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DENSE INFINITE B_h SEQUENCES

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Abstract: For $h = 3$ and $h = 4$ we prove the existence of infinite B_h sequences $\mathcal{B}$ with counting function

$$\mathcal{B}(x) = x^{\sqrt{(h-1)^2+1}-(h-1)+o(1)}.$$

This result extends a construction of I. Ruzsa for B_2 sequences.

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1. Introduction

Let $h \geq 2$ be an integer. We say that a sequence $\mathcal{B}$ of positive integers is a B_h sequence if all the sums

$$b_1 + \cdots + b_h \quad (b_k \in \mathcal{B},\ 1 \leq k \leq h)$$

are distinct subject to $b_1 \leq b_2 \leq \cdots \leq b_h$. The study of the size of finite B_h sets or of the growing function of infinite B_h sequences is a classic topic in combinatorial number theory. A simple counting argument proves that if $\mathcal{B} \subset [1, n]$ is a B_h set then $|\mathcal{B}| \leq (C_h + o(1))n^{1/h}$ for a constant C_h (see [**2**] and [**4**] for non trivial upper bounds for C_h) and consequently that $\mathcal{B}(x) \ll x^{1/h}$ when $\mathcal{B}$ is an infinite B_h sequence.

Erdős conjectured the existence, for all $\epsilon > 0$, of an infinite B_h sequence $\mathcal{B}$ with counting function $\mathcal{B}(x) \gg x^{1/h-\epsilon}$. It is believed that ϵ cannot be removed from the last exponent, a fact that has only been proved for h even. On the other hand, the *greedy* algorithm produces an infinite B_h sequence $\mathcal{B}$ with

$$\mathcal{B}(x) \gg x^{\frac{1}{2h-1}} \quad (h \geq 2). \tag{1.1}$$

Up to now the exponent $1/(2h-1)$ is the largest known for the growth of a B_h sequence when $h \geq 3$. For further information about B_h sequences see [**5**, §II.2] or [**7**].

For the case $h = 2$, Ajtai, Komlós, and Szemerédi [**1**] proved that there exists a B_2 sequence (also called Sidon sequence) with $\mathcal{B}(x) \gg$

$(x\log x)^{1/3}$, improving by a power of a logarithm the lower bound (1.1). So far the largest improvement of (1.1) for the case $h=2$ was achieved by Ruzsa [**8**]. He constructed, in a clever way, an infinite Sidon sequence $\mathcal{B}$ satisfying

$$\mathcal{B}(x)=x^{\sqrt{2}-1+o(1)}.$$

Our aim is to adapt Ruzsa's ideas to build dense infinite B_3 and B_4 sequences so to improve the lower bound (1.1) for $h=3$ and $h=4$.

Theorem 1.1. *For $h=2,3,4$ there is an infinite B_h sequence $\mathcal{B}$ with counting function*

$$\mathcal{B}(x)=x^{\sqrt{(h-1)^2+1}-(h-1)+o(1)}.$$

The starting point in Ruzsa's construction were the numbers $\log p$, p prime, which form an infinite Sidon set of *real* numbers. Instead we start from the arguments of the Gaussian primes, which also have the same B_h property with the additional advantage of being a bounded sequence. This idea was suggested in [**3**] to simplify the original construction of Ruzsa and was written in detail for B_2 sequences in [**6**].

Since $\sqrt{(h-1)^2+1}-(h-1)\sim 1/(2(h-1))$ for $h\to\infty$ the construction is really meaningful for small values of h and perhaps not so for large ones.

2. The Gaussian arguments

For each rational prime $p\equiv 1 \pmod 4$ we consider the Gaussian prime $\mathfrak{p}$ of $\mathbb{Z}[i]$ such that

$$\mathfrak{p}:=a+bi,\quad p=a^2+b^2,\quad a>b>0,$$

so the argument $\theta(\mathfrak{p})$ of $\mathfrak{p}=\sqrt{p}e^{2\pi i\,\theta(\mathfrak{p})}$ is a real number in the interval $(0,1/8)$. We will use several times throughout the paper the following lemma that can be seen as a measure of the quality of the B_h property of this sequence of real numbers.

Lemma 2.1. *Let $\mathfrak{p}_1,\dots,\mathfrak{p}_h,\mathfrak{p}'_1,\dots,\mathfrak{p}'_h$ be distinct Gaussian primes satisfying $0<\theta(\mathfrak{p}_r),\theta(\mathfrak{p}'_r)<1/8$, $r=1,\dots,h$. The following inequality holds:*

$$\left|\sum_{r=1}^{h}(\theta(\mathfrak{p}_r)-\theta(\mathfrak{p}'_r))\right|>\frac{1}{7|\mathfrak{p}_1\cdots\mathfrak{p}_h\mathfrak{p}'_1\cdots\mathfrak{p}'_h|}.$$

Proof: It is clear that

$$\sum_{r=1}^{h}(\theta(\mathfrak{p}_r)-\theta(\mathfrak{p}'_r))\equiv\theta(\mathfrak{p}_1\cdots\mathfrak{p}_h\overline{\mathfrak{p}'_1\cdots\mathfrak{p}'_h}) \pmod 1. \tag{2.1}$$

Since $\mathbb{Z}[i]$ is a unique factorization domain, all the primes are in the first octant and they are all distinct, the Gaussian integer $\mathfrak{p}_1 \cdots \mathfrak{p}_h \overline{\mathfrak{p}'_1 \cdots \mathfrak{p}'_h}$ cannot be a real integer. Using this fact and the inequality $\arctan(1/x) > 0.99/x$ for $x \geq \sqrt{5 \cdot 13}$ (observe that 5 and 13 are the two smallest primes $p \equiv 1 \pmod 4$) we have

$$\begin{aligned} |\theta(\mathfrak{p}_1 \cdots \mathfrak{p}_h \overline{\mathfrak{p}'_1 \cdots \mathfrak{p}'_h})| &\geq \|\theta(\mathfrak{p}_1 \cdots \mathfrak{p}_h \overline{\mathfrak{p}'_1 \cdots \mathfrak{p}'_h})\| \\ &\geq \frac{1}{2\pi} \arctan\left(\frac{1}{|\mathfrak{p}_1 \cdots \mathfrak{p}_h \overline{\mathfrak{p}'_1} \cdots \overline{\mathfrak{p}'_h}|}\right) \qquad (2.2) \\ &> \frac{1}{7|\mathfrak{p}_1 \cdots \mathfrak{p}_h \overline{\mathfrak{p}'_1} \cdots \overline{\mathfrak{p}'_h}|}, \end{aligned}$$

where $\|\cdot\|$ means the distance to $\mathbb{Z}$. The lemma follows from (2.1) and (2.2). □

We illustrate the B_h property of the arguments of the Gaussian primes with a quick construction of a finite B_h set which is only a $\log x$ factor below the optimal bound. Unfortunately this simple construction cannot be used for infinite B_h sequences because the elements of $\mathcal{A}$ depend on x.

Theorem 2.2. *The set*

$$\mathcal{A} = \left\{ \lfloor x\theta(\mathfrak{p}) \rfloor : |\mathfrak{p}| \leq \left(\frac{x}{7h}\right)^{\frac{1}{2h}} \right\} \subset [1, x]$$

is a B_h set with $|\mathcal{A}| \gg x^{1/h}/\log x$.

Proof: When

$$\lfloor x\theta(\mathfrak{p}_1)\rfloor + \cdots + \lfloor x\theta(\mathfrak{p}_h)\rfloor = \lfloor x\theta(\mathfrak{p}'_1)\rfloor + \cdots + \lfloor x\theta(\mathfrak{p}'_h)\rfloor$$

then

$$x|\theta(\mathfrak{p}_1) + \cdots + \theta(\mathfrak{p}_h) - \theta(\mathfrak{p}'_1) - \cdots - \theta(\mathfrak{p}'_h)| \leq h.$$

If the Gaussian primes are distinct, then Lemma 2.1 implies that

$$|\theta(\mathfrak{p}_1) + \cdots + \theta(\mathfrak{p}_h) - \theta(\mathfrak{p}'_1) - \cdots - \theta(\mathfrak{p}'_h)| > \frac{1}{7|\mathfrak{p}_1 \cdots \mathfrak{p}_h \mathfrak{p}'_1 \cdots \mathfrak{p}'_h|} \geq h/x,$$

which is a contradiction.

We observe that for each prime $p \equiv 1 \pmod 4$ there is a Gaussian prime $\mathfrak{p}$ with $|\mathfrak{p}| = \sqrt{p}$ and $\theta(\mathfrak{p}) \in (0, 1/8)$. Thus,

$$|\mathcal{A}| = \#\left\{ p : p \equiv 1 \pmod 4,\ p \leq \left(\frac{x}{7h}\right)^{\frac{1}{h}} \right\}$$

and the Prime Number Theorem for arithmetic progressions implies that

$$|\mathcal{A}| \sim \frac{\left(\frac{x}{7h}\right)^{\frac{1}{h}}}{2\log\left(\left(\frac{x}{7h}\right)^{\frac{1}{h}}\right)} \gg x^{1/h}/\log x. \qquad \square$$

3. Proof of Theorem 1.1

We start following the lines of [**8**] with several adjustments. In the sequel we will write $\mathfrak{p}$ for a Gaussian prime in the first octant ($0 < \theta(\mathfrak{p}) < 1/8$).

We fix a number $c_h > h$ which will determine the growth of the sequence we construct. Indeed $c_h = \sqrt{(h-1)^2+1} + (h-1)$ will be taken in the last step of the proof.

3.1. The construction. We will construct for each $\alpha \in [1,2]$ a sequence of positive integers indexed with the Gaussian primes

$$\mathcal{B}_\alpha := \{b_\mathfrak{p}\},$$

where each $b_\mathfrak{p}$ will be built using the expansion in base 2 of $\alpha\,\theta(\mathfrak{p})$:

$$\alpha\,\theta(\mathfrak{p}) = \sum_{i=1}^{\infty} \delta_{i\mathfrak{p}} 2^{-i} \quad (\delta_{i\mathfrak{p}} \in \{0,1\}).$$

The role of the parameter α will be clear at a later stage, for the moment it is enough to note that the set $\{\alpha\,\theta(\mathfrak{p})\}$ obviously keeps the same B_h property as the set $\{\theta(\mathfrak{p})\}$.

To organize the construction we describe the sequence $\mathcal{B}_\alpha$ as a union of finite sets according to the sizes of the primes:

$$\mathcal{B}_\alpha = \bigcup_{K \geq h+1} \mathcal{B}_{\alpha,K},$$

where K is an integer and

$$\mathcal{B}_{\alpha,K} = \{b_\mathfrak{p} : \mathfrak{p} \in P_K\},$$

with

$$P_K := \left\{\mathfrak{p} : 2^{\frac{(K-2)^2}{c_h}} < |\mathfrak{p}|^2 \leq 2^{\frac{(K-1)^2}{c_h}}\right\}.$$

Now we build the positive integers $b_\mathfrak{p} \in \mathcal{B}_{\alpha,K}$. For any $\mathfrak{p} \in P_K$ let $\widehat{\alpha\,\theta(\mathfrak{p})}$ denote the truncated series of $\alpha\,\theta(\mathfrak{p})$ at the K^2-place:

$$\widehat{\alpha\,\theta(\mathfrak{p})} := \sum_{i=1}^{K^2} \delta_{i\mathfrak{p}} 2^{-i}. \tag{3.1}$$

Combining the digits at places $(j-1)^2+1,\ldots,j^2$ into a single number

$$\Delta_{j\mathfrak{p}} = \sum_{i=(j-1)^2+1}^{j^2} \delta_{i\mathfrak{p}} 2^{j^2-i} \quad (j=1,\ldots,K),$$

we can write

$$\widehat{\alpha\,\theta(\mathfrak{p})} = \sum_{j=1}^{K} \Delta_{j\mathfrak{p}} 2^{-j^2}. \tag{3.2}$$

We observe that if $\mathfrak{p} \in P_K$ then

$$|\widehat{\alpha\,\theta(\mathfrak{p})} - \alpha\,\theta(\mathfrak{p})| \le 2^{-K^2}. \tag{3.3}$$

The definition of $b_{\mathfrak{p}}$ is informally outlined as follows. We consider the series of blocks $\Delta_{1\mathfrak{p}},\ldots,\Delta_{K\mathfrak{p}}$ and re-arrange them opposite to the original left to right arrangement. Then we insert at the left of each $\Delta_{j\mathfrak{p}}$ an additional filling block of $2d+1$ digits, with $d=\lceil\log_2 h\rceil$. At the filling blocks the digits will be always 0 but for an only exception: the leftmost filling block contains one digit 1 which marks the subset P_K the prime $\mathfrak{p}$ belongs to. Namely

$$\alpha\,\theta(\mathfrak{p}) = 0.\overset{\Delta_1}{1}\,\overbrace{001}^{\Delta_2}\ldots\overbrace{1\ldots0}^{\Delta_j}\ldots\;\overbrace{01\ldots1\underset{\underset{K^2}{\uparrow}}{1}}^{\Delta_K}\ldots$$

$$b_{\mathfrak{p}} \leftrightarrow 0^{(d)}10^{(d)}\Delta_K 0^{(2d+1)}\Delta_{K-1}\cdots 0^{(2d+1)}\Delta_2 0^{(2d+1)}\Delta_1,$$

where $0^{(m)}$ means a string of m consecutive zeroes and Δ_i denotes the sequence of digits in the definition of $\Delta_{i\mathfrak{p}}$. The reason to add the blocks of zeroes and the value of d will be clarified just before Lemma 3.2.

More formally, for $\mathfrak{p}\in P_K$ we define

$$t_{\mathfrak{p}} = 2^{K^2+(2d+1)(K-1)+(d+1)} \tag{3.4}$$

and

$$b_{\mathfrak{p}} = t_{\mathfrak{p}} + \sum_{j=1}^{K} \Delta_{j\mathfrak{p}} 2^{(j-1)^2+(2d+1)(j-1)}.$$

Furthermore we define $\Delta_{j\mathfrak{p}}=0$ for $j>K$.

Remark 3.1. The construction in [**8**] was based on the numbers $\alpha\log p$, with p rational prime, hence the digits of their integral parts had to be also included in the corresponding integers b_p. Ruzsa solved this problem by reserving fixed places for these digits. Since in our construction the integral part of $\alpha\,\theta(\mathfrak{p})$ is zero there is no need to care about it.

We observe that distinct primes $\mathfrak{p}$, $\mathfrak{q}$ provide distinct $b_{\mathfrak{p}}$, $b_{\mathfrak{q}}$. Indeed if $b_{\mathfrak{p}} = b_{\mathfrak{q}}$ then $\Delta_{i\mathfrak{p}} = \Delta_{i\mathfrak{q}}$ for all $i \le K$. Also $t_{\mathfrak{p}} = t_{\mathfrak{q}}$ which means $\mathfrak{p}, \mathfrak{q} \in P_K$, and so

$$|\theta(\mathfrak{p}) - \theta(\mathfrak{q})| = \alpha^{-1} \cdot \sum_{j>K} (\Delta_{j\mathfrak{p}} - \Delta_{j\mathfrak{q}}) < 2^{-K^2}.$$

Now if $\mathfrak{p} \ne \mathfrak{q}$ then Lemma 2.1 implies that $|\theta(\mathfrak{p}) - \theta(\mathfrak{q})| > \frac{1}{7|\mathfrak{p}\mathfrak{q}|} > 2^{-\frac{1}{c}(K-1)^2-3}$. Combining both inequalities we have a contradiction for $K \ge h+1$.

Since all the integers $b_{\mathfrak{p}}$ are distinct, we have that

$$|\mathcal{B}_{\alpha,K}| = |P_K| = \pi\big(2^{\frac{(K-1)^2}{c_h}}; 1, 4\big) - \pi\big(2^{\frac{(K-2)^2}{c_h}}; 1, 4\big) \gg K^{-2} 2^{\frac{K^2}{c_h}}, \tag{3.5}$$

where $\pi(x; 1, 4)$ counts the primes not greater than x that are congruent with 1 modulus 4. Note also that

$$b_{\mathfrak{p}} < 2^{K^2+(2d+1)K+(d+1)+1}.$$

Using these estimates we can easily prove that $\mathcal{B}_\alpha(x) = x^{\frac{1}{c_h}+o(1)}$. Indeed, if K is the integer such that

$$2^{K^2+(2d+1)K+(d+1)+1} < x \le 2^{(K+1)^2+(2d+1)(K+1)+(d+1)+1},$$

then we have

$$\mathcal{B}_\alpha(x) \ge |\mathcal{B}_{\alpha,K}| = 2^{\frac{1}{c_h}K^2(1+o(1))} = x^{\frac{1}{c_h}+o(1)}. \tag{3.6}$$

For the upper bound we have

$$\mathcal{B}_\alpha(x) \le \#\left\{\mathfrak{p} : |\mathfrak{p}|^2 \le 2^{\frac{K^2}{c_h}}\right\} \le 2^{\frac{K^2}{c_h}} = x^{\frac{1}{c_h}+o(1)}.$$

There is a tradeoff in the choice of a particular value of c_h for the construction. On one hand larger values of c_h capture more information from the Gaussian arguments which brings the sequence $\mathcal{B}_\alpha = \{b_{\mathfrak{p}}\}$ closer to being a B_h sequence. On the other hand smaller values of c_h provide higher growth of the counting function of $\mathcal{B}_\alpha$.

Clearly $\mathcal{B}_\alpha$ would be a B_h sequence if for all $l = 2, \dots, h$ it does not contain $b_{\mathfrak{p}_1}, \dots, b_{\mathfrak{p}_l}, b_{\mathfrak{p}'_1}, \dots, b_{\mathfrak{p}'_l}$ satisfying

$$b_{\mathfrak{p}_1} + \dots + b_{\mathfrak{p}_l} = b_{\mathfrak{p}'_1} + \dots + b_{\mathfrak{p}'_l}, \tag{3.7}$$

$$\{b_{\mathfrak{p}_1}, \dots, b_{\mathfrak{p}_l}\} \cap \{b_{\mathfrak{p}'_1}, \dots, b_{\mathfrak{p}'_l}\} = \emptyset,$$

$$b_{\mathfrak{p}_1} \ge \dots \ge b_{\mathfrak{p}_l} \quad \text{and} \quad b_{\mathfrak{p}'_1} \ge \dots \ge b_{\mathfrak{p}'_l}. \tag{3.8}$$

We say that $(\mathfrak{p}_1, \dots, \mathfrak{p}_l, \mathfrak{p}'_1, \dots, \mathfrak{p}'_l)$ is a *bad* $2l$-tuple if the equation (3.7) is satisfied by the corresponding $b_{\mathfrak{p}_r}$, $b_{\mathfrak{p}'_r}$ $(1 \le r \le l)$.

The sequence $\mathcal{B}_\alpha = \{b_\mathfrak{p}\}$ we have constructed so far is not a B_h sequence yet. Some repeated sums as in (3.7) will eventually appear, however the precise way how the elements $b_\mathfrak{p}$ are built will allow us to study these bad $2l$-tuples in order to prove that there are not too many repeated sums. Then after removing the bad elements involved in these bad $2l$-tuples we will obtain a true B_h sequence.

Now we will see why blocks of zeroes were added to the binary expansion of $b_\mathfrak{p}$. We can identify each $b_\mathfrak{p}$, with $\mathfrak{p} \in P_K$, with a vector as follows:

$$b_\mathfrak{p} \leftrightarrow (0^\infty, 1, 0^{(d)}, \Delta_K, 0^{(2d+1)}, \Delta_{K-1}, \dots, 0^{(2d+1)}, \Delta_2, 0^{(2d+1)}, \Delta_1),$$

where $0^{(m)}$ means a string of m consecutive zeroes and Δ_i denotes the sequence of digits in the definition of $\Delta_{i\mathfrak{p}}$. Note that the leftmost part of each vector is null. The value of $d = \lceil \log_2 h \rceil$ has been chosen to prevent the propagation of the carry between any two consecutive coordinates separated by a comma in the above identification. So when we sum no more than h integers $b_\mathfrak{p}$ we can just sum the corresponding vectors coordinate-wise. This fact is used in the following lemma.

Lemma 3.2. *Let* $(\mathfrak{p}_1, \dots, \mathfrak{p}_l, \mathfrak{p}'_1, \dots, \mathfrak{p}'_l)$ *be a bad* $2l$*-tuple. Then there are integers* $K_1 \ge \cdots \ge K_l$ *such that* $\mathfrak{p}_1, \mathfrak{p}'_1 \in P_{K_1}, \dots, \mathfrak{p}_l, \mathfrak{p}'_l \in P_{K_l}$*, and we have*

$$\widehat{\alpha\,\theta(\mathfrak{p}_1)} + \cdots + \widehat{\alpha\,\theta(\mathfrak{p}_l)} = \widehat{\alpha\,\theta(\mathfrak{p}'_1)} + \cdots + \widehat{\alpha\,\theta(\mathfrak{p}'_l)}. \tag{3.9}$$

Proof: Note that (3.7) implies $t_{\mathfrak{p}_1} + \cdots + t_{\mathfrak{p}_l} = t_{\mathfrak{p}'_1} + \cdots + t_{\mathfrak{p}'_l}$ and $\Delta_{j\mathfrak{p}_1} + \cdots + \Delta_{j\mathfrak{p}_l} = \Delta_{j\mathfrak{p}'_1} + \cdots + \Delta_{j\mathfrak{p}'_l}$ for each j. Using (3.2) we conclude (3.9). As the bad $2l$-tuple satisfies condition (3.8) we deduce that $\mathfrak{p}_r$, $\mathfrak{p}'_r$ belong to the same P_{K_r} for all r. □

According to the previous lemma we will write $E_{2l}(\alpha; K_1, \dots, K_l)$ for the set of bad $2l$-tuples $(\mathfrak{p}_1, \dots, \mathfrak{p}'_l)$ with $\mathfrak{p}_r, \mathfrak{p}'_r \in P_{K_r}$, $1 \le r \le l$ and

$$E_{2l}(\alpha; K) = \bigcup_{K_l \le \cdots \le K_1 = K} E_{2l}(\alpha; K_1, \dots, K_l),$$

where $K = K_1$. Also we define the set

$$\mathrm{Bad}_{\alpha,K} = \{b_\mathfrak{p} \in \mathcal{B}_{\alpha,K} : b_\mathfrak{p} \text{ is the largest element involved in some equation (3.7)}\}.$$

It is clear that $\sum_{l\le h}|E_{2l}(\alpha,K)|$ is an upper bound for $|\mathrm{Bad}_{\alpha,K}|$, the number of elements we need to remove from each $\mathcal{B}_{\alpha,K}$ to get a B_h sequence:

$$|\mathrm{Bad}_{\alpha,K}| \le \sum_{l\le h}|E_{2l}(\alpha,K)|. \tag{3.10}$$

We do not know how to obtain a good upper bound for $|E_{2l}(\alpha,K)|$ for a particular α, however we can do it for almost all α.

Lemma 3.3. *For $l=2,3,4$ and $c_h>h\ge l$ we have*

$$\int_1^2 |E_{2l}(\alpha,K)|\,\mathrm{d}\alpha \ll K^{m_l}2^{\left(\frac{2(l-1)}{c_h-1}-1\right)(K-1)^2-2K},$$

for some m_l.

The proof of this lemma is involved and we postpone it to §4.

3.2. Last step in the proof of Theorem 1.1: For $h=2,3,4$ we use (3.10) and (3.5) to get

$$\begin{aligned}\int_1^2 \frac{|\mathrm{Bad}_{\alpha,K}|}{|\mathcal{B}_{\alpha,K}|}\,\mathrm{d}\alpha &\ll \frac{\sum_{l\le h}\int_1^2|E_{2l}(\alpha,K)|\,\mathrm{d}\alpha}{K^{-2}2^{\frac{1}{c_h}(K-1)^2}}\\ &\ll \frac{\sum_{l\le h}K^{m_l}2^{\left(\frac{2(l-1)}{c_h-1}-1\right)(K-1)^2-2K}}{K^{-2}2^{\frac{1}{c_h}(K-1)^2}}\\ &\ll K^{m_l+2}2^{\left(\frac{2(h-1)}{c_h-1}-1-\frac{1}{c_h}\right)(K-1)^2-2K}\\ &\ll K^{m_l+2}2^{-2K}\end{aligned}$$

for $c_h=\sqrt{(h-1)^2+1}+(h-1)$ which is the smallest number c satisfying the inequality $\frac{2(h-1)}{c-1}-1-\frac{1}{c}\le 0$. So for this c_h the sum $\sum_K\int_1^2\frac{|\mathrm{Bad}_{\alpha,K}|}{|\mathcal{B}_{\alpha,K}|}\,\mathrm{d}\alpha$ is convergent and then we have that $\int_1^2\sum_K\frac{|\mathrm{Bad}_{\alpha,K}|}{|\mathcal{B}_{\alpha,K}|}\,\mathrm{d}\alpha$ is finite. So $\sum_K\frac{|\mathrm{Bad}_{\alpha,K}|}{|\mathcal{B}_{\alpha,K}|}$ is convergent for almost all $\alpha\in[1,2]$. We take one of these α, say α_0, and consider the sequence

$$\mathcal{B}=\bigcup_K(\mathcal{B}_{\alpha_0,K}\setminus\mathrm{Bad}_{\alpha_0,K}).$$

We claim that this sequence satisfies the condition of the theorem. On one hand this sequence clearly is a B_h sequence because we have destroyed all the repeated sums of h elements of $\mathcal{B}_{\alpha_0}$ by removing one element from each bad $2l$-tuple.

On the other hand the convergence of $\sum_K \frac{|\mathrm{Bad}_{\alpha_0,K}|}{|\mathcal{B}_{\alpha_0,K}|}$ implies that $|\mathrm{Bad}_{\alpha_0,K}| = o(|\mathcal{B}_{\alpha_0,K}|)$. We proceed as in (3.6) to estimate the counting function of $\mathcal{B}$. For any x let K be the integer such that

$$2^{K^2+(2d+1)K+(d+1)+1} < x \le 2^{(K+1)^2+(2d+1)(K+1)+(d+1)+1}.$$

We have

$$\mathcal{B}(x) \ge |\mathcal{B}_{\alpha_0,K}| - |\mathrm{Bad}_{\alpha_0,K}| = |\mathcal{B}_{\alpha_0,K}|(1+o(1)) \gg K^{-2}2^{\frac{1}{c_h}K^2} = x^{\frac{1}{c_h}+o(1)}.$$

For the upper bound, we have

$$\mathcal{B}(x) \le \mathcal{B}_{\alpha_0}(x) = x^{\frac{1}{c_h}+o(1)}.$$

Note that $1/c_h = \sqrt{(h-1)^2+1} - (h-1)$. Hence

$$\mathcal{B}(x) = x^{\sqrt{(h-1)^2+1}-(h-1)+o(1)}. \qquad \square$$

4. Proof of Lemma 3.3

The proof of Lemma 3.3 will be a consequence of Propositions 4.5, 4.6, and 4.7. Before proving these propositions we need some properties of the bad $2l$-tuples and an auxiliary lemma about visible lattice points.

4.1. Some properties of the $2l$-tuples. For any $2l$-tuple $(\mathfrak{p}_1, \dots, \mathfrak{p}_l, \mathfrak{p}'_1, \dots, \mathfrak{p}'_l)$ we define the numbers $\omega_s = \omega_s(\mathfrak{p}_1, \dots, \mathfrak{p}_l, \mathfrak{p}'_1, \dots, \mathfrak{p}'_l)$ by

$$\omega_s = \sum_{r=1}^{s} (\theta(\mathfrak{p}_r) - \theta(\mathfrak{p}'_r)) \quad (s \le l).$$

The next two lemmas show several properties of the bad $2l$-tuples.

Lemma 4.1. *Let $(\mathfrak{p}_1, \dots, \mathfrak{p}_l, \mathfrak{p}'_1, \dots, \mathfrak{p}'_l) \in E_{2l}(\alpha; K_1, \dots, K_l)$ be a bad $2l$-tuple. We have*

i) $|\omega_l| \le l2^{-K_l^2}$,

ii) $|\omega_{l-1}| \ge 2^{-\frac{1}{c_h}(K_l-1)^2-4}$,

iii) $(K_l-1)^2 \le \dfrac{(K_1-1)^2 + \cdots + (K_{l-1}-1)^2}{c_h - 1}$.

Proof: i) This is a consequence of (3.9) and (3.3):

$$|\omega_l| = \frac{1}{\alpha}\left|\sum_{r=1}^{l}(\alpha\,\theta(\mathfrak{p}_r) - \alpha\,\theta(\mathfrak{p}'_r))\right| \le \frac{1}{\alpha}\left(2^{-K_1^2} + \cdots + 2^{-K_l^2}\right) \le l2^{-K_l^2}.$$

ii) Lemma 2.1 implies

$$|\theta(\mathfrak{p}_l)-\theta(\mathfrak{p}_l')|\geq \frac{1}{7|\mathfrak{p}_l\mathfrak{p}_l'|}\geq 2^{-3-\frac{1}{c_h}(K_l-1)^2}, \tag{4.1}$$

and so

$$\begin{aligned}|\omega_{l-1}|&=|\omega_l+\theta(\mathfrak{p}_l')-\theta(\mathfrak{p}_l)|\geq|\theta(\mathfrak{p}_l')-\theta(\mathfrak{p}_l)|-|\omega_l|\\&\geq 2^{-\frac{1}{c_h}(K_l-1)^2-3}-l2^{-K_l^2}\geq 2^{-\frac{1}{c_h}(K_l-1)^2-4},\end{aligned}$$

since $K_l\geq h+1\geq l+1$.

iii) Lemma 2.1 also implies that

$$|\omega_l|=\left|\sum_{r=1}^{l}(\theta(\mathfrak{p}_r)-\theta(\mathfrak{p}_r'))\right|>\frac{1}{7|\mathfrak{p}_1\cdots\mathfrak{p}_l'|}>2^{-3-\frac{1}{c_h}\sum_{r=1}^{l}(K_r-1)^2}.$$

Combining this with i) we obtain

$$(K_l-1)^2\leq\frac{1}{c_h-1}\left((K_1-1)^2+\cdots+(K_{l-1}-1)^2\right)+\frac{\log_2 l-2K_l+4}{1-1/c_h}.$$

The last term is negative because $K_l\geq h+1\geq l+1$ and $l\geq 2$. □

Lemma 4.2. *Let* $(\mathfrak{p}_1,\dots,\mathfrak{p}_l,\mathfrak{p}_1',\dots,\mathfrak{p}_l')\in E_{2l}(\alpha;K_1,\dots,K_l)$ *be a bad* $2l$*-tuple. Then for any* $\omega_s=\sum_{r=1}^{s}(\theta(\mathfrak{p}_r)-\theta(\mathfrak{p}_r'))$ *with* $1\leq s\leq l-1$ *we have*

$$\|\alpha 2^{K_{s+1}^2}\omega_s\|\leq s2^{K_{s+1}^2-K_s^2}\quad(s=1,\dots,l-1), \tag{4.2}$$

where $\|\cdot\|$ *means the distance to the nearest integer.*

Proof: Since $0\leq\alpha\,\theta(\mathfrak{p})-\widehat{\alpha\,\theta(\mathfrak{p})}\leq 2^{-K^2}$ when $\mathfrak{p}\in P_K$, then

$$\left|(\theta(\mathfrak{p}_r)-\theta(\mathfrak{p}_r'))-\left(\widehat{\alpha\,\theta(\mathfrak{p}_r)}-\widehat{\alpha\,\theta(\mathfrak{p}_r')}\right)\right|\leq 2^{-K_s^2}$$

for any $\mathfrak{p}_r,\mathfrak{p}_r'\in K_r$ with $r\leq s$ and we can write

$$2^{K_{s+1}^2}\alpha\sum_{r=1}^{s}(\theta(\mathfrak{p}_r)-\theta(\mathfrak{p}_r'))=2^{K_{s+1}^2}\sum_{r=1}^{s}\left(\widehat{\alpha\,\theta(\mathfrak{p}_r)}-\widehat{\alpha\,\theta(\mathfrak{p}_r')}\right)+\epsilon_s,$$

with $|\epsilon_s|\leq s2^{K_{s+1}^2-K_s^2}$. By the definition (3.1) of $\widehat{\alpha\,\theta(\mathfrak{p})}$ we have

$$2^{K_{s+1}^2}\sum_{r=s+1}^{l}\left(\widehat{\alpha\,\theta(\mathfrak{p}_r')}-\widehat{\alpha\,\theta(\mathfrak{p}_r)}\right)=\sum_{r=s+1}^{l}\sum_{i=1}^{K_r^2}2^{K_{s+1}^2-i}(\delta_{i\mathfrak{p}_r'}-\delta_{i\mathfrak{p}_r})$$

which is an integer. By Lemma 3.2 we know that

$$\sum_{r=1}^{l}\left(\widehat{\alpha\,\theta(\mathfrak{p}_r)} - \widehat{\alpha\,\theta(\mathfrak{p}'_r)}\right) = 0.$$

It follows that

$$\|2^{K_{s+1}^2}\omega_s\| = |\epsilon_s| \le s2^{K_{s+1}^2-K_s^2},$$

as claimed. □

Lemma 4.3.

$$\int_1^2 |E_{2l}(\alpha; K_1,\dots,K_l)|\,\mathrm{d}\alpha \ll 2^{K_l^2-K_1^2} \sum_{\substack{(\mathfrak{p}_1,\dots,\mathfrak{p}'_l)\\ |\omega_l|<l\cdot 2^{-K_l^2}}} \frac{|\omega_{l-1}|}{|\omega_1|}\prod_{j=1}^{l-2}\left(\frac{|\omega_j|}{|\omega_{j+1}|}+1\right).$$

Proof: We know by Lemma 4.1 i) that if $(\mathfrak{p}_1,\dots,\mathfrak{p}'_l)\in E_{2l}(\alpha;K_1,\dots,K_l)$, then $|\omega_l| < l2^{-K_l^2}$. Thus

$$\int_1^2 |E_{2l}(\alpha; K_1,\dots,K_l)|\,\mathrm{d}\alpha \le \sum_{\substack{(\mathfrak{p}_1,\dots,\mathfrak{p}'_l)\\ |\omega_l|<l\cdot 2^{-K_l^2}}} \mu\{\alpha : (\mathfrak{p}_1,\dots,\mathfrak{p}'_l)\in E_{2l}(\alpha;K_1,\dots,K_l)\}. \tag{4.3}$$

We have seen that if $(\mathfrak{p}_1,\dots,\mathfrak{p}'_l)\in E_{2l}(\alpha;K_1,\dots,K_l)$, then

$$\|\alpha 2^{K_{s+1}^2}\omega_s\| \le s2^{K_{s+1}^2-K_s^2},\quad s=1,\dots,l-1. \tag{4.4}$$

Then there exist integers j_s, $s=1,\dots,l-1$ such that

$$|\alpha 2^{K_{s+1}^2}\omega_s - j_s| \le s2^{K_{s+1}^2-K_s^2}, \tag{4.5}$$

so

$$\left|\alpha - \frac{j_s}{2^{K_{s+1}^2}\omega_s}\right| \le \frac{s2^{-K_s^2}}{|\omega_s|}. \tag{4.6}$$

Writing $I_{j_1},\dots,I_{j_s}$ for the intervals defined by the inequalities (4.6), we have

$$\begin{aligned}\mu\{\alpha : (\mathfrak{p}_1,\dots,\mathfrak{p}'_l)\in E_{2l}(\alpha;K_1,\dots,K_l)\}&\\ &\le \sum_{j_1,\dots,j_{l-1}} |I_{j_1}\cap\dots\cap I_{j_{l-1}}|\\ &\le \frac{2^{-K_1^2+1}}{|\omega_1|}\#\left\{(j_1,\dots,j_{l-1}) : \bigcap_{i=1}^{l-1} I_{j_i}\ne\emptyset\right\}.\end{aligned} \tag{4.7}$$

To estimate this last cardinal note that for all $s = 1, \dots, l-2$ we have

$$\left|\frac{j_s}{2^{K_{s+1}^2}\omega_s} - \frac{j_{s+1}}{2^{K_{s+2}^2}\omega_{s+1}}\right| < \left|\alpha - \frac{j_s}{2^{K_{s+1}^2}\omega_s}\right| + \left|\alpha - \frac{j_{s+1}}{2^{K_{s+2}^2}\omega_{s+1}}\right|$$
$$< \frac{s2^{-K_s^2}}{|\omega_s|} + \frac{(s+1)2^{-K_{s+1}^2}}{|\omega_{s+1}|}.$$

Thus

$$\left|j_s - j_{s+1}\frac{2^{K_{s+1}^2}\omega_s}{2^{K_{s+2}^2}\omega_{s+1}}\right| < s2^{-K_s^2+K_{s+1}^2} + \frac{(s+1)|\omega_s|}{|\omega_{s+1}|}. \tag{4.8}$$

We observe that for each $s = 1, \dots, l-2$ and for each j_{s+1}, the number of j_s satisfying (4.8) is bounded by $2\left(s2^{-K_s^2+K_{s+1}^2} + \frac{(s+1)|\omega_s|}{|\omega_{s+1}|}\right) + 1 \ll \frac{|\omega_s|}{|\omega_{s+1}|} + 1$.

Note also that (4.5) for $s = l-1$ implies

$$\begin{aligned}|j_{l-1}| &\le \alpha 2^{K_l^2}\omega_{l-1} + (l-1)2^{K_l^2 - K_{l-1}^2}\\ &\le 2^{K_l^2+1}\omega_{l-1} + (l-1)\\ &\ll 2^{K_l^2}\omega_{l-1}.\end{aligned}$$

Thus,

$$\#\left\{(j_1, \dots, j_{l-1}) : \bigcap_{i=1}^{l-1} I_{j_i} \neq \emptyset\right\} \ll 2^{K_l^2}\omega_{l-1}\prod_{s=1}^{l-2}\left(\frac{|\omega_s|}{|\omega_{s+1}|} + 1\right). \tag{4.9}$$

The proof can be completed putting (4.9) in (4.7) and then in (4.3). □

4.2. Visible points. We will denote by $\mathcal{V}$ the set of points in the integer two dimensional lattice $\mathbb{Z}^2$ visible from the origin except $(1,0)$. In the next subsection we will use several times the following lemma.

Lemma 4.4. *The number of points in $\mathcal{V}$ that are contained in a circular sector centred at the origin of radius R and angle ϵ is at most $\epsilon R^2 + 1$. In other words, for any real number t*

$$\#\{\nu \in \mathcal{V},\ |\nu| < R,\ \|\theta(\nu) + t\| < \epsilon\} \le \epsilon R^2 + 1.$$

Furthermore,

$$\#\{\nu \in \mathcal{V},\ |\nu| < R,\ \|\theta(\nu)\| < \epsilon\} \le \epsilon R^2.$$

Proof: We order the N points inside de sector $\nu_1, \nu_2, \ldots, \nu_N \in \mathcal{V}$ by the value of their argument so that $\theta(\nu_i) < \theta(\nu_j)$ for $1 \le i < j \le N$. For each $i = 1, \ldots, N-1$ the three lattice points O, ν_i, ν_{i+1} define a triangle T_i with $\mathrm{Area}(T_i) \ge 1/2$, that does not contain any other lattice point.

Since all T_i are inside the circular sector their union covers at most the area of the sector. Their interiors are pairwise disjoint, thus

$$N - 1 \le \sum_{i=1}^{N} 2 \cdot \mathrm{Area}(T_i) = 2 \cdot \mathrm{Area}\left(\bigcup_{i=1}^{N} T_i\right) \le R^2 \epsilon.$$

For the last statement we add $\nu_0 = (1, 0)$ to the points $\nu_1, \ldots, \nu_N$ and we repeat the argument. □

4.3. Estimates for the number of bad $2l$-tuples ($l = 2, 3, 4$). We start with the case $l = 2$ which was considered by Ruzsa for B_2 sequences. In the sequel all lattice points ν appearing in the proofs belong to $\mathcal{V}$ and Lemma 4.4 applies.

Proposition 4.5. *For any $c_h > 2$ we have*

$$\int_1^2 |E_4(\alpha; K)| \, \mathrm{d}\alpha \ll K \cdot 2^{\left(\frac{2}{c_h - 1} - 1\right)(K-1)^2 - 2K}.$$

Proof: Lemma 4.3 implies that

$$\int_1^2 |E_4(\alpha; K_1, K_2)| \, \mathrm{d}\alpha \ll 2^{K_2^2 - K_1^2} \# \left\{ (\mathfrak{p}_1, \mathfrak{p}_1', \mathfrak{p}_2, \mathfrak{p}_2') : |\omega_2| \le 2 \cdot 2^{-K_2^2} \right\}.$$

We get an upper bound for the second factor here by using Lemma 4.4 to estimate the number of lattice points of the form $\nu_2 = \mathfrak{p}_1 \mathfrak{p}_1' \overline{\mathfrak{p}_2 \mathfrak{p}_2'}$ such that

$$|\omega_2| = \|\theta(\nu_2)\| < \epsilon, \; |\nu_2| < R \quad \text{with} \quad \epsilon = 2 \cdot 2^{-K_2^2}$$
$$\text{and} \quad R = 2^{\frac{1}{c_h}((K_1-1)^2 + (K_2-1)^2)}.$$

We have

$$\int_1^2 |E_4(\alpha; K_1, K_2)| \, \mathrm{d}\alpha \ll 2^{K_2^2 - K_1^2} \cdot 2^{\frac{2}{c_h}((K_1-1)^2 + (K_2-1)^2) - K_2^2}$$
$$\ll 2^{\frac{2}{c_h}((K_1-1)^2 + (K_2-1)^2) - K_1^2}.$$

By Lemma 4.1 iii) we also have $(K_2-1)^2 \le \frac{(K_1-1)^2}{c_h-1}$, thus

$$\int_1^2 |E_4(\alpha;K_1,K_2)|\,\mathrm{d}\alpha \ll 2^{\left(\frac{2}{c_h-1}-1\right)K_1^2-2K_1}$$

and

$$\int_1^2 |E_4(\alpha;K)|\,\mathrm{d}\alpha = \sum_{K_2\le K}\int_1^2 |E_4(\alpha;K,K_2)|\,\mathrm{d}\alpha \ll K\cdot 2^{\left(\frac{2}{c_h-1}-1\right)(K-1)^2-2K}. \qquad \square$$

Proposition 4.6. *For any $c_h>3$ we have*

$$\int_1^2 |E_6(\alpha;K)|\,\mathrm{d}\alpha \ll K^4 2^{\left(\frac{4}{c_h-1}-1\right)(K-1)^2-2K}.$$

Proof: Lemma 4.3 says that

$$\int_1^2 |E_6(\alpha;K_1,K_2,K_3)|\,\mathrm{d}\alpha \ll 2^{K_3^2-K_1^2}\sum_{\substack{(\mathfrak{p}_1,\dots,\mathfrak{p}_3')\\ |\omega_3|\le 3\cdot 2^{-K_3^2}}}\frac{1}{|\omega_1|}.$$

Since $|\omega_1| = \|\theta(\mathfrak{p}_1\overline{\mathfrak{p}_1'})\| \ge 2^{-3-\frac{(K_1-1)^2}{c_h}}$ we split the sum above according $|\omega_1|\le 2^{-m}$ for $m\le M = 3+(K_1-1)^2/c_h$. Summing for all m in this range and applying Lemma 4.4 with $\nu_1 = \mathfrak{p}_1\overline{\mathfrak{p}_1'}$ and $\nu_2 = \mathfrak{p}_2\mathfrak{p}_3\overline{\mathfrak{p}_2'\mathfrak{p}_3'}$, we have that

$$\begin{aligned}\sum_{\substack{(\mathfrak{p}_1,\dots,\mathfrak{p}_3')\\ |\omega_3|\le 3\cdot 2^{-K_3^2}}}\frac{1}{|\omega_1|} &\ll \sum_{m\le M} 2^m\#\left\{(\mathfrak{p}_1,\dots,\mathfrak{p}_3') : |\omega_1|\le 2^{-m}, |\omega_3|\le 3\cdot 2^{-K_3^2}\right\}\\ &\ll \sum_{m\le M} 2^m\#\Big\{(\nu_1,\nu_2) : \|\theta(\nu_1)\|\le 2^{-m},\\ &\qquad\qquad \|\theta(\nu_1)+\theta(\nu_2)\|\le 3\cdot 2^{-K_3^2}\Big\}\\ &\ll \sum_{m\le M} 2^m \sum_{|\theta(\nu_1)|\le 2^{-m}}\#\left\{\nu_2 : \|\theta(\nu_1)+\theta(\nu_2)\|\le 3\cdot 2^{-K_3^2}\right\}\\ &\ll \sum_{m\le M} 2^m\cdot 2^{\frac{2}{c_h}(K_1-1)^2-m}\left(2^{\frac{2}{c_h}\left((K_2-1)^2+(K_3-1)^2\right)-K_3^2}+1\right).\end{aligned}$$

Hence using the inequalities $K_3 \le K_2 \le K_1$ and $(K_3-1)^2 \le \frac{(K_2-1)^2+(K_1-1)^2}{c_h-1}$ (property iii) in Lemma 4.1) we have

$$\begin{aligned}
&\int_1^2 |E_6(\alpha; K_1, K_2, K_3)| \,\mathrm{d}\alpha \\
&\quad \ll K_1^2 2^{K_3^2-K_1^2+\frac{2}{c_h}(K_1-1)^2}\left(2^{\frac{2}{c_h}\left((K_2-1)^2+(K_3-1)^2\right)-K_3^2}+1\right) \\
&\quad \ll K_1^2 2^{-K_1^2+\frac{2}{c_h}\left((K_1-1)^2+(K_2-1)^2+(K_3-1)^2\right)} + K_1^2 2^{K_3^2-K_1^2+\frac{2}{c_h}(K_1-1)^2} \\
&\quad \ll K_1^2 2^{-(K_1-1)^2+\frac{2}{c_h}\left((K_1-1)^2+(K_2-1)^2+(K_3-1)^2\right)-2K_1} \\
&\qquad + K_1^2 2^{(K_3-1)^2-(K_1-1)^2+\frac{2}{c_h}(K_1-1)^2} \\
&\quad \ll K_1^2 2^{\left(\frac{4}{c_h-1}-1\right)(K_1-1)^2-2K_1} + K_1^2 2^{\left(\frac{4}{c_h-1}-1\right)(K_1-1)^2-\frac{2}{c_h(c_h-1)}(K_1-1)^2} \\
&\quad \ll K_1^2 2^{\left(\frac{4}{c_h-1}-1\right)(K_1-1)^2-2K_1}.
\end{aligned}$$

Then we can write

$$\begin{aligned}
&\int_1^2 |E_6(\alpha; K)| \,\mathrm{d}\alpha \\
&\quad = \sum_{K_3 \le K_2 \le K} \int_1^2 |E_6(\alpha; K, K_2, K_3)| \,\mathrm{d}\alpha \ll K^4 2^{\left(\frac{4}{c-1}-1\right)(K-1)^2-2K},
\end{aligned}$$

as claimed. □

Proposition 4.7. *For any $c_h > 4$ we have*

$$\int_1^2 |E_8(\alpha; K)| \,\mathrm{d}\alpha \ll K^5 2^{(\frac{6}{c_h-1}-1)(K-1)^2-2K}.$$

Proof: Considering the two possibilities $|\omega_1| < |\omega_2|$ and $|\omega_1| \ge |\omega_2|$ we get the inequality

$$\frac{|\omega_3|}{|\omega_1|}\left(\frac{|\omega_1|}{|\omega_2|}+1\right)\left(\frac{|\omega_2|}{|\omega_3|}+1\right) \ll \frac{|\omega_3|}{|\omega_1|}\left(\frac{|\omega_1|}{|\omega_2|}+1\right)\frac{1}{|\omega_3|} \ll \max\left(\frac{1}{|\omega_1|}, \frac{1}{|\omega_2|}\right).$$

This combined with Lemma 4.3 implies that

$$\begin{aligned}
&\int_1^2 |E_8(\alpha, K_1, K_2, K_3, K_4)| \,\mathrm{d}\alpha \\
&\qquad \ll 2^{-K_1^2+K_4^2}\left(\sum_{\substack{(\mathfrak{p}_1,\dots,\mathfrak{p}_4') \\ |\omega_4| \le 4\cdot 2^{-K_4^2}}} \frac{1}{|\omega_1|} + \sum_{\substack{(\mathfrak{p}_1,\dots,\mathfrak{p}_4') \\ |\omega_4| \le 4\cdot 2^{-K_4^2}}} \frac{1}{|\omega_2|}\right).
\end{aligned}$$

Applying Lemma 4.4 with the notation $\nu_1=\mathfrak{p}_1\overline{\mathfrak{p}_1'}$ and $\nu_2=\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_4\overline{\mathfrak{p}_2'\mathfrak{p}_3'\mathfrak{p}_4'}$ and taking again $M=3+(K_1-1)^2/c_h$, we have that

$$\begin{aligned}\sum_{\substack{(\mathfrak{p}_1,\dots,\mathfrak{p}_4')\\ |\omega_4|\le 4\cdot 2^{-K_4^2}}}\frac{1}{|\omega_1|} &\ll \sum_{m\le M}2^m\#\left\{(\mathfrak{p}_1,\dots,\overline{\mathfrak{p}_4}) : |\omega_1|<2^{-m},\ |\omega_4|\le 4\cdot 2^{-K_4^2}\right\}\\
&\ll \sum_{m\le M}2^m\#\Big\{(\nu_1,\nu_2):\|\theta(\nu_1)\|\le 2^{-m},\\
&\qquad\qquad \|\theta(\nu_1)+\theta(\nu_2)\|\le 4\cdot 2^{-K_4^2}\Big\}\\
&\ll \sum_{m\le M}\sum_{\|\theta(\nu_1)\|<2^{-m}}\#\left\{\nu_2:\|\theta(\nu_1)+\theta(\nu_2)\|\le 4\cdot 2^{-K_4^2}\right\}\\
&\ll \sum_{m\le M}2^{\frac{2}{c_h}(K_1-1)^2}\Big(2^{\frac{2}{c_h}((K_2-1)^2+(K_3-1)^2+(K_4-1)^2)-K_4^2}+1\Big)\\
&\ll K_1^2 2^{\frac{2}{c_h}((K_1-1)^2+(K_2-1)^2+(K_3-1)^2+(K_4-1)^2)-K_4^2}\\
&\quad + K_1^2 2^{\frac{2}{c_h}(K_1-1)^2}.\end{aligned}$$

Similarly, but writing now $\nu_1=\mathfrak{p}_1\mathfrak{p}_2\overline{\mathfrak{p}_1'\mathfrak{p}_2'}$ and $\nu_2=\mathfrak{p}_3\mathfrak{p}_4\overline{\mathfrak{p}_3'\mathfrak{p}_4'}$ we have

$$\begin{aligned}\sum_{\substack{(\mathfrak{p}_1,\dots,\mathfrak{p}_4')\\ |\omega_4|\le 4\cdot 2^{-K_4^2}}}\frac{1}{|\omega_2|} &\ll \sum_{m\le M}2^m\#\left\{(\mathfrak{p}_1,\dots,\overline{\mathfrak{p}_4}) : |\omega_2|\le 2^{-m},\ |\omega_4|\le 4\cdot 2^{-K_4^2}\right\}\\
&\ll \sum_{m\le K_4^2}2^m\#\Big\{(\nu_1,\nu_2):\|\theta(\nu_1)\|\le 2^{-m},\\
&\qquad\qquad \|\theta(\nu_1)+\theta(\nu_2)\|\le 4\cdot 2^{-K_4^2}\Big\}\\
&\quad + \sum_{m>K_4^2}2^m\#\Big\{(\nu_1,\nu_2):\|\theta(\nu_1)\|\le 2^{-m},\\
&\qquad\qquad \|\theta(\nu_1)+\theta(\nu_2)\|\le 4\cdot 2^{-K_4^2}\Big\}\\
&= S_1+S_2.\end{aligned}$$

We observe that if $m \le K_4^2$ then $\|\theta(\nu_2)\| \le \|\theta(\nu_1)+\theta(\nu_2)\| + \|\theta(\nu_1)\| \le 5\cdot 2^{-m}$. Thus

$$\begin{aligned}
S_1 &\ll \sum_{m\le K_4^2} 2^m \# \left\{(\nu_1,\nu_2) : \|\theta(\nu_2)\| \le 5\cdot 2^{-m},\ \|\theta(\nu_1)+\theta(\nu_2)\| \le 4\cdot 2^{-K_4^2}\right\} \\
&\ll \sum_{m\le K_4^2} 2^m \sum_{\|\theta(\nu_2)\|\le 5\cdot 2^{-m}} \# \left\{\nu_1 : \|\theta(\nu_1)+\theta(\nu_2)\| \le 4\cdot 2^{-K_4^2}\right\} \\
&\ll \sum_{m\le K_4^2} 2^m \cdot 2^{\frac{2}{c_h}\left((K_3-1)^2+(K_4-1)^2\right)-m}\left(2^{\frac{2}{c_h}\left((K_1-1)^2+(K_2-1)^2\right)-K_4^2}+1\right) \\
&\ll K_4^2 2^{\frac{2}{c_h}((K_1-1)^2+(K_2-1)^2+(K_3-1)^2+(K_4-1)^2)-K_4^2} \\
&\quad + K_4^2 2^{\frac{2}{c_h}\left((K_3-1)^2+(K_4-1)^2\right)}.
\end{aligned}$$

To estimate S_2, we observe that if $m > K_4^2$ then $\|\theta(\nu_2)\| \le \|\theta(\nu_1) + \theta(\nu_2)\| + \|\theta(\nu_1)\| \le 5\cdot 2^{-K_4^2}$. Thus

$$\begin{aligned}
S_2 &\ll \sum_{K_4^2<m\le M} 2^m \# \left\{(\nu_1,\nu_2) : \|\theta(\nu_1)\| \le 2^{-m},\ \|\theta(\nu_2)\| \le 5\cdot 2^{-K_4^2}\right\} \\
&\ll \sum_{K_4^2<m\le M} 2^m \cdot 2^{\frac{2}{c_h}\left((K_1-1)^2+(K_2-1)^2\right)-m} \cdot 2^{\frac{2}{c_h}\left((K_3-1)^2+(K_4-1)^2\right)-K_4^2} \\
&\ll K_1^2 2^{\frac{2}{c_h}((K_1-1)^2+(K_2-1)^2+(K_3-1)^2+(K_4-1)^2)-K_4^2}.
\end{aligned}$$

Putting together the estimates we have obtained for $\sum \frac{1}{|\omega_1|}$ and $\sum \frac{1}{|\omega_2|}$ we get

$$\begin{aligned}
&\int_1^2 |E_8(\alpha, K_1, K_2, K_3, K_4)|\, \mathrm{d}\alpha \\
&\qquad \ll K_1^2 2^{\frac{2}{c_h}((K_1-1)^2+(K_2-1)^2+(K_3-1)^2+(K_4-1)^2)-K_1^2} \\
&\qquad\quad + K_1^2 2^{-K_1^2+K_4^2+\frac{2}{c_h}(K_1-1)^2} + K_1^2 2^{K_4^2-K_1^2+\frac{2}{c_h}\left((K_3-1)^2+(K_4-1)^2\right)} \\
&\qquad = T_1 + T_2 + T_3.
\end{aligned}$$

Using the inequalities $(K_4-1)^2 \le \frac{1}{c_h-1}\left((K_1-1)^2+(K_2-1)^2+(K_3-1)^2\right)$ and $K_4 \le K_3 \le K_2 \le K_1$ we have

$$T_1 \ll K_1^2 2^{\left(-1+\frac{6}{c_h-1}\right)(K_1-1)^2-2K_1},$$

$$\begin{aligned} T_2 &\ll K_1^2 2^{-(K_1-1)^2+(K_4-1)^2+\frac{2}{c_h}(K_1-1)^2} \\ &\ll K_1^2 2^{\left(-1+\frac{3}{c_h-1}+\frac{2}{c_h}\right)(K_1-1)^2} \\ &\ll K_1^2 2^{\left(-1+\frac{6}{c_h-1}\right)(K_1-1)^2-2K_1}, \end{aligned}$$

and

$$\begin{aligned} T_3 &\ll K_1^2 2^{(K_4-1)^2-(K_1-1)^2+\frac{2}{c_h}\left((K_3-1)^2+(K_4-1)^2\right)} \\ &\ll K_1^2 2^{\left(1+\frac{2}{c_h}\right)\frac{1}{c_h-1}\left((K_1-1)^2+(K_2-1)^2+(K_3-1)^2\right)-(K_1-1)^2+\frac{2}{c_h}(K_3-1)^2} \\ &\ll K_1^2 2^{\left(\left(1+\frac{2}{c_h}\right)\frac{3}{c_h-1}-1+\frac{2}{c_h}\right)(K_1-1)^2} \\ &\ll K_1^2 2^{\left(-1+\frac{6}{c_h-1}\right)(K_1-1)^2-2K_1}, \end{aligned}$$

since $c_h > 4$. Finally

$$\begin{aligned} \int_1^2 |E_8(\alpha, K)| \, \mathrm{d}\alpha &\ll \sum_{K_4 \le K_3 \le K_2 \le K} K^2 2^{\left(-1+\frac{6}{c_h-1}\right)(K-1)^2-2K} \\ &\ll K^5 2^{\left(\frac{6}{c_h-1}-1\right)(K-1)^2-2K}, \end{aligned}$$

as claimed. □

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