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Strichartz inequalities with weights in Morrey–Campanato classes

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Abstract

We prove some weighted refinements of the classical Strichartz inequalities for initial data in the Sobolev spaces $\dot{H}^s(\mathbb{R}^n)$. We control the weighted L^2-norm of the solution of the free Schrödinger equation whenever the weight is in a Morrey–Campanato type space adapted to that equation. Our partial positive results are complemented by some necessary conditions based on estimates for certain particular solutions of the free Schrödinger equation.

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1. Introduction

Consider the initial value problem associated to the free Schrödinger equation:

$$\begin{cases} i\partial_t u + \Delta_x u = 0 & (x,t) \in \mathbb{R}^n \times \mathbb{R} \\ u(x,0) = f(x) \end{cases} \tag{1}$$

where $f \in \dot{H}^s(\mathbb{R}^n)$, the usual homogeneous L^2-Sobolev space.

As usual, we denote the solution u of (1) by $e^{it\Delta}f(x)$. The purpose of this note is to establish some weighted refinements of the classical Strichartz inequalities*

$$\|e^{it\Delta}f\|_{L^r_{x,t}(\mathbb{R}^n\times\mathbb{R})} \lesssim \|f\|_{\dot{H}^s(\mathbb{R}^n)}, \tag{2}$$

where $0 \leq s < \frac{n}{2}$ and $r = \frac{2(n+2)}{n-2s}$. This inequality follows in a straightforward manner from the Hardy–Littlewood–Sobolev inequality along with well-known mixed-norm Strichartz estimates (see [16, 8, 11] and [13]). The range $0 \leq s < \frac{n}{2}$ is the best-possible as one can see by using classical counterexamples.

In order to describe the refinements that we have in mind, we must first recast the inequalities (2) in the L^2-weighted form

$$\|e^{it\Delta}f\|^2_{L^2_{x,t}(V)} \lesssim \|V\|_{L^p_{x,t}(\mathbb{R}^n\times\mathbb{R})}\|f\|^2_{\dot{H}^s(\mathbb{R}^n)}, \tag{3}$$

where V is an arbitrary element of $L^p_{x,t}(\mathbb{R}^n \times \mathbb{R})$ with $p = \frac{n+2}{2s+2}$ and $0 \leq s < \frac{n}{2}$. This weighted formulation, which is more adapted to the study of perturbations of the equation given in (1) by time-dependent potentials, naturally leads one to consider the possibility of the existence of more subtle positive functionals $V \mapsto C_s(V)$, for which one might have control of $e^{it\Delta}f$ in the form

$$\|e^{it\Delta}f\|^2_{L^2_{x,t}(V)} \lesssim C_s(V)\|f\|^2_{\dot{H}^s(\mathbb{R}^n)}, \tag{4}$$

for certain $0 \leq s < \frac{n}{2}$. Questions of this nature have been posed many times before in related settings; see for example [15, 3] and [4].

The functionals that we shall introduce here are variants of the so-called Morrey–Campanato norms, and are motivated by the closely related work of Ruiz and Vega in [15] in the context of the stationary Schrödinger operator, and also by the works in [9, 10] and [17]. For $\alpha > 0$ and $1 \leq p \leq \frac{n+2}{\alpha}$ let

$$\mathcal{L}^{\alpha,p}_{\text{par}} = \left\{F \in L^p_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}) : \|F\|_{\mathcal{L}^{\alpha,p}_{\text{par}}} < \infty\right\},$$

where

$$\|F\|_{\mathcal{L}^{\alpha,p}_{\text{par}}} = \sup_{(x,t)\in\mathbb{R}^n\times\mathbb{R}, r>0} r^\alpha \left(\frac{1}{r^{n+2}}\int_{C(x,t,r)} |F(y,s)|^p dy\, ds\right)^{1/p},$$

* For non-negative quantities X and Y we use $X \lesssim Y$ ($X \gtrsim Y$) to denote the existence of a positive constant C, depending on at most n, p and s, such that $X \leq CY$ ($X \geq CY$). We write $X \sim Y$ if both $X \lesssim Y$ and $X \gtrsim Y$.

and $C(x,t,r)$ denotes the "parabolic box" $B(x,r)\times(t-r^2,t+r^2)$. We observe that these norms have the parabolic homogeneity

$$\|F(\lambda\,\cdot,\lambda^2\,\cdot)\|_{\mathcal{L}^{\alpha,p}_{\mathrm{par}}}=\lambda^{-\alpha}\|F\|_{\mathcal{L}^{\alpha,p}_{\mathrm{par}}}.$$

We notice also that $\mathcal{L}^{\alpha,p}_{\mathrm{par}}=L^p(\mathbb{R}^n\times\mathbb{R})$ when $p=\frac{n+2}{\alpha}$, and for $p<\frac{n+2}{\alpha}$ the Lorentz space $L^{p,\infty}(\mathbb{R}^n\times\mathbb{R})\subset\mathcal{L}^{\alpha,p}_{\mathrm{par}}$. Finally, for $q<p$ we have the strict inclusion $\mathcal{L}^{\alpha,p}_{\mathrm{par}}\subset\mathcal{L}^{\alpha,q}_{\mathrm{par}}$. These observations and scaling considerations raise the possibility that the inequalities

$$\|e^{it\Delta}f\|^2_{L^2_{x,t}(V)}\lesssim\|V\|_{\mathcal{L}^{2s+2,p}_{\mathrm{par}}(\mathbb{R}^n\times\mathbb{R})}\|f\|^2_{\dot{H}^s(\mathbb{R}^n)},\tag{5}$$

might hold for some $p<\frac{n+2}{2s+2}$ and $0\le s<\frac{n}{2}$, thus yielding improvements on (3).

Our main theorem states that these inequalities (5) do indeed hold, at least when the number of derivatives s is sufficiently large.

Theorem 1.1

The estimate (5) *holds if* $\frac{n}{4}\le s<\frac{n}{2}$ *and* $1<p\le\frac{n+2}{2s+2}$.

Remark. Theorem 1.1 allows potentials strongly depending on time of the form $V(x,t)=|x|^{-a}|t|^{-b}$ with $ap<n$, $bp<1$, $a+2b=2s+2+\frac{n+2}{p}$, and p and s under the conditions of the theorem. Notice that these potentials are never in Lebesgue spaces.

In Section 2 we give a proof of Theorem 1.1 using a bilinear interpolation technique due Keel and Tao [13] in the context of Strichartz mixed-norm inequalities.

In Section 3 we use the special solutions of the Schrödinger equation from [1] to give the following necessary conditions on the exponents p and s for (5) to hold.

Proposition 1.2

The estimate (5) *is false if* $0\le s<\frac{n}{4}$ *and* $p<\frac{n+4}{4(s+1)}$.

Figure 1 summarises all the results that we present here. The non-convexity of the regions $(s,1/p)$ may be possibly due to the bad interpolation properties of Morrey–Campanato classes, see [5].

In particular, we leave unanswered the perhaps more difficult question of whether (5) might hold for functions $f\in H^s$ for $0\le s<\frac{n}{4}$ and some $\frac{n+4}{4s+4}\le p<\frac{n+2}{2s+2}$.

Remark. As we have already mentioned, the inequalities (5) were inspired by similar inequalities that are known to hold for the stationary Schrödinger operator, with weights belonging to the classical Morrey–Campanato classes $\mathcal{L}^{\alpha,p}$ (see [15]). As is well-known (again see for example [15]), norm estimates for the stationary Schrödinger operator imply related norm estimates for the time-dependent operator, which in this setting leads to the validity of

$$\|e^{it\Delta}f\|^2_{L^2_{x,t}(V)}\lesssim\|V\|_{\mathcal{L}^{2s+2,p}_x(L^\infty_t)}\|f\|^2_{\dot{H}^s(\mathbb{R}^n)},\tag{6}$$

within a certain range of the exponents p and s (see [15] for the specific exponents corresponding to $s=0$, and the methods that extend in a straightforward manner to $s\ne0$). It should be pointed out that although the parabolic norm $\|\cdot\|_{\mathcal{L}^{\alpha,p}_{\mathrm{par}}}$ is in

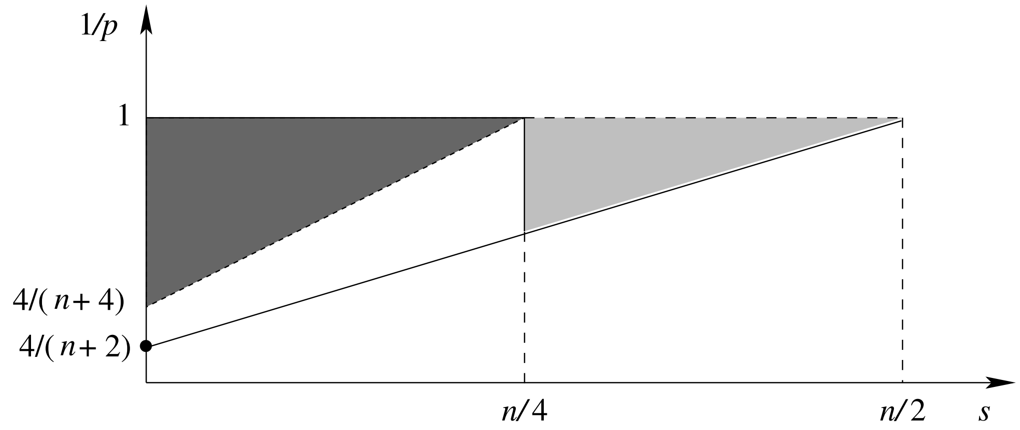

Figure 1: The region with the lightest shade of grey corresponds to where the estimates (5) are known to be true, and the remaining darker shade corresponds to where it is known to be false.

general much smaller than the mixed norm $\|\cdot\|_{\mathcal{L}_x^{\alpha,p}(L_t^\infty)}$, the interesting feature of the latter is that it continues to be relevant for certain $s<0$, yielding smoothing estimates for $e^{it\Delta}f$ for $f\in L^2(\mathbb{R}^n)$.

We remark that the examples we give in Section 3, yielding necessary conditions for (5) to hold, are also effective in producing necessary conditions for (6) to hold (see [2]).

2. The proof of Theorem 1.1

We begin by observing that (5) is equivalent to the inequality

$$\|I_s e^{it\Delta} f\|^2_{L^2_{x,t}(V)} \lesssim \|V\|_{\mathcal{L}^{2s+2,p}_{\text{par}}(\mathbb{R}^n\times\mathbb{R})}\|f\|^2_{L^2(\mathbb{R}^n)}, \tag{7}$$

where I_s denotes the fractional integral operator of order s. By duality, this is in turn equivalent to the bilinear inequality

$$\left|\left\langle I_{2s}\int_{-\infty}^{+\infty} e^{i(t-\tau)\Delta}F(\cdot,\tau)\,d\tau, G\right\rangle\right| \lesssim \|V\|_{\mathcal{L}^{2s+2,p}_{\text{par}}}\|F\|_{L^2_{x,t}(V^{-1})}\|G\|_{L^2_{x,t}(V^{-1})}.$$

Using the Fourier transform we can express the linear operator in the previous inequality as the convolution on $\mathbb{R}^{n+1}$ of F with the kernel K, where

$$K(x,t)=\int_{\mathbb{R}^n} e^{-it|\xi|^2+ix\cdot\xi}\frac{d\xi}{|\xi|^{2s}}. \tag{8}$$

If we take a partition of unity by smooth functions $\{\varphi_j\}_{j\in\mathbb{Z}}$ defined on $\mathbb{R}^{n+1}$ such that for each j

$$\text{supp}\,(\varphi_{\rm j})\subseteq \mathrm{B}(0,2^{\rm j})\times(-2^{2\rm j},2^{2\rm j})\setminus \mathrm{B}(0,2^{\rm j-2})\times(-2^{2(\rm j-2)},2^{2(\rm j-2)}),$$

it is enough to prove that

$$\sum_{j\in\mathbb{Z}} |\langle(\varphi_j K) * F, G\rangle| \lesssim \|V\|_{\mathcal{L}^{2s+2,p}_{\mathrm{par}}} \|F\|_{L^2_{x,t}(V^{-1})} \|G\|_{L^2_{x,t}(V^{-1})}.$$

If we write $\{\langle(\varphi_j K) * F, G\rangle\}_{j\in\mathbb{Z}} = \mathcal{T}(F,G)$, then the previous inequality is equivalent to a bound for

$$\mathcal{T} \ : \ L^2_{x,t}(V^{-1}) \times L^2_{x,t}(V^{-1}) \to \ell^0_1(\mathbb{C}). \tag{9}$$

Here, in general, for $a \in \mathbb{R}$ and $1 \le p \le \infty$, $\ell^a_p(\mathbb{C})$ denotes the Banach space of sequences $\bar{x} = \{x_k\}_{k\in\mathbb{Z}}$ such that

$$\|\bar{x}\|_{\ell^a_p} = \begin{cases} \left(\sum_{k\in\mathbb{Z}} \left(2^{ka}|x_k|\right)^p\right)^{1/p} < \infty, & \text{if } p \neq \infty \\ \sup_{k\in\mathbb{Z}} \left(2^{ka}|x_k|\right) < \infty, & \text{if } p = \infty. \end{cases}$$

Following the ideas of Keel and Tao [13], we now appeal to a bilinear interpolation result (see [6, Section 3.13, Exercise 5(b)]).

Lemma 2.1

If A_0, A_1, B_0, B_1, C_0 and C_1 are Banach spaces, and T is a bilinear operator such that

$$T : A_0 \times B_0 \longrightarrow C_0,$$
$$T : A_0 \times B_1 \longrightarrow C_1,$$
$$T : A_1 \times B_0 \longrightarrow C_1,$$

then, whenever $0 < \theta_0, \theta_1 < \theta = \theta_0 + \theta_1 < 1$, $1 \le p, q \le \infty$ and $1 \le \frac{1}{p} + \frac{1}{q}$, we have that

$$T : (A_0, A_1)_{\theta_0,p} \times (B_0, B_1)_{\theta_1,q} \longrightarrow (C_0, C_1)_{\theta,1}.$$

Here $(C_0, C_1)_{\theta,1}$ denotes the Banach space obtained from C_0 and C_1 by the real interpolation method. Implicit in this is that the bounds interpolate nicely.

We will prove that whenever $\frac{n}{4} \le s < \frac{n}{2}$ and $1 < p \le \frac{n+2}{2s+2}$, the bilinear vector-valued operator $\mathcal{T}$ satisfies

$$\|\mathcal{T}(F,G)\|_{\ell^{\beta_0}_\infty(\mathbb{C})} \lesssim \|V\|^p_{\mathcal{L}^{2s+2,p}_{\mathrm{par}}(\mathbb{R}^n\times\mathbb{R})} \|F\|_{L^2_{x,t}(V^{-p})} \|G\|_{L^2_{x,t}(V^{-p})}, \tag{10}$$

$$\|\mathcal{T}(F,G)\|_{\ell^{\beta_1}_\infty(\mathbb{C})} \lesssim \|V\|^{p/2}_{\mathcal{L}^{2s+2,p}_{\mathrm{par}}(\mathbb{R}^n\times\mathbb{R})} \|F\|_{L^2_{x,t}(V^{-p})} \|G\|_{L^2_{x,t}}, \tag{11}$$

$$\|\mathcal{T}(F,G)\|_{\ell^{\beta_1}_\infty(\mathbb{C})} \lesssim \|V\|^{p/2}_{\mathcal{L}^{2s+2,p}_{\mathrm{par}}(\mathbb{R}^n\times\mathbb{R})} \|F\|_{L^2_{x,t}} \|G\|_{L^2_{x,t}(V^{-p})}, \tag{12}$$

with $\beta_0 = (2s+2)(p-1)$ and $\beta_1 = (2s+2)\left(\frac{p}{2} - 1\right)$.

Using these estimates (10), (11) and (12), we may apply Lemma 2.1 to obtain an appropriate interpolated bound for

$$\mathcal{T} : \left(L^2_{x,t}(V^{-p}), L^2_{x,t}\right)_{\theta_0,q_0} \times \left(L^2_{x,t}(V^{-p}), L^2_{x,t}\right)_{\theta_1,q_1} \to \left(\ell^{\beta_0}_\infty(\mathbb{C}), \ell^{\beta_1}_\infty(\mathbb{C})\right)_{\theta,1},$$

whenever $0 < \theta_0, \theta_1 < \theta = \theta_0 + \theta_1 < 1$ and $1 \le \frac{1}{q_0} + \frac{1}{q_1}$. The proof will then be completed on taking $\theta_0 = \theta_1 = 1 - \frac{1}{p}$, $q_0 = q_1 = 2$, and using the following real interpolation space identities (see [6, Theorems 5.6.1 and 5.4.1]), *(i)* $(\ell_\infty^{\beta_0}(\mathbb{C}), \ell_\infty^{\beta_1}(\mathbb{C}))_{\theta,1} = \ell_1^0(\mathbb{C})$ if $\beta_0 \neq \beta_1$ and $(1-\theta)\beta_0 + \theta\beta_1 = 0$ and

(ii) $(L^2_{x,t}(V^{-p}), L^2_{x,t})_{\theta_0,q_0} = L^2_{x,t}(V^{-1})$ and $(L^2_{x,t}(V^{-p}), L^2_{x,t})_{\theta_1,q_1} = L^2_{x,t}(V^{-1})$.

We now turn to the proofs of (10), (11) and (12). In order to obtain (10) we have to prove that for each $j \in \mathbb{Z}$,

$$\big|\langle(\varphi_j K) * F, G\rangle\big| \lesssim 2^{-j(2s+2)(p-1)} \|V\|^p_{\mathcal{L}^{2s+2,p}_{\text{par}}(\mathbb{R}^n\times\mathbb{R})} \|F\|_{L^2_{x,t}(V^{-p})} \|G\|_{L^2_{x,t}(V^{-p})}.$$

For j fixed we decompose $\mathbb{R}^{n+1}$ into a grid of rectangles $\{Q_\nu\}_{\nu\in\mathbb{Z}^{n+1}}$ with disjoint interiors and dimensions $2^j \times \cdots \times 2^j \times 2^{2j}$, so that

$$\big|\langle(\varphi_j K) * F, G\rangle\big| \leq \sum_{\nu\in\mathbb{Z}^{n+1}} \sum_{\mu\in\mathbb{Z}^{n+1}} \big|\langle(\varphi_j K) * (F\chi_{Q_\nu}), G\chi_{Q_\mu}\rangle\big|.$$

Due to the disjointness of the supports of $(\varphi_j K) * F\chi_{Q_\nu}$ and $G\chi_{Q_\mu}$, we have that

$$\big|\langle(\varphi_j K) * F, G\rangle\big| \leq \sum_{\nu\in\mathbb{Z}^{n+1}} \big|\langle(\varphi_j K) * (F\chi_{Q_\nu}), G\chi_{\tilde{Q}_\nu}\rangle\big|,$$

where $\tilde{Q}_\nu$ denotes the rectangle with the same centre as Q_ν but dilated three times about its centre. From here, using Young's inequality, the definition of K given in (8), the case $\gamma \geq n/2$ of Lemma 2.2 (see below) and the Cauchy–Schwarz inequality, we obtain that

$$\begin{aligned}\big|\langle(\varphi_j K) * F, G\rangle\big| &\leq \sum_{\nu\in\mathbb{Z}^{n+1}} \|\varphi_j K\|_{L^\infty_{x,t}} \|F\chi_{Q_\nu}\|_{L^1_{x,t}} \|G\chi_{\tilde{Q}_\nu}\|_{L^1_{x,t}} \\ &\lesssim 2^{-j(n-2s)} \sum_{\nu\in\mathbb{Z}^{n+1}} \|F\chi_{Q_\nu}\|_{L^1_{x,t}} \|G\chi_{\tilde{Q}_\nu}\|_{L^1_{x,t}} \qquad (13) \\ &\lesssim 2^{-j(2s+2)(p-1)} \|V\|^p_{\mathcal{L}^{2s+2,p}_{\text{par}}} \|F\|_{L^2_{x,t}(V^{-p})} \|G\|_{L^2_{x,t}(V^{-p})}.\end{aligned}$$

This completes the proof of (10). Finally, inequalities (11) and (12) (which are equivalent by symmetry) follow by a similarly straightforward application of the Cauchy–Schwarz inequality in (13). □

Lemma 2.2

Let $0 < \gamma < n$. Then if $t \neq 0$

$$\left|\int_{\mathbb{R}^n} e^{-\pi i t|\xi|^2 + 2\pi i x\cdot\xi} \frac{d\xi}{|\xi|^\gamma}\right| \lesssim \begin{cases} \dfrac{|t|^{-(n/2-\gamma)}}{(|x|^2+|t|)^{\gamma/2}} & \text{if } 0 \leq \gamma \leq \dfrac{n}{2}, \\[2ex] \dfrac{1}{(|x|^2+|t|)^{(n-\gamma)/2}} & \text{if } \dfrac{n}{2} \leq \gamma < n. \end{cases}$$

This estimate can be reduced to the case $t = 1$ by a change of variables, and is then the content of [7, Lemma A.1].

Remarks

(1) Following the same argument used to prove (10) and (11), but using Young's inequality in (13) with more general exponents, we obtain similar such estimates when s is smaller than $\frac{n}{4}$, with the space $L^2_{x,t}(V^{-p})$ replaced by $L^2_{x,t}(V^{-q})$ for

$$0 \le q < p, \beta_0 = (2s+2)(q-1) + n\Big(1 - \frac{q}{p}\Big)$$

and

$$\beta_1 = (2s+2)\Big(\frac{q}{2} - 1\Big) + n\Big(1 - \frac{q}{p}\Big).$$

However, the desired estimate (5) for $s < \frac{n}{4}$ fails to follow from this since the interpolation method that we use requires that $(1-\theta)\beta_0 + \theta\beta_1 = 0$ with $\theta = 2 - \frac{2}{q}$, and thus $q = p$. We include the relevant oscillatory integral estimate (for $\gamma < \frac{n}{2}$) in the statement of Lemma 2.2 merely for the sake of completeness.

(2) We note that for $\gamma = \frac{n}{2}$ in Lemma 2.2 we have an exact formula,

$$\int_{\mathbb{R}^n} e^{-\pi i t|\xi|^2 + 2\pi i x\cdot\xi} \,\frac{d\xi}{|\xi|^{n/2}} = \frac{C_n}{|t|^{1/2}|x|^{(n-2)/2}} e^{i|x|^2/|t|} J_{(n-2)/4}\left(\frac{|x|^2}{8|t|}\right).$$

This follows using polar coordinates and the identities (6.686-1-2) given on [12, page 759]. Lemma 2.2, in this case, is easily seen to be sharp by appealing to standard Bessel function asymptotics.

3. The proof of Proposition 1.2

Following [1], let $0 < \delta << 1$ and $0 < \sigma < \frac{1}{2}$. We consider the function of one variable

$$g = \sum_{\ell\in\mathbb{N},\ 1\le \ell\le \delta^{-\sigma}} \chi_{(\ell\delta^\sigma - \delta, \ell\delta^\sigma + \delta)}. \tag{14}$$

Note that g is simply the characteristic function of a union of disjoint, equally spaced subintervals of $[0, 1]$ of equal size. (If preferred, these characteristic functions may be replaced by smooth compactly supported bump functions.) We now define f by

$$\widehat{f}(\xi) = \prod_{j=1}^{n} g(\xi_j), \tag{15}$$

where $\xi = (\xi_1, \ldots, \xi_n)$.

We now set $X = \{p\delta^{-\sigma} : p \in \mathbb{N} \text{ with } p \lesssim \delta^{\sigma-1}\}$, Ω to be an $O(1)$-neighbourhood of

$$\Lambda = \big\{(x,t) \in \mathbb{R}^n \times \mathbb{R} : x \in X^n \text{ and } t = 2q\delta^{-2\sigma} \text{ where } q \in \mathbb{N} \text{ and } q \lesssim \delta^{2\sigma-1}\big\},$$

and V the characteristic function of the set Ω.

Arguing as in [1], for $(x,t) \in \Omega$ we have the uniform bound

$$\begin{aligned} \left|e^{it\Delta} f(x)\right| &= \left|\int_{\mathbb{R}^n} e^{-it|\xi|^2 + ix\cdot\xi} \widehat{f}(\xi)\, d\xi\right| \\ &\sim \int_{\mathbb{R}^n} \widehat{f}(\xi)\, d\xi \sim \delta^{n(1-\sigma)}, \end{aligned} \tag{16}$$

and hence,

$$\|e^{it\Delta}f\|_{L^2_{x,t}(V)} \sim \delta^{n(1-\sigma)}|\Omega|^{1/2} = \delta^{(1-\sigma)n/2+\sigma-1/2}. \tag{17}$$

On the other hand,

$$\|f\|_{\dot{H}^s(\mathbb{R}^n)} = \||\cdot|^s\widehat{f}\|_{L^2(\mathbb{R}^n)} \lesssim \delta^{(1-\sigma)n/2}, \tag{18}$$

and for $1 \leq p \leq \frac{n+2}{2s+2}$,

$$\|V\|_{\mathcal{L}^{2s+2,p}_{\text{par}}} \sim \max\left\{1, \delta^{(\sigma(n+2))/p-s-1}, \delta^{(\sigma(n+2)+1)/p-2s-2}\right\}.$$

Letting $\delta \to 0$ now leads the required necessary condition for (7) (and hence (5)). □

References

1. J.A. Barceló, J.M. Bennett, A. Carbery, A. Ruiz, and M.C. Vilela, Some special solutions of the Schrödinger equation, *Indiana Univ. Math. J.* **56** (2007), 1581–1593.
2. J.A. Barceló, J.M. Bennett, A. Carbery, A. Ruiz, and M.C. Vilela, A note on weighted estimates for the Schrödinger operator, *Rev. Mat. Complut.* **21** (2008), 481–488.
3. J.A. Barceló, A. Ruiz, and L. Vega, Weighted estimates for the Helmholtz equation and some applications, *J. Funct. Anal.* **150** (1997), 356–382.
4. J.M. Bennett, A. Carbery, F. Soria, and A. Vargas, A Stein conjecture for the circle, *Math. Ann.* **336** (2006), 671–695.
5. O. Blasco, A. Ruiz, and L. Vega, Non-interpolation in Morrey–Campanato and block spaces, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **28** (1999), 31–40.
6. J. Bergh and J. Löfström, *Interpolation Spaces, An Introduction,* Springer-Verlag, Berlin-New York, 1976.
7. T. Cazenave, L. Vega, and M.C. Vilela, A note on the nonlinear Schrödinger equation in weak L^p spaces, *Commun. Contemp. Math.* **3** (2001), 153–162.
8. T. Cazenave and F.B. Weissler, The Cauchy problem for the nonlinear Schrödinger equation in H^1, *Manuscripta Math.* **61** (1988), 477–494.
9. S. Chanillo and E. Sawyer, Unique continuation for $\Delta + V$ and the C. Fefferman–Phong class, *Trans. Amer. Math. Soc.* **318** (1990), 275–300.
10. C. Fefferman and D.H. Phong, Lower bounds for Schrödinger equations, *Conference on Partial Differential Equations (Saint Jean de Monts)*, Conf. No. 7, Soc. Math. France, Paris, 1982.
11. J. Ginibre and G. Velo, The global Cauchy problem for the nonlinear Schrödinger equation revisited, *Ann. Inst. H. Poincaré Anal. Non Linéare* **2** (1985), 309–327.
12. A. Jeffrey (Ed.) *Table of Integrals, Series and Products,* Academic Press, New York-London, 1965.
13. M. Keel and T. Tao, Endpoint Strichartz estimates, *Amer. J. Math.* **120** (1998), 955–980.
14. A. Ruiz and L. Vega, Unique continuation for Schrödinger operators with potential in Morrey spaces, Conference on Mathematical Analysis (El Escorial, 1989), *Publ. Mat.* **35** (1991), 291–298.
15. A. Ruiz and L. Vega, Local regularity of solutions to wave equations with time-dependent potentials, *Duke Math. J.* **76** (1994), 913–940.
16. R.S. Strichartz, Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations, *Duke Math. J.* **44** (1977), 705–714.
17. T.H. Wolff, Unique continuation for $|\Delta u| \leq V|\nabla u|$ and related problems, *Rev. Mat. Iberoamericana* **6** (1990), 155–200.