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On existence of solutions of a neutral differential equation with deviating argument

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Abstract

We establish theorems on the existence and asymptotic characterization of solutions of a differential equation of neutral type with deviated argument on neutral type. The mentioned differential equation admits both delayed and advanced arguments. In our considerations we use technique linking measures of noncompactness with the Tikhonov fixed point principle in suitable Frechet space. This approach admits us to improve and extend some results.

1. Introduction

The aim of the present paper is an improvement to the results contained in papers [4, 11], concerning the existence and asymptotic behaviour of solutions of the following neutral differential equation with deviating argument

$$y'(t) = f(t, y(H(t)), y'(h(t))), \qquad \text{where} \;\; t \in \mathbb{R} = [0, \infty) \tag{1}$$

with the initial condition of the form

$$y(0) = 0. \tag{2}$$

The problem (1)-(2) was considered in [4, 11] (see also [1, 2, 3, 6, 7, 8, 9, 12, 13] under strong assumptions. Authors obtained their results by using two-component measure of noncompactness in the Banach space $C_p(\mathbb{R}_+)$ consisting of all continuous functions on $\mathbb{R}_+$ and tempered by p.

Keywords: Neutral differential equation with deviating argument, asymptotic behaviour, measure of noncompactness, Tikhonov fixed point theorem.

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In this paper we improve those existence theorems and formulate our assumptions in more concise form. This aim is achieved in virtue of conjunction of an appropriate one-component measure of noncompactness in a Frechet space and Tikhonov fixed point principle.

2. Notation and auxiliary facts

For further purposes, we collect in this section a few auxiliary results which will be needed in the sequel.

Consider

$$C(\mathbb{R}_+) = \{x : \mathbb{R}_+ \to \mathbb{R}, \quad x \text{ continuous}\},$$

equipped with the family of seminorms $|x|_n = \sup\{|x(t)| : t \in [0, n]\}, \quad n \in \mathbb{N}$.
$C(\mathbb{R}_+)$ becomes a Fréchet space furnished with the distance

$$d(x, y) = \sup\left\{2^{-n}\frac{|x-y|_n}{1+|x-y|_n} : \quad n \in \mathbb{N}\right\}.$$

It is known that $C(\mathbb{R}_+)$ is a locally convex space.

Let us recall the following fact:

(A) a sequence (x_n) is convergent to x in $C(\mathbb{R}_+)$ if and only if (x_n) is uniformly convergent to x on compact subsets of $\mathbb{R}_+$.

If X is a subset of $C(\mathbb{R}_+)$, then $\overline{X}$ and Conv$X$ denote the closure and convex closure of X, respectively. We use the symbols λX and $X + Y$ to denote the algebraic operations on sets. Moreover, the symbol $\mathfrak{M}_C$ denotes the family of all nonempty subsets of $C(\mathbb{R}_+)$ consisting of functions uniformly bounded on compact intervals of $\mathbb{R}_+$ i.e.

$$\mathfrak{M}_C = \left\{X \subset C(\mathbb{R}_+) : X \neq \emptyset \text{ and } \forall_{T>0} \sup\left\{|x(t)| : x \in X, \ t \leq T\right\} < \infty\right\},$$

while $\mathfrak{N}_C$ stands for its subfamily consisting of all relatively compact sets.

Now, we recall the definition of quantities which will be used in our further investigations. These ones was introduced and studied in [5].

Let $X \in \mathfrak{M}_C$. Fix $T > 0, \ \varepsilon > 0$ and let us denote

$$\omega^T(x, \varepsilon) = \sup\left\{|x(t) - x(s)| : \ t, s \in [0, T], \ |t - s| \leq \varepsilon\right\}.$$

Further, let us put:

$$\omega^T(X, \varepsilon) = \sup\left\{\omega^T(x, \varepsilon) : x \in X\right\},$$

$$\omega_0^T(X) = \lim_{\varepsilon \to 0} \omega^T(X, \varepsilon),$$

$$\omega_0(X) = \lim_{T \to \infty} \omega_0^T(X).$$

The mapping $\omega_0 : \mathfrak{M}_C \to [0, \infty]$ is called a *measure of noncompactness* and it satisfies the following conditions (see [10]):

1° the family $\ker \omega_0 = \{X \in \mathfrak{M}_C : \omega_0(X) = 0\} = \mathfrak{N}_C$,

2° $X \subset Y \Rightarrow \omega_0(X) \leq \omega_0(Y)$,

3° $\omega_0(\overline{X}) = \omega_0(\mathrm{Conv}X) = \omega_0(X)$,

4° $\omega_0(\lambda X + (1-\lambda)Y) \leq \lambda\omega_0(X) + (1-\lambda)\omega_0(Y)$ for $\lambda \in [0,1]$.

5° If (X_n) is a sequence of closed sets from $\mathfrak{M}_C$ such that $X_{n+1} \subset X_n$ $(n = 1, 2, ...)$ and if $\lim\limits_{n\to\infty} \omega_0(X_n) = 0$ then the set $X_\infty = \bigcap\limits_{n=1}^{\infty} X_n$ is nonempty.

The family ker μ defined in 1° is called *the kernel of the measure of noncompactness* μ.

Other facts concerning measures of noncompactness may be found in [5].

3. Main result

This section is devoted to the study of the differential equation with deviating argument (1) with the initial condition (2). The problem (1)-(2) will be investigated under the following assumptions:

(H_1) the function $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a constant $k < 1$ such that
$$|f(t,x,y_1) - f(t,x,y_2)| \leq k|y_1 - y_2|$$
for all $t \in \mathbb{R}_+$ and $x, y_1, y_2 \in \mathbb{R}$,

(H_2) $h : \mathbb{R}_+ \to \mathbb{R}_+$ and $H : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions such that $h(t) \leq t$,

(H_3) there exist a continuous functions A and $B : \mathbb{R}_+ \to (0,\infty)$ such that the following inequality
$$|f(t,x,0)| \leq A(t) + B(t)|x|$$
holds for each $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$,

(H_4) there is a nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following inequality
$$A(t) + B(t)\int_0^{H(t)} \psi(\tau)d\tau \leq (1-k)\psi(t). \tag{3}$$

Now, let us put $x(t) = y'(t)$. Then the problem (1)-(2) can be replaced equivalently by the following functional-integral equation
$$x(t) = f\Big(t, \int_0^{H(t)} x(s)ds,\ x(h(t))\Big), \quad t \in \mathbb{R}_+. \tag{4}$$

In the sequel we will examine the equation (4).

Now, we formulate the first general result of the paper.

Theorem 3.1

Under the assumptions (H_1)-(H_4) *the equation* (4) *has at least one solution* $x \in C(\mathbb{R}_+)$ *such that* $|x(t)| \le \psi(t)$ *for* $t \ge 0$.

Proof. Let us consider the operator F defined on the space $C(\mathbb{R}_+)$ by the formula

$$(Fx)(t) = f\Big(t, \int_0^{H(t)} x(s)ds,\ x(h(t))\Big), \quad t \ge 0.$$

Observe that in view of assumptions (H_1) and (H_2), the function Fx is continuous on $\mathbb{R}_+$ for any $x \in C(\mathbb{R}_+)$, i.e. F transforms the space $C(\mathbb{R}_+)$ into itself.

Let us denote: $\overline{H}(t) = \sup_{s \le t} H(s)$.

In what follows let us fix $T > 0$ and $\varepsilon > 0$. Next, take any $t, s \in [0,T]$ with $|t-s| \le \varepsilon$. Then, for arbitrarily fixed $x \in C(\mathbb{R}_+)$ such that $|x(t)| \le \psi(t)$ for $t \ge 0$ we get

$$\begin{aligned}
|(Fx)(t) - (Fx)(s)| \le & \left| f\Big(t, \int_0^{H(t)} x(\tau)d\tau, x(h(t))\Big) - f\Big(s, \int_0^{H(t)} x(\tau)d\tau, x(h(t))\Big) \right| \\
& + \left| f\Big(s, \int_0^{H(t)} x(\tau)d\tau, x(h(t))\Big) - f\Big(s, \int_0^{H(s)} x(\tau)d\tau, x(h(t))\Big) \right| \\
& + \left| f\Big(s, \int_0^{H(s)} x(\tau)d\tau, x(h(t))\Big) - f\Big(s, \int_0^{H(s)} x(\tau)d\tau, x(h(s))\Big) \right|.
\end{aligned}$$

Hence, in view of the assumption (H_1) we get

$$\omega^T(Fx,\varepsilon) \le \nu_1^T(f,\varepsilon) + \nu_2^T\big(f, \omega^T(H,\varepsilon)\cdot \psi(T)\big) + k\omega^T\big(x, \omega^T(h,\varepsilon)\big), \tag{5}$$

where we denoted

$$\nu_1^T(f,\varepsilon) = \sup\Big\{ |f(s,x,y) - f(t,x,y)| : \quad t,s \in [0,T], \quad |t-s| \le \varepsilon\ ,$$
$$|x| \le \int_0^{\overline{H}(T)} \psi(\tau)d\tau, \ |y| \le \psi(T)\Big\},$$

$$\nu_2^T(f,\varepsilon) = \sup\Big\{ |f(s,x_1,y) - f(s,x_2,y)| : \ s \in [0,T], \quad |x_1 - x_2| \le \varepsilon,$$
$$|x_1|, |x_2| \le \int_0^{\overline{H}(T)} \psi(\tau)d\tau, \quad |y| \le \psi(T)\Big\}.$$

Let us observe that the uniform continuity of the functions $f(t,x,y), h(t)$ and $H(t)$ on compact subsets of $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$ and $\mathbb{R}_+$ implies that the functions $\nu_1^T(f,\varepsilon)$, $\nu_2^T(f,\varepsilon)$, $\omega^T(h,\varepsilon)$, $\omega^T(H,\varepsilon) \to 0$ when $\varepsilon \to 0$ for fixed arbitrary $T > 0$.

Next we denote

$$\xi_0(T,\varepsilon) = \nu_1^T(f,\varepsilon) + \nu_2^T\big(f, \omega^T(H,\varepsilon)\cdot\psi(T)\big),$$

and we put

$$\xi_{n+1}(T,\varepsilon) = \xi_n\big(T, \omega^T(h,\varepsilon)\big) \quad \text{for} \quad n = 0,1,2,\dots\,.$$

Let us consider the series

$$\sum_{n=0}^{\infty} k^n \xi_n(T,\varepsilon).$$

For fixed $T > 0$ all functions $\xi_n(T,\varepsilon)$ are uniformly bounded for $\varepsilon \geq 0$ and $n = 0,1,2,\dots$. This implies that the series is convergent for any $T > 0$ and $\varepsilon \geq 0$. Obviously $\xi_n(T,\varepsilon) \to 0$ when $\varepsilon \to 0$.

Next, we define the set Ω in the space $C(\mathbb{R}_+)$ as follows

$$\Omega = \Big\{ x \in C(\mathbb{R}_+) : |x(t)| \leq \psi(t) \text{ for } t \geq 0 \ \text{ and } \ \omega^T(x,\varepsilon) \leq \sum_{n=0}^{\infty} k^n \xi_n(T,\varepsilon) \text{ for T} \geq 0 \Big\}.$$

Keeping in mind the fact (A) and the convexity of $\omega^T(x,\varepsilon)$ we deduce that Ω is closed, convex and nonempty subset of $C(\mathbb{R}_+)$.

We now proceed to show that $F : \Omega \to \Omega$. Let $x \in \Omega$. Taking into account the assumptions (H_1)-(H_4) we obtain the following chain of inequalities

$$\begin{aligned}
|(Fx)(t)| \leq{}& \left| f\Big(t, \int_0^{H(t)} x(\tau)d\tau, x(h(t))\Big) - f\Big(t, \int_0^{H(t)} x(\tau)d\tau, 0\Big) \right| \\
&+ \left| f\Big(t, \int_0^{H(t)} x(\tau)d\tau, 0\Big) \right| \leq k|x(h(t))| + A(t) + B(t) \int_0^{H(t)} |x(\tau)|d\tau \\
&\leq k|\psi(h(t))| + A(t) + B(t) \int_0^{H(t)} \psi(\tau)d\tau \leq k\psi(t) + (1-k)\psi(t) = \psi(t)
\end{aligned}$$

i.e. $|(Fx)(t)| \leq \psi(t)$.

Moreover, in the light of (5), above notations and the definition of the set Ω we derive

$$\begin{aligned}
\omega^T(Fx,\varepsilon) &\leq \nu_1^T(f,\varepsilon) + \nu_2^T(f, \omega^T(H,\varepsilon)\cdot\psi(T)) + k\omega^T(x, \omega^T(h,\varepsilon)) \\
&\leq \xi_0(T,\varepsilon) + k\sum_{n=0}^{\infty} k^n \xi_n(T, \omega^T(h,\varepsilon)) = \xi_0(T,\varepsilon) \\
&+ \sum_{n=0}^{\infty} k^{n+1} \xi_{n+1}(T,\varepsilon) = \sum_{n=0}^{\infty} k^n \xi_n(T,\varepsilon)
\end{aligned}$$

so $\omega^T(Fx,\varepsilon) \leq \sum\limits_{n=0}^{\infty} k^n \xi_n(T,\varepsilon)$ and $F:\Omega \to \Omega$.

Next, let us notice that $\omega^T(\Omega,\varepsilon) \leq \sum\limits_{n=0}^{\infty} k^n \xi_n(T,\varepsilon)$, and when $\varepsilon \to 0$ we obtain $\omega_0^T(\Omega) = 0$ and $\omega_0(\Omega) = 0$, hence, in view of 1^o we infer that Ω is compact.

Finally, we prove that F is continuous on Ω. Let $x, x_n \in \Omega$ and $x_n \to x$ in $C(\mathbb{R}_+)$ i.e. $x_n \to x$ uniformly on every bounded interval . Let $T > 0$. Keeping in mind the assumptions (H_1) and previous notations we obtain

$$\begin{aligned}
|(Fx)(t) - (Fx_n)(t)| &\leq \left| f\Big(t, \int_0^{H(t)} x(\tau)d\tau, x(h(t))\Big) - f\Big(t, \int_0^{H(t)} x(\tau)d\tau, x_n(h(t))\Big) \right| \\
&\quad + \left| f\Big(t, \int_0^{H(t)} x(\tau)d\tau, x_n(h(t))\Big) - f\Big(t, \int_0^{H(t)} x_n(\tau)d\tau, x_n(h(t))\Big) \right| \\
&\leq k \sup_{t\leq T} \big| x(h(t)) - x_n(h(t)) \big| + \nu_2^T\big(f, \overline{H}(T) \cdot \sup_{t \leq T} |x(t) - x_n(t)|\big).
\end{aligned}$$

Hence, in view of $\lim\limits_{n\to\infty} \sup\limits_{t\leq T} |x(t) - x_n(t)| = 0$ we get continuity of F on Ω.

Finally, taking into account the properties of the operator $F:\Omega \to \Omega$ established above and applying the classical Tikhonov fixed point theorem we infer that the operator F has at least one fixed point x in the set Ω. Obviously the function $x(t)$ is a solutions of the equation (4).

This completes the proof. $\square$

The assumption (H_4) of the above theorem seems to be too uncomfortable for direct applications therefore we will formulate this condition in more useful form. We start with the following lemma.

Lemma 3.2

If there exists a nondecreasing function $\psi_0 : \mathbb{R}_+ \to (0,\infty)$ satisfying two condition

$$\sup\left\{\frac{A(t)}{\psi_0(t)} : t \geq 0\right\} < \infty \tag{6}$$

and

$$\sup\left\{B(t)H(t)\frac{\psi_0(H(t))}{\psi_0(t)} : t \geq 0\right\} < 1-k \tag{7}$$

then the inequality (3) *has a solution.*

Proof. Let us take $C > 0$ so big that

$$\sup\left\{\frac{A(t)}{C\psi_0(t)} : t \geq 0\right\} + \sup\left\{B(t)H(t)\frac{\psi_0(H(t))}{\psi_0(t)} : t \geq 0\right\} \leq 1-k \tag{8}$$

and put $\psi(t) = C\psi_0(t)$.

Taking into account the inequality

$$\int_0^{H(t)} \psi(\tau)d\tau \leq H(t)\psi(H(t))$$

and the estimate (8) we obtain

$$A(t) + B(t) \int_0^{H(t)} \psi(\tau)d\tau \leq A(t) + B(t)H(t)C\psi_0(H(t)) \leq (1-k)C\psi_0(t).$$

This completes the proof. $\square$

Putting various functions for $\psi_0(t)$ we can generate many conditions ensuring solvability of the inequality (3). First let us consider the following assumption.

(H_5) there exist a nondecreasing continuous function $A : \mathbb{R}_+ \to (0, \infty)$ and a continuous function $B : \mathbb{R}_+ \to (0, \infty)$ such that

$$\sup \left\{ B(t)H(t)\frac{A(H(t))}{A(t)} : t \geq 0 \right\} < 1 - k$$

and the following inequality

$$|f(t, x, 0)| \leq A(t) + B(t)|x|$$

holds for each $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$.

Observe, that the assumption (H_5) fulfills the conditions (6) and (7) of Lemma 3.2 for $\psi_0(t) = A(t)$, so applying Theorem 3.1 we get the following theorem.

Theorem 3.3

Under the assumptions (H_1), (H_2) and (H_5) the equation (4) *has at least one solution $x \in C(\mathbb{R}_+)$.*

Remark 3.4 This Theorem 3.3 improves [11, Theorem 1]. Indeed, applying notations from this paper and putting

$$A(t) = \sup_{s \leq t} a(s) \cdot \exp h(t), \;\; B(t) = \exp(-H(t))$$

we can easy show that (H_5) is fulfilled. Moreover, let us observe that following conditions in [11] are needless:

$$\lim_{t\to\infty} h(t) = \infty, \quad \limsup_{t\to\infty}(t - h(t)) = \infty, \quad H(0) = 0,$$

and condition $a(t) \to 0$ when $t \to \infty$ can be replaced by boundedness of $a(t)$ for $t \geq 0$.

Now we give next sufficient condition for solvability of the inequality (3).

(H_6) There exist a nondecreasing continuous function $A : \mathbb{R}_+ \to (0, \infty)$ and a continuous function $B : \mathbb{R}_+ \to (0, \infty)$ such that

$$\sup\left\{B(t)H(t)\exp(A(H(t)) - A(t)) : t \geq 0\right\} < 1 - k$$

and the following inequality

$$|f(t, x, 0)| \leq A(t) + B(t)|x|$$

holds for each $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$.

Taking $\psi_0(t) = \exp(A(t))$ and using Lemma 3.2 and Theorem 3.1 we get.

Theorem 3.5

Under the assumptions (H_1), (H_2) and (H_6) the equation (4) *has at least one solution $x \in C(\mathbb{R}_+)$.*

Lemma 3.6

If there exists a nondecreasing function $\psi_0 : \mathbb{R}_+ \to (0, \infty)$ satisfying two conditions

$$\sup\left\{\frac{A(t)}{\psi_0(t)} : t \geq 0\right\} < \infty \text{ and } \sup\left\{\frac{B(t)}{\psi_0(t)} \int\limits_0^{H(t)} \psi_0(\tau))d\tau : t \geq 0\right\} < 1 - k \qquad (9)$$

then the inequality (3) *has a solution.*

Proof. Let us take $C > 0$ so large that

$$\sup\left\{\frac{A(t)}{C\psi_0(t)} : t \geq 0\right\} + \sup\left\{\frac{B(t)}{\psi_0(t)} \int\limits_0^{H(t)} \psi_0(\tau)d\tau : t \geq 0\right\} \leq 1 - k$$

and put $\psi(t) = C\psi_0(t)$. This implies

$$\frac{A(t)}{\psi(t)} + \frac{B(t)}{\psi_0(t)} \int\limits_0^{H(t)} \psi_0(\tau)d\tau \leq 1 - k \qquad \text{for } \ t \geq 0$$

and therefore

$$A(t) + B(t) \int\limits_0^{H(t)} \psi(\tau)d\tau \leq (1 - k)\psi \qquad \text{for } \ t \geq 0$$

what confirm that $\psi(t)$ is a solution of the inequality (3). This completes the proof. $\square$

Let us put $\psi_0(t) = \exp(ML(t))$ where $L(t) = \int\limits_0^t (A(\tau) + B(\tau))d\tau$ and M is enough big. The first part of (9) is satisfied if

$$\sup\left\{A(t)\exp\Big(-\int\limits_0^t \exp(A(\tau))d\tau\Big) : t \geq 0\right\} < \infty \text{ and } M > 1.$$

The second part of (9) takes form

$$\sup\left\{B(t)\exp(-ML(t))\int\limits_0^{H(t)}\exp(ML(\tau))d\tau : t\geq 0\right\} < 1-k \text{ for some } M>1.$$

Now we can formulate next version of the assumption ensuring the solvability of (3).

(H_7) There exist continuous functions A and $B:\mathbb{R}_+\to(0,\infty)$ such that the inequality

$$|f(t,x,0)|\leq A(t)+B(t)|x|$$

holds for each $t\in\mathbb{R}_+$, $x\in\mathbb{R}$ and such that

$$\sup\left\{A(t)\exp\Big(-\int\limits_0^t\exp(A(\tau))d\tau\Big) : t\geq 0\right\}<\infty$$

and

$$\inf_{M>1}\sup\left\{B(t)\exp(-ML(t))\int\limits_0^{H(t)}\exp(ML(\tau))d\tau : t\geq 0\right\}<1-k$$

where $L(t)=\int\limits_0^t(A(\tau)+B(\tau))d\tau$.

Applying Lemma 3.6 for $\psi_0(t)=\exp(ML(t))$ and Theorem 3.1 we have

Theorem 3.7

Under the assumptions (H_1), (H_2) *and* (H_7) *the equation* (4) *has at least one solution* $x\in C(\mathbb{R}_+)$.

Remark 3.8 Notice that Theorem 3.7 generalizes [4, Theorem 1]. Applying the notations from this paper and putting $A(t)=L_0(t), B(t)=\exp L_1(t)$ we can show, arguing similarly as in [4], that (H_7) is fulfilled. Moreover, the following assumptions in mentioned paper are unnecessary:

$$\lim_{t\to\infty} t\exp L_1(t)=0,\qquad \lim_{t\to\infty}(t-h(t))=0,\qquad H(t)\geq t,$$

and the conditions

$$\lim_{t\to\infty}L_0(t)\exp\Big(-\int\limits_0^t L_0(s)ds\Big)=0\quad\text{and}\quad\lim_{t\to\infty}(H(t)-t)=0$$

can be weaken by boundedness of functions:

$$L_0(t)\exp\Big(-\int\limits_0^t L_0(s)ds\Big)\ \text{ and }\ H(t)-t\ \text{ for }\ t\geq 0\,.$$

EXAMPLE 3.9 Let us consider the following neutral differential equation

$$y'(t) = \frac{x^3(\sqrt{t})}{(2t+3)(x^2(\sqrt{t})+1)} + \frac{t}{t^2+2} y(\exp(t^2)) + \frac{1}{t+1} \tag{10}$$

with the initial condition $y(0) = 0$. Observe that this problem is a particular case of the problem (1)-(2), where

$$f(t,x,y) = \frac{x^3}{(2t+3)(x^2+1)} + \frac{t}{t^2+2} y + \frac{1}{t+1}$$

and $h(t) = \sqrt{t}, H(t) = \exp(t^2)$. Let us put $A(t) = 1/(t+1), B(t) = 1/(2t+3)$.

We show that there are satisfied the assumptions of Theorem 3.3. Indeed, we have

$$|f(t,x,0)| \leq \frac{|x|^3}{(2t+3)(x^2+1)} + \frac{1}{t+1} \leq A(t) + B(t)|x|.$$

Moreover, the function $f = f(t,x,y)$ is continuous on the set $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$ and for arbitrary $t \in \mathbb{R}_+, x, y_1, y_2 \in \mathbb{R}$ we obtain

$$|f(t,x,y_1) - f(t,x,y_2)| \leq \frac{t}{t^2+2}|y_1 - y_2| \leq \frac{1}{2\sqrt{2}}|y_1 - y_2|.$$

This shows that the function $f(t,x,y)$ satisfies the Lipshitz condition with respect to y with the constant $k = \frac{1}{2\sqrt{2}}$. Notice that

$$\sup\left\{B(t)H(t)\frac{A(H(t))}{A(t)} : t \geq 0\right\} = \sup\left\{\frac{\exp(t^2)(t+1)}{(2t+3)(\exp(t^2)+1)} : t \geq 0\right\} = \frac{1}{2} < 1 - k.$$

This confirms that (H_5) is satisfied. Summing up all the above facts, in view of Theorem 3.3 we conclude that the problem (10) with $y(0) = 0$ has a solution $y = y(t)$.

Let us observe that the function $H(t)$ and $h(t)$ do not satisfy the condition $H(t) \leq m\exp(h(t)), m > 0$ from [11, Theorem 1] and the condition $\lim_{t\to\infty}(H(t) - t) = 0$ from [4, Theorem 1].

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