGW}$ in two halves,
and $\overline{GH}$ divides $\angle\ \overline{BGD}$ in two halves.

[Lines 12 – 14 of the Arabic text]

Then the arc intercepting $\angle\ \overline{BGH}$ = 1/192 * the perimeter of circle $\overline{ATB}$,
and $\overline{BH}$ = 1/2 * a side of the polygon of 96 sides circumscribed around circle $\overline{ATB}$.

[Lines 15 – 28 of the Arabic text]

We make $\overline{GZ}$ = 306, to facilitate further computing.

[See the next chapter]

Then if $\overline{GZ}$ = 306, $\overline{GZ}^2$ = 93.636,
and $\overline{BZ}$ = 153, because $\angle\ \overline{BGZ}$ = 1/3 * 90º, and $\angle\ \overline{GBZ}$ = 90º.
And $\overline{BZ}^2$ = 23.409, and $\overline{GB}^2$ = 70.227.
Thus $\overline{GB} > 265$.
But $\dfrac{\overline{BG}+\overline{GZ}}{\overline{BZ}} = \dfrac{\overline{GB}}{\overline{BE}}$, so that $\overline{GE}$ divides $\angle\ \overline{BGZ}$ in two halves,
and $\overline{BG} + \overline{GZ} > 571$, and $\overline{BZ}$ = 153.

Thus $\frac{\overline{GB}}{\overline{BE}} > \frac{571}{153}$.

Thus $\overline{GB} > 571$.

If we make $\overline{BE} = 153$, and $\overline{GB}^2 > 326.041$, and $\overline{BE}^2 = 23.409$,

Thus $\overline{GE}^2 > 349.450$.

Thus $\overline{GE} > 591 + 1/8$.

[Lines 29 – 35 of the Arabic text]

And in the same way,

$\frac{\overline{GB}}{\overline{BW}} > \frac{1.162 + 1/8}{153}$.

Thus if we make $\overline{BW} = 153$, $\overline{GB} > 1.162 + 1/8$,

and $\overline{GB}^2 > 1.350.534$, and $\overline{BW}^2 = 23.409$, and $\overline{GW}^2 > 1.373.943$.

Thus $\overline{GW} > 1.172 + 1/8$.

[Lines 36 – 42 of the Arabic text]

And in the same way,

$\frac{\overline{GB}}{\overline{BD}} > \frac{2.334 + 1/4}{153}$.

Thus if we make $\overline{BD} = 153$, $\overline{GB} > 2.334 + 1/4$,

and $\overline{GB}^2 > 5.448.723$, and $\overline{BD}^2 = 23.409$, and $\overline{GD}^2 > 5.472.032$.

[The last square should be 5.472.132, but the text contains 5.472.032]

Thus $\overline{GD} > 2.339 + 1/4$.

[Lines 43 – 50 of the Arabic text]

And in the same way,

$\frac{\overline{GB}}{\overline{BH}} > \frac{4.673 + 1/2}{153}$.

Thus if we make $\overline{HB} = 153$, $\overline{GB} > 4.673 + 1/2$.

And that is the ratio of the side of a polygon of 96 sides to the diameter.

Thus the ratio of the diameter to the sum of the sides of a polygon of 96 sides circumscribed around the circle is $> \frac{4.673+1/2}{14.688}$.

Thus the ratio of the sum of the 96 sides of the polygon to the diameter is $<$ 3 + 1/7.

[Lines 51 – 54 of the Arabic text]

We now draw in the circle $\overline{ATB}$ the cord of a sixth of the circle, that is $\overline{TB}$,

and we draw $\overline{AT}$,

and $\overline{AOV}$ cuts $\angle\ \overline{TAB}$ in two halves,
and we draw the cord $\overline{VB}$,
and $\overline{AK}$ cuts $\angle\ \overline{VAB}$ in two halves,
and we draw the cord $\overline{KB}$,
and $\overline{AL}$ cuts $\angle\ \overline{KAB}$ in two halves,
and we draw the cord $\overline{LB}$,
and $\overline{AM}$ cuts $\angle\ \overline{LAB}$ in two halves,
and we draw the cord $\overline{MB}$.

[Lines 55 – 68 of the Arabic text]

Then the cord $\overline{MB}$ is the side of a polygon of 96 sides inscribed in the circle.
Then we make $\overline{AB}$ = 1.560, to facilitate further computing.
Then the cord $\overline{TB}$ = 780, and $\overline{AB}^2$ = 2.433.600, and $\overline{TB}^2$ = 680.400.
And $\overline{AT}^2$ = 1.825.200.
Thus $\overline{TA}$ < 1.351.
But $\frac{\overline{TA}+\overline{AB}}{\overline{\overline{TB}}} = \frac{\overline{AT}}{\overline{\overline{TO}}}$, and $\frac{\overline{AT}}{\overline{\overline{TO}}} = \frac{\overline{AV}}{\overline{\overline{VB}}}$,
and $\overline{TA}$ + $\overline{AB}$ < 2.911.
Then $\frac{\overline{AT}}{\overline{\overline{TO}}} = \frac{2.911}{780}$.
And if $\overline{VB}$ = 780,
$\overline{AV}$ < 2.911, and $\overline{AV}^2$ < 8.473.921, and $\overline{VB}^2$ = 608.400, and $\overline{AB}^2$ < 9.082.021.

[The last square should be 9.082.321, but the text contains 9.082.021]

Thus $\overline{AB}$ < 3.013 + 3/4.

[Lines 69 – 78 of the Arabic text]

And in the same way,
$\frac{\overline{AK}}{\overline{\overline{KB}}} < \frac{5.924+3/4}{780}$.
Then if $\overline{KB}$ = 780, $\overline{AK}$ < 5.924 + 3/4.
And $\frac{5.924+3/4}{780} = \frac{1.823}{240}$.
Then if $\overline{KB}$ = 240,
$\overline{AK}$ < 1.823, and $\overline{AK}^2$ < 3.323.329, and $\overline{KB}^2$ = 57.600, and $\overline{AB}^2$ < 3.380.929.

Thus $\overline{AB} < 1.838 + 9/11$.

[Lines 79 – 84 of the Arabic text]

And in the same way,

$\frac{\overline{AL}}{\overline{LB}} < \frac{3.661 + 9/11}{240}$, which is a ratio equal to $\frac{1.007}{66}$.

Thus if $\overline{LB} = 66$,

$\overline{AL} < 1.007$, and $\overline{AL}^2 < 1.014.049$.

And $\overline{LB}^2 = 4.356$, and $\overline{AB}^2 < 1.018.405$.

Thus $\overline{AB} < 1.009 + 1/6$.

[Lines 85 – 93 of the Arabic text]

And in the same way,

$\frac{\overline{AM}}{\overline{MB}} < \frac{2.016 + 1/6}{66}$.

Then if $\overline{MB} = 66$,

$\overline{AM} < 2.016 + 1/6$, and $\overline{AM}^2 < 4.064.928$, and $\overline{AB}^2 < 4.069.280$.

[The last square should be 4.069.284, but the text contains 4.069.280]

Thus $\overline{AB} < 2.017 + 1/4$.

But $\overline{MB} = 66$,

and $\overline{MB}$ is the side of a polygon of 96 sides inscribed in the circle.

Then the ratio of the diameter to the sum of the sides of a polygon of 96 sides inscribed in the circle, is $< \frac{2.017}{6.336}$.

[The numerator of the fraction should be 2.017 + 1/4, but the text contains 2.017]

[Lines 94 – 97 of the Arabic text]

Thus the ratio of the sum of the sides of a polygon of 96 sides, inscribed in the circle, to its diameter is $> 3 + \frac{10}{71}$.

And the perimeter of the circle is longer than the sum of the sides of a polygon of 96 sides, inscribed in the circle, and it is shorter than the sum of the sides of a polygon of 96 sides, circumscribed around the circle.

[Lines 98 – 101 of the Arabic text]

Thus the ratio of the perimeter of the circle to its diameter is $> 3 + \frac{10}{71}$, and it is $< 3 + \frac{10}{70}$.

The arithmetical values used in the proposition

The reader may find it surprising that the proposition assigns a precise arithmetical value to the length of the first segment to be considered: in the case of the polygon circumscribed around the circle, the author assigns to GZ a length of 306.

GZ is the hypotenuse of the right-angled triangle GBZ. This triangle being the $\frac{1}{2}*\frac{1}{6}$ part of the hexagon circumscribed around the circle, its two other angles are equal to 30º and 60º.

Thus BZ measures 306*sin 30º = 153, and GB measures 306*cos 30º > 265.

But why these values, and not any other ones?

Sin 30º = $\frac{1}{2}$, and poses no problem. But cos 30º = $\frac{\sqrt{3}}{2}$.

Archimedes had found a very accurate formula that sets bounds to the value of $\sqrt{3}$:

$$\frac{265}{153} < \sqrt{3} < \frac{1351}{780}$$

We do not know the method he used to arrive at this result. Davies speculates with several possibilities in his article, and concludes that Archimedes probably used Hero's method, based on an algorithm for calculating square roots which relies on the fact that the arithmetical mean of two numbers is greater than their geometric mean, which in turn is greater than their harmonic mean. Davies' article (Davies, 2011) is very instructive on this subject, and provides a thorough analysis and comparison of several other possibilities.

Archimedes establishes the values for the sides of right-angled triangle GBZ, that is the $\frac{1}{2}*\frac{1}{6}$ part of the circumscribed hexagon, with the figures

GZ = 306 BZ = 153 GB > 265

which have obviously been taken from the formula that gives a lower bound to $\sqrt{3}$.

After this he divides the central angle by two several times (four times, in fact), and generates circumscribed regular polygons of successively 12, 24, 48 and 96 sides. Each regular polygon gives an increasing approximation to the upper bound to the value of π.

For the five iterations, al-Mu'taman gives the values of the lower bounds:

First iteration, for BZ = 153, GZ = 306 GB > 265

Second iteration, for BE = 153, GE > 591+$\frac{1}{8}$ GB > 571

Third iteration, for BW = 153, GW > 1172+$\frac{1}{8}$ GB > 1162+$\frac{1}{8}$

Fourth iteration, for BD = 153, GD > 2339+$\frac{1}{4}$ GB > 2334+$\frac{1}{4}$

Fifth iteration, for BH = 153, GH > 4673+$\frac{1}{2}$ GB > 4673+$\frac{1}{2}$

To deal with an angle that is half the angle of the previous iteration, Archimedes uses geometry to establish the equation which, in the second iteration, is expressed by $\frac{\overline{BG}+\overline{GZ}}{\overline{BZ}}=\frac{\overline{GB}}{\overline{BE}}$, where GE divides $\angle$ BGZ in two halves. No explanation is given either by Archimedes or by his medieval followers, so it seems that the reader is expected to be familiar with this particular modus operandi. And again, as in the first iteration, the value 153 is arbitrarily given to one of the sides of every new right-angled triangle, to be able to continue using the figures that have their origin in the lower bound to the value of $\sqrt{3}$.

Interestingly, there is an arithmetical discrepancy in the fourth iteration. The square of GD is not 5.472.082, as al-Mu'taman states in his text, but 5.472.182. If we take al-Mu'taman's square, then GD is not greater than 2339+$\frac{1}{4}$, but slightly smaller.

It may be that the copyist has omitted one word, "and one hundred". However, the two manuscripts (Leiden and Copenhagen) show the same omission; furthermore, in the second part of the procedure, in which the polygons are inscribed in the circle to determine a lower bound to the value of π, three numerical words are also omitted.

So it is fair to infer that the author "copied" the procedure, and did not present results evaluated by himself.

Van Lit has analysed in his paper (Van Lit, 2008) the same procedure as it was written by al-Ṭūsī in the thirteenth century, and has also found a few omitted words. This makes it likely that the omissions were made by the brothers Banū Mūsā in their book, which was the bridge between Archimedes and the medieval mathematicians.

Comparison of the texts

From a mathematical point of view, the three medieval texts, written between the eleventh and the thirteenth centuries in Arabic and in Latin, are very similar. They take up the geometrical operations of Archimedes' proposition, which starts with the regular hexagon circumscribed around the circle, and the regular hexagon inscribed in the circle, and then increases the number of the sides of the polygons up until a value of 96, finally obtaining a useful approximation of the ratio between the perimeter and the diameter of the circle.

Given that the text by al-Mu'taman ibn Hūd precedes in time the two texts by Gerard of Cremona and by al-Ṭūsī, here we offer a textual comparison of the different versions of the proposition. The comparison is presented in a table on the next page, in which a certain number of sentences have been selected.

The table has five columns. The empty second column symbolizes the text of the brothers Banū Mūsā, which has not survived. The first column contains Archimedes' Greek text, and the last three columns contain the Arabic, Latin, and again Arabic texts by al-Mu'taman, Gerard of Cremona and al-Ṭūsī.

		مجموعين إلى ب ز كنسبة ب جـ إلى ب هـ		مجموعين إلى ب د كنسبة جـ ب إلى ب هـ
Then the sum of ZE and EΓ is to ZΓ like EΓ is to ΓH.		But the ratio of the two lines $\overline{BG}$ and $\overline{GZ}$ aggregated to $\overline{BZ}$ is equal to the ratio of $\overline{GB}$ to $\overline{BE}$.	But the ratio of the two lines BG and GZ aggregated to BZ is equal to the ratio of GB to BE.	But the ratio of $\overline{BG}$ and $\overline{GD}$ aggregated to $\overline{BD}$ is equal to the ratio of $\overline{BG}$ to $\overline{BE}$
ὥστε ἡ ΓE πρός ΓH μείζονα λόγον ἔχει ἤπερ φοα πρός ρνγ.		فنسبة جـ ب إلى ب هـ أعظم من نسبة خمس مائة وواحد وسبعين إلى مائة وثلاثة وخمسين.	Ergo proportio GB ad BE is maior proportione quingentorum and septuaginta unius ad centum and quinquaginta tria.	فنسبة جـ ب إلى ب هـ أعظم من نسبة 571 إلى 153.
thus the ratio of ΓE to ΓH is greater than the ratio of 571 to 153.		thus the ratio of $\overline{GB}$ to $\overline{BE}$ is greater than the ratio of 571 to 153.	thus the ratio of GB to BE is greater than the ratio of 571 to 153.	thus the ratio of $\overline{GB}$ to $\overline{BE}$ is greater than the ratio of 571 to 153.
συναμφότερος ἡ ΓAB πρός BΓ		ولكن نسبة خطّي ط ا ا ب مجموعين إلى ط ب	Sed proportio duarum linearum TA, AB coniuctarum ad TB	ولكن نسبة ط ا ا ب معًا إلى ط ب
the sum of ΓA and AB is to BΓ		But the ratio of the two lines $\overline{TA}$ and $\overline{AB}$ aggregated to $\overline{TB}$	But the ratio of the two lines TA and AB aggregated to TB	But the ratio of $\overline{TA}$ and $\overline{AB}$ together to $\overline{TB}$
-------		وذلك ما أردنا أن نبيّن.	Et illud is quod declarare voluimus.	وذلك ما أردناه.
		And that's what we wanted to demonstrate.	And that's what we wanted to demonstrate.	And that's what we wanted.

1 **Archimedes text** **(third century BC)**	2 **Banū Mūsā (ninth)**	3 **al-Mu'taman text** **(eleventh century)**	4 **Gerard of Cremona text** **(twelfth century)**	5 **al-Ṭūsī text** **(thirteenth century)**
Εστω κύκλος καί διάμετρος ή ΑΓ καί κέντρον τό Ε καί,		فنخطّ دائرة با ط ، وقطرها ا ب ، ومركزها نقطة جـ	Lineemus ergo circulum ATB, cuius diameter sit AB, and ipsius centrum sit punctum G.	وليكن لبيانه دائرة ا ط ب ، وقطرها ا ب ، ومركزها جـ
Let a circle be, ΑΓ its diameter, E its centre,		Then we draw circle $\overline{ATB}$, and its diameter is $\overline{AB}$, and its centre is point $\overline{G}$,	Then we will draw circle ATB, the diameter of which is AB, and its centre is point G,	And be it for the proof circle $\overline{ATB}$, and its diameter is $\overline{AB}$, and its centre is $\overline{G}$,
ή ύπό ΖΕΓ τρίτου όρθής,		ونخرج خطّ جـ ز يُحيط مع خطّ ب جـ بثُلث زاوية قائمة،	Et protraham ex centro G lineam GZ continentem cum linea GB tertiam anguli recti.	ونخرج من جـ خطّ جـ د يُحيط مع ب جـ بثُلث قائمة،
Let angle ZEΓ be equal to the third of a square angle,		And we draw line $\overline{GZ}$ that contains with line $\overline{BG}$ a third of a square angle ,	And I will draw from the centre G line GZ that contains with line GB a third of a square angle.	And we draw from $\overline{G}$ line $\overline{GD}$ that contains with line $\overline{BG}$ a third of a square,
-------		ونقيم على نقطة ب من خطّ ب جـ خطًا على زاوية قائمة، وهو خطّ ب ز.	Et erigam super punctum B linee GB lineam BZ orthogonaliter.	ونخرج من ب عمود ب د على جـ ب .
		And we erect on point $\overline{B}$ of line $\overline{BG}$ a line on a square angle, and it is line $\overline{BZ}$.	And I will erect on point B of line GB line BZ, [that is] perpendicular.	And we draw from $\overline{B}$ the perpendicular $\overline{BD}$ on $\overline{GB}$.

Τετμήσθω ούν ή ύπό ΖΕΓ δίχα τή ΕΗ.		ونقسم زاوية ب جـ ز بنصفين بخطّ جـ هـ.	Et dividam angulum BGZ in duo media cum linea GE.	وننصف زاوية ب جـ د بخطّ جـ هـ.
Let's bisect angle ΖΕΓ by ΕΗ.		And we divide angle $\overline{BGZ}$ in two halves with line $\overline{GE}$.	And I will divide angle BGZ in two halves with line GE.	And we divide in two angle $\overline{BGD}$ with line $\overline{GE}$.
-------		وعلى هذا المثال الذي وصفنا	Et secundum exemplum quod narravimus,	وعلى ذلك المثال
		And after this example that we have described	And after the example that we have described	And after this example
-------		فأنا نصير خطّ جـ ز ثلاث مائة وستّة لسهولة استعمال هذا العدد فيما كسب.	tunc nos ponemus lineam GZ trecentum and sex propter facilitatem usus huius numeri in eo quod computatur.	ولنجعل جـ د 306 لسهولة العمل كما تبيّن.
		then we give to line $\overline{GZ}$ [a length of] 306 to facilitate the usage of this number where its is convenient.	then we give to line GZ [a length of] 306 to facilitate the usage of this number when it is computed.	And we make $\overline{GD}$ [equal to] 306 to facilitate the work of demonstration.
Ως άρα συναμφότερος ή ΖΕ, ΕΓ πρός ΖΓ, ή ΕΓ πρός ΓΗ.		ولكن بنسبة خطّي ب جـ جـ ز	Sed proportio duarum linearum BG, GZ agregatarum ad BZ is sicut proportio GB ad BE.	ولكن بنسبة ب جـ جـ د

The first column underlines the synthetic nature of Archimedes' proposition in Greek. He has only written the results of the operations, and in general he leaves out the intermediate steps. Several sentences that serve as introductory formulae in the language of the mathematics do not appear.

The comparison of morphology and syntax in columns 3 and 4, (al-Mu'taman and Gerard of Cremona) reveals the coincidences between the terms, which are particularly striking in view of the considerable differences between the two languages, Arabic and Latin.

Thus we have « wa-markazu-hā nuqṭat G »,
and « et ipsius centrum sit punctum G »,
that we can translate « and its centre is point G ».
And also « walakin nisbat khaṭṭay TA AB majmuʿayn ilā TB » ,
and « sed proportio duarum linearum TA AB coniuctarum ad TB »,
that we can translate « but the ratio of the two lines TA AB together to TB ».
The parallelism of the semantic choice of words is remarkable :
For « erect » (a perpendicular on a point of a line),
Gerard uses the Latin verb « et erigam »,
and al-Mu'taman uses the verb « wa-nuqīm »,
that are equivalent.

The last example especially shows that both al-Mu'taman and Gerard of Cremona, worked on an analysis of the same text, and that they obtained parallel results for Archimedes' proposition.

Rashed (Rashed, 1996, pages 74 - 83) presents a comparison of the texts by Gerard of Cremona (column 4) and al-Ṭūsī (column 5).

Column 5 shows the simplified language used by al-Ṭūsī. This distinguished thirteenth-century mathematician shortens the syntax of every sentence, seemingly aiming to say as much as possible in very short statements:
« a square angle » becomes, for him, « a square »,
« a perpendicular line » becomes « a perpendicular », etc.

It is remarkable that, whereas al-Mu'taman and Gerard of Cremona use the expression « divide in two halves »,
in Arabic « naqsim bi-niṣfayn »,
and in Latin « dividam in duo media »,

al-Ṭūsī draws on the resources of Arabic lexicology to use the verb « nunaṣṣif », which derives from the name « niṣf » (half), and means exactly the same, though he uses only one word instead of two or three. But this is not a simple lexicological difference: the language of al-Mu'taman is purely geometrical (« qasama » is a geometrical operation), while the language of al-Ṭūsī may have been borrowed from arithmetic: « to halve » or « to divide by two ».

The texts of al-Mu'taman and of Gerard are so similar that we may conclude that both authors worked from the text of the brothers Banū Mūsā.

We might also wonder whether Gerard did or did not make his copy from the text of al-Mu'taman.

But this is not possible: in their version of the proposition, and using very different languages, both Gerard of Cremona and after him al-Ṭūsī copied a very long introduction, praising the scientific goals of Archimedes, and explaining the objective of the proposition, which would have come at the beginning of Banū Mūsā's text. Gerard's introduction in Latin can be found in the edition by Clagett (Clagett, 1964, pages 264 - 279), and al-Ṭūsī's introduction in Arabic in the edition by Rashed (Rashed, 1996, pages 74 - 83).

However, the prince of Saragossa omitted it from his copy of the proposition, a justifiable decision, since he was composing an encyclopaedia. But this shows conclusively that Gerard and al-Ṭūsī were working with the book by the Banū Mūsā, and not with the book by al-Mu'taman.

Therefore we can conclude with some confidence that all three authors worked from the book of the brothers Banū Mūsā.

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