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Volterra composition operators between weighted Bergman spaces and weighted Bloch type spaces

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Abstract

We characterize boundedness and compactness of Volterra composition operators acting between weighted Bergman spaces $A_{v,p}$ and weighted Bloch type spaces B_w.

1. Introduction

Let $H(D)$ be the set of all analytic functions on the open unit disk D of the complex plane.

Moreover, let v and w be strictly positive bounded continuous functions (*weights*) on D. Then the weighted Bergman space $A_{v,p}$ is defined as follows

$$A_{v,p} := \left\{ f \in H(D);\ \|f\|_{v,p} := \Big(\int_D |f(z)|^p v(z)\, dA(z) \Big)^{1/p} < \infty \right\},\ 1 \le p < \infty,$$

where $dA(z)$ is the area measure on D normalized so that area of D is 1. Moreover we consider the weighted Bloch type spaces B_w of functions $f \in H(D)$ satisfying $\|f\|_{B_w} := \sup_{z\in D} w(z)|f'(z)| < \infty$. Provided we identify functions that differ by a constant, $\|.\|_{B_w}$ becomes a norm and B_w a Banach space.

An analytic self-map ϕ of D induces the composition operator C_ϕ defined by $C_\phi f = f \circ \phi$. For an analytic map $g : D \to \mathbb{C}$ and a map $f \in H(D)$ the Volterra type operator or the Riemann–Stieltjes operator J_g is defined as

$$J_g f(z) := \int_0^z f(\xi) g'(\xi)\, d\xi,\ z \in D$$

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(see [15, 17]). In this paper we consider the Volterra composition operator which is defined as follows

$$(J_{g,\phi}f)(z) := \int_0^z (f \circ \phi)(\xi)(g \circ \phi)'(\xi)\, d\xi.$$

Recently composition operators acting on various spaces of analytic functions have been of much interest, see e.g. [5, 4, 6, 8, 7, 13]. Also operators of type J_g have been studied by many authors, see e.g. [1, 2, 3, 15, 17]. Boundedness and compactness of the Volterra composition operator acting between Bergman spaces weighted with the standard weights and Bloch type spaces have been characterized by Li in [12]. In this article we want to generalize his results for more general weighted Bergman spaces and weighted Bloch type spaces as described above.

2. Preliminaries

In the sequel we consider the following weights. Let ν be a holomorphic function on D, non-vanishing, strictly positive on $[0,1[$ and satisfying $\lim_{r\to 1}\nu(r) = 0$. Then we define the weight v as follows $v(z) := \nu(|z|^2)$ for every $z \in D$.

Next, we give some illustrating examples of weights of this type:

(i) Consider $\nu(z) = (1-z)^\alpha$, $\alpha \geq 1$. Then the corresponding weight is the so-called standard weight $v(z) = (1-|z|^2)^\alpha$.

(ii) Select $\nu(z) = e^{-1/(1-z)^\alpha}$, $\alpha \geq 1$. Then we obtain the weight $v(z) = e^{-1/(1-|z|^2)^\alpha}$.

(iii) Choose $\nu(z) = \sin(1-z)$ and the corresponding weight is given by $v(z) = \sin(1-|z|^2)$.

(iv) Let $\nu(z) = (1-\log(1-z))^\beta$ for some $\beta < 0$. Then we get $v(z) = (1-\log(1-|z|^2))^\beta$.

For a fixed point $a \in D$ we introduce a function $v_a(z) := \nu(\overline{a}z)$ for every $z \in D$. Since ν is holomorphic on D, the function v_a is also holomorphic on D.

Furthermore, we need some geometric data of the open unit disk. Fix $\alpha \in D$ and consider the authomorphism $\varphi_\alpha(z) := \frac{\alpha - z}{1-\overline{\alpha}z}$, $z \in D$, which interchanges 0 and α. Moreover we use the fact that

$$\varphi'_\alpha(z) = \frac{|\alpha|^2 - 1}{(1-\overline{\alpha}z)^2}, z \in D.$$

Let us finish the preliminaries with stating a very useful lemma, which can be easily derived from [9, Proposition 3.11].

Lemma 1

Let v and w be weights. Then the operator $J_{g,\phi} : A_{v,p} \to B_w$ is compact if and only if it is bounded and for every bounded sequence $(f_n)_n$ in $A_{v,p}$ which converges to zero uniformly on the compact subsets of D, $J_{g,\phi}f_n$ tends to zero in B_w if $n \to \infty$.

3. Results

We first need the following auxiliary result. The following lemma is well-known for standard weights (see [10, 11]) and was given in its present form in [16], but for a better understanding we give the full proof here.

Lemma 2

Let v be a weight as defined in the previous section (i.e. $v(z) := \nu(|z|^2)$ for every $z \in D$) such that

$$\sup_{a\in D}\sup_{z\in D}\frac{v(z)|v_a(\varphi_a(z))|}{v(\varphi_a(z))} \le C < \infty.$$

Then

$$|f(z)| \le \frac{C^{1/p}}{v(0)^{1/p}(1-|z|^2)^{2/p}v(z)^{1/p}}\|f\|_{v,p}$$

for all $z \in D$, $f \in A_{v,p}$.

Proof. Let $\alpha \in D$ be an arbitrary point. Consider the map

$$T_\alpha : A_{v,p} \to A_{v,p},\ T_\alpha(f(z)) = f(\varphi_\alpha(z))\varphi'_\alpha(z)^{2/p}v_\alpha(\varphi_\alpha(z))^{1/p}.$$

Then a change of variables yields

$$\begin{aligned}
\|T_\alpha f\|^p_{v,p} &= \int_D v(z)|f(\varphi_\alpha(z))|^p|\varphi'_\alpha(z)|^2|v_\alpha(\varphi_\alpha(z))|\,dA(z)\\
&= \int_D \frac{v(z)|v_\alpha(\varphi_\alpha(z))|}{v(\varphi_\alpha(z))}|f(\varphi_\alpha(z))|^p|\varphi'_\alpha(z)|^2 v(\varphi_\alpha(z))\,dA(z)\\
&\le \sup_{z\in D}\frac{v(z)|v_\alpha(\varphi_\alpha(z))|}{v(\varphi_\alpha(z))}\int_D |f(\varphi_\alpha(z))|^p|\varphi'_\alpha(z)|^2 v(\varphi_\alpha(z))\,dA(z)\\
&\le C\int_D v(t)|f(t)|^p\,dA(t) = C\|f\|^p_{v,p}.
\end{aligned}$$

Now put $g(z) := T_\alpha(f(z))$ for every $z \in D$. By the mean-value property we obtain

$$v(0)|g(0)|^p \le \int_D v(z)|g(z)|^p\,dA(z) = \|g\|^p_{v,p} \le C\|f\|^p_{v,p}.$$

Hence

$$v(0)|g(0)|^p = v(0)|f(\alpha)|^p(1-|\alpha|^2)^2v(\alpha) \le C\|f\|^p_{v,p}.$$

Thus

$$|f(\alpha)| \le C^{1/p}\frac{\|f\|_{v,p}}{v(0)^{1/p}(1-|\alpha|^2)^{2/p}v(\alpha)^{1/p}}.$$

Since α was arbitrary, the claim follows. □

Calculations show that the examples (i)-(iv) which were listed up above satisfy the assumptions of the previous lemma.

Theorem 3

Let w be a weight and v be a weight as in Lemma 2 with

$$M := \sup_{a\in D}\sup_{z\in D}\frac{v(z)}{|\nu(\overline{a}z)|} < \infty.$$

Then the operator $J_{g,\phi} : A_{v,p} \to B_w$ is bounded if and only if

$$\sup_{z\in D}\frac{w(z)|\phi'(z)||g'(\phi(z)|}{(1-|\phi(z)|^2)^{2/p}v(\phi(z))^{1/p}} < \infty.$$

Proof. We start with assuming that the operator $J_{g,\phi}$ is bounded. Fix a point $a \in D$ and set

$$f_a(z) := \frac{\varphi_a'(z)^{2/p}}{\nu(\overline{a}z)^{1/p}} \text{ for every } z \in D.$$

Then

$$\begin{aligned}\|f\|_{v,p}^p &= \int_D \frac{|\varphi_a'(z)|^2}{|\nu(\overline{a}z)|}v(z)\,dA(z)\\ &\le \sup_{z\in D}\frac{v(z)}{|\nu(\overline{a}z)|}\int_D |\varphi_a'(z)|^2\,dA(z)\\ &\le \sup_{z\in D}\frac{v(z)}{|\nu(\overline{a}z)|} \le M,\end{aligned}$$

and the constant M is independent of the choice of the point a. Hence we can find a constant $C^* > 0$ such that

$$\begin{aligned}\frac{w(a)|\phi'(a)||g'(\phi(a))|}{(1-|\phi(a)|^2)^{2/p}v(\phi(a))^{1/p}} &= |f_{\phi(a)}(\phi(a))|w(a)|g'(\phi(a))||\phi'(a)|\\ &= |(J_{g,\phi}f_{\phi(a)})'(a)|w(a) \le C^*\|J_{g,\phi}\|\|f_{\phi(a)}\|_{v,p}.\end{aligned}$$

Conversely, we suppose that

$$\sup_{z\in D}\frac{w(z)|\phi'(z)||g'(\phi(z)|}{(1-|\phi(z)|^2)^{2/p}v(\phi(z))^{1/p}} < \infty.$$

An application of Lemma 2 yields for $f \in A_{v,p}$

$$\begin{aligned}\sup_{z\in D}|(J_{g,\phi}f)'(z)|w(z) &= \sup_{z\in D}|f(\phi(z))||g'(\phi(z))||\phi'(z)|w(z)\\ &\le \sup_{z\in D}\frac{C^{1/p}\|f\|_{v,p}w(z)|g'(\phi(z))||\phi'(z)|}{v(0)^{1/p}(1-\phi(z)|^2)^{2/p}v(\phi(z))^{1/p}}.\end{aligned}$$

Hence the claim follows. □

Theorem 4

Let w be a weight and v be a weight as in Theorem 3. Then the operator $J_{g,\phi}$: $A_{v,p} \to B_w$ is compact if and only if

$$\sup_{z\in D} w(z)|g'(\phi(z))||\phi'(z)| < \infty \tag{0.1}$$

and

$$\lim_{|\phi(z)|\to 1} \frac{w(z)|g'(\phi(z))||\phi'(z)|}{(1-|\phi(z)|^2)^{2/p}v(\phi(z))^{1/p}} = 0. \tag{0.2}$$

Proof. Assume that the operator $J_{g,\phi} : A_{v,p} \to B_w$ is compact. Then obviously $J_{g,\phi}$ is bounded. Taking $f = 1$, we get (0.1). To show (0.2) let $(z_n)_n$ be a sequence with $|\phi(z_n)| \to 1$ and put

$$f_k(z) := \frac{\varphi'_{\phi(z_k)}(z)^{2/p}}{\nu(\overline{\phi(z_k)}z)^{1/p}} \text{ for every } z \in D \text{ and every } k \in \mathbb{N}.$$

Analogously to the proof of Theorem 3 we can show that $(f_n)_n$ is a bounded sequence which tends to zero uniformly on the compact subsets of D. Since $J_{g,\phi}$ is compact, by Lemma 1

$$\|J_{g,\phi}f_n\|_{B_w} \to 0 \text{ if } n \to \infty.$$

Thus,

$$\|J_{g,\phi}f_n\|_{B_w} \geq \frac{w(z_n)|\phi'(z_n)||g'(\phi(z_n)|}{(1-|\phi(z_n)|^2)^{2/p}v(\phi(z_n))^{1/p}},$$

and condition (0.2) follows.

Conversely, suppose that (0.1) and (0.2) are satisfied. Let $(f_n)_n$ be a bounded sequence in $A_{v,p}$ such that $\|f_n\|_{v,p} \leq M_1 < \infty$ for every $n \in \mathbb{N}$ and such that $(f_n)_n$ converges uniformly to zero on the compact subsets of D if $n \to \infty$. For a fixed $\varepsilon > 0$ we can find $0 < r_0 < 1$ such that if $|\phi(z)| > r_0$, then

$$\frac{w(z)|g'(\phi(z))||\phi'(z)|}{(1-|\phi(z)|^2)^{2/p}v(\phi(z))^{1/p}} < \frac{\varepsilon v(0)^{1/p}}{2C^{1/p}M_1}.$$

Moreover, we can find $M_2 > 0$ such that

$$\sup_{|\phi(z)|\leq r_0} w(z)|g'(\phi(z))||\phi'(z)| \leq M_2.$$

There is $n_0 \in \mathbb{N}$ such that

$$\sup_{|\phi(z)|\leq r_0} |f_n(\phi(z))| \leq \frac{\varepsilon}{2M_2} \text{ for every } n \geq n_0.$$

Furthermore, from (0.2) we can easily derive that

$$\sup_{z\in D} \frac{w(z)|\phi'(z)||g'(\phi(z)|}{(1-|\phi(z)|^2)^{2/p}v(\phi(z))^{1/p}} < \infty.$$

Thus, the operator $J_{g,\phi}$ must be bounded. We obtain applying Lemma 2

$$\begin{aligned}
\sup_{z\in D} |(J_{g,\phi}f_n)'(z)|w(z) &= \sup_{z\in D} w(z)|f_n(\phi(z))||g'(\phi(z))||\phi'(z)| \\
&\leq \sup_{|\phi(z)|\leq r_0} w(z)|f_n(\phi(z))||g'(\phi(z))||\phi'(z)| \\
&\quad + \sup_{|\phi(z)|> r_0} w(z)|f_n(\phi(z))||g'(\phi(z))||\phi'(z)| \\
&\leq \sup_{|\phi(z)|\leq r_0} |f_n(\phi(z))| \sup_{|\phi(z)|\leq r_0} w(z)|g'(\phi(z))||\phi'(z)| \\
&\quad + \sup_{|\phi(z)|> r_0} \frac{C^{1/p}\|f_n\|_{v,p} w(z)|g'(\phi(z))||\phi'(z)|}{v(0)^{1/p}(1-|\phi(z)|^2)^{2/p} v(\phi(z))^{1/p}} \\
&\leq \varepsilon,
\end{aligned}$$

and the claim follows. □

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