Publ. Mat. (2014), 333–352
Proceedings of *New Trends in Dynamical Systems.* Salou, 2012.
DOI: 10.5565/PUBLMAT_Extra14_18

INTEGRABLE SYSTEMS ON $\mathbb{S}^3$

José Martínez-Alfaro and Regilene Oliveira

Abstract: We classify the links of basic periodic orbits of integrable vector fields on $\mathbb{S}^3$ generalizing results on two degree of freedom Hamiltonian systems. We also study the case of completely integrable systems and define invariants for the two classes of vector fields.

2010 Mathematics Subject Classification: Primary: 37J35; Secondary: 37D15, 34C25, 57R30.

Key words: Morse–Bott functions, integrability, topological invariants, vector fields on $\mathbb{S}^3$.

1. Introduction

The study of two degrees of freedom Hamiltonian systems defined on a symplectic manifold is one of the areas where mathematicians and physicists have devoted a lot of efforts. In particular Hamiltonian systems with an additional integral are useful models for a wide range of physical phenomena. But in many applications systems have a first integral that is not a Hamiltonian function of the vector field.

The knowledge of the first integrals is of particular interest in mathematics and physics because of the possibility to have explicit expressions for the solutions of the system. However, it can be interesting sometimes to know if the system can have an invariant.

Here we generalize basic results on Hamiltonian integrable systems ([**8**] and [**9**]) to the case of a Morse–Bott integral.

Given a vector field ν on a manifold $\mathbb{M}$, a first integral F of ν will be a real C^r, $r \geq 1$ function $F\colon \mathbb{M} \to \mathbb{R}$ that is constant on trajectories of ν but not identically constant on open sets of $\mathbb{M}$.

Denote by

$$I_b(F) = \{p \in \mathbb{M} \mid F(p) = b\}$$

the level sets of F. In the case where F is a first integral of the vector field, $I_b(F)$ is an invariant set for the flow.

An integrable Hamiltonian system is a Hamiltonian system with another first integral F independent of the H. A particular aspect in the study of integrable Hamiltonian systems is the decomposition of $I_b(H)$ in level sets of F or in other words the study of the foliation defined on $I_b(H)$ by F. These foliations are singular. By a singular k-foliation on $\mathbb{M}$ we understand a regular foliation on a dense set $\mathbb{M} - S$, where S is a finite union of closed connected components where the level sets contain leaves with dimension $< k$.

Recall that a point p of a manifold $\mathbb{M}_1$ is said to be a singular point of a smooth map $f\colon \mathbb{M}_1 \to \mathbb{M}_2$ if the rank of df at p is less than the maximum possible value; $f(p)$ is called a critical value.

Given $F\colon \mathbb{M} \to \mathbb{R}$, F is a Morse–Bott function (MB function from now on) if its singular points are organized as non degenerate smooth critical submanifolds (see [**6**]). Here a critical submanifold of F is called non-degenerate if the Hessian of F is non-degenerate on normal planes to this submanifold. The index of F at $p \in \mathbb{M}$ is the index of $F|_{TN}$, where TN is the normal plane of $\mathbb{M}$ at p. If all the singular points are isolated the function F will be called a Morse function.

We will assume that ν is C^r vector field $r \geq 2$, that admits a MB function F as a first integral. The set of these vector fields will be denoted by $\psi_{\mathrm{MB}}(\mathbb{M})$. We will say that $v \in \psi_{\mathrm{MB}}(\mathbb{M})$ is non singular if it has no equilibrium points. Therefore we will assume that F does not have critical level set components that consist of a point. The corresponding set will be denoted by $\psi_{\mathrm{NMB}}(\mathbb{M})$. Given a vector field if it has a first integral F all function functionally related with F are also first integrals; therefore, if needed we will denote by (ν, F) the pair of a vector field and a particular first integral.

We will say that (ν, F) has saddle connection if there exists an orbit that leaves one saddle periodic orbit and ends in another saddle periodic orbit.

The main result in this paper is Theorem 12 which says that the structure of the level sets for vector fields on $\psi_{\mathrm{NMB}}(\mathbb{S}^3)$ is similar to the case of Hamiltonian systems. Although the structure is the same the tools used are different; for instance we do not have the Arnold–Liouville Theorem. In particular we introduce in Definition 14 an invariant that is a graph whose vertex are in correspondence with singular knotted periodic orbits of the system.

The study of invariants for orbital equivalent flows and topological obstructions to the integrability are main items in this area. See [**4**], [**5**], [**9**], and [**10**]. For a symplectic point of view see [**13**]. The knowledge of the structure of the integrable systems implies integrability criterion. In particular we found that, as in the Hamiltonian case, all periodic orbits in a MB integrable system must be generalized torus knots.

The characterization of Morse–Bott foliations without codimension one singularities is also analyzed in [**16**] and [**17**].

In the second part of the paper we focus on systems that admit another MB first integral independent of the former one.

To our knowledge the structure corresponding to a vector field with two first independent integrals has not been studied in the context of foliations. For Hamiltonian systems it is assumed that all basic information is contained in the foliation defined on $I_b(H)$ by F. Here we see that in fact the second integral is also important. More concretely, we introduce in Definition 22 a new invariant where the edges of the graph are associated with knotted periodic orbits. Therefore in the case of one first integral we associate a knotted periodic orbit to each vertex. In the case of two first integrals, we also associate a knotted periodic orbit to each edge.

This paper is organized as follows. In Section 2 we study the level set of MB first integrals defined on 3-spheres. In Section 3 we present a handle decomposition of $\mathbb{S}^3$, this decomposition is done using a MB first integral. Their local characterization is given. The main theorem of this paper (Theorem 12) is in Section 4. It characterizes the link of singular periodic orbits of a non singular Morse Bott integrable vector field on $\mathbb{S}^3$. We focus on the MB completely integrable systems in Section 4. Finally, in last section, there are two examples of integrable and completely integrable MB systems.

2. Level sets of Morse–Bott first integral

We will denote $\mathcal{F}(\mathbb{M}^n)$ the (singular) foliation defined on $\mathbb{M}^n$ by the level sets of F. Here we will assume that $\mathbb{M}^n = \mathbb{S}^3$ but the results of this section hold for any orientable 3-manifold that can not contain an embedded Klein bottle as for instance, $\mathbb{M}^2 \times \mathbb{S}^1$, $\mathbb{M}^2 \neq \mathbb{S}^2$, a lens space $L(p,q)$ with p odd or $L(2,q)$, see [**7**].

The fact that we have a vector field on $\mathbb{S}^3$ that admits a MB first integral imposes restrictions on the type of singularities.

- In Figure 1 we have a leaf $\mathcal{L}(\mathcal{F})$ asymptotic to a torus τ and to a circular singular set. The transversal structure is of Morse type but any first integral F will be constant on the solid torus bounded by τ, therefore is not an admissible singularity in our case. See also [**16**].
- In [**18**] we have a necessary and sufficient condition that a Reeb graph must fulfill in order to have a Morse first integral of a vector field v on a surface S. The direct product of v with the vector field

defined on $\mathbb{S}^1$ by $\dot{\alpha} = 1$ is a three dimensional example of system where all singularities are admissible but with a global structure incompatible with a MB first integral.

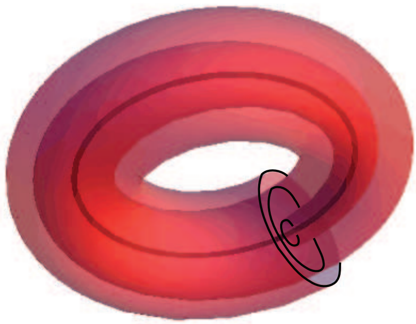

Figure 1. Not admissible singularity.

Now we will focus the level set of a Morse–Bott function F.

Proposition 1. *Given a pair (ν, F), with $v \in \psi_{\mathrm{NMB}}(\mathbb{S}^3)$, assume that b is a regular value of F, then each connected component of $I_b(F)$ is homeomorphic to a two dimensional torus.*

Proof: If b is a regular value of F then $I_b(F)$ is a 2-dimensional orientable, compact manifold on $\mathbb{S}^3$ without boundary. As it is defined a non singular vector field on it, its Euler characteristic is zero. Therefore $I_b(F)$ could be only a torus. $\square$

$\mathrm{Sing}(\mathcal{F}(\mathbb{S}^3))$ will be the set of singularities of $\mathcal{F}(\mathbb{S}^3)$. According to the index of F the singularities of it are classified as $\mathrm{Sad}(\mathcal{F})$ saddle singularities and $\mathrm{Cen}(\mathcal{F})$, center singularities. A center singularity of dimension m will be denoted by ς^m and an m-dimensional saddle singularity by σ^m.

Let b be a critical value of F. $S_b(F)$ will be the set of critical points of F contained in $I_b(F)$.

Proposition 2. *Let F be a* MB *first integral for a vector field ν on $\mathbb{S}^3$, then $S_b(F)$ is an invariant set of ν.*

Proof: Assume that $b = 0$ and $p \in S_b(F)$ then, in a neighborhood of p the function F can be written as

$$\sum_{i=1}^{k} \pm x_i^2, \quad k \leq 3$$

and the critical level set:

$$\sum_{i=1}^{k} \pm x_i^2 = 0, \quad k \leq 3.$$

If $k = 3$ then $S_0(F) = I_0(F)$. It is an equilibrium point of the vector field.

If $k = 2$ and $F(x_1, x_2, x_3) = x_1^2 - x_2^2$ the singular level set decomposes in two invariant subsets: $x_1 = \pm x_2$. Its intersection is invariant and coincides with $S_0(F)$.

If $k = 2$ or $k = 1$ and the index is zero, then $S_0(F) = I_0(F)$ and the singularity is a center. □

From now on we will assume that ν is non singular, i.e. $v \in \psi_{\mathrm{NMB}}(\mathbb{S}^3)$.

Proposition 3. *Given a pair (ν, F), with $v \in \psi_{\mathrm{NMB}}(\mathbb{S}^3)$, and a singular value b of F then each connected component of $S_b(F)$ is homeomorphic to a circle or a two dimensional torus.*

Proof: The function F cannot have isolated critical points. Therefore by the definition of a MB function, $S_b(F)$ must be a circle or a compact surface with Euler characteristic equal to zero. Since a Klein bottle can not be contained in $\mathbb{S}^3$, the surface must be a $\mathbb{T}^2$. □

It follows from Proposition 3 that $S_b(F)$ is a union of circles and tori but $I_b(F)$ can have a complicated topology.

If $S_b(F)$ is homeomorphic to a disjoint union of tori only, then it is a critical manifold of F but it is not a singularity of the foliation since all nearby leaves are tori.

Any invariant set diffeomorphic to $\mathbb{S}^1$ is a period orbit of ν. Otherwise we will have equilibrium points on the circle.

Definition 4. A singularity of the vector field $\mathrm{Sing}(\nu)$ will be a center equilibrium point or a center periodic orbit and any subset that is the limit set of some other trajectories.

The vector field ν can have singularities that are not singularities of the MB first integral, as periodic orbits on regular 2-torus.

Now we will describe the dynamics on a regular torus that can be richer than in the case of Liouville integrable systems. The dynamic on a torus depends on the rotation number if it is defined. If it is irrational the trajectories are dense on the torus. In the other cases the flow consist in a set of annulus bounded by periodic separatrices. Inside the annulus one has a periodic parallel flow or open trajectories that tend to the separatrices. In case the separatrices have reverse orientation with respect to time we will say that the annulus is a Reeb component called *Kneser ring*. See Figure 2.

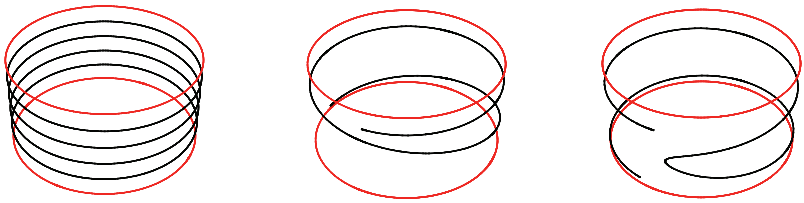

FIGURE 2. Parallel, asymptotic and Kneser ring.

Proposition 5 ([**2**]). *Let φ_t a flow on a torus without equilibrium states. Then the Poincaré rotation number υ exists and 1) if υ is rational or ∞, then there is at least one closed trajectory on the torus; 2) if υ is irrational, then the flow φ_t has no closed trajectory and contains exactly one non-trivial (that is, not an equilibrium state or a closed trajectory) minimal set which is either the whole torus (transitive flow) or a nowhere dense on the torus (singular flow).*

The last case of Proposition 5 is possible only if the vector field is not twice differentiable. A global section is a simple closed curve C that is a section and such that any positive half-trajectory starting on it reaches C again for $t > 0$.

If the flow on the torus has neither equilibrium states nor a global section, then the rotation number υ of this flow is either rational or ∞ and the flow contains at least one Kneser ring.

Proposition 6 ([**2**]). *Two transitive flows on a torus without equilibrium states are topologically equivalent if and only if their rotation numbers are commensurable.*

3. Handles

The usual definition of handle of dimension n and index k, $h_k = D^k \times D^{n-k}$ with $\partial_- h_k = \partial D^k \times D^{n-k}$ and $\partial_+ h_k = D^k \times \partial D^{n-k}$ is not suitable for the study of MB functions and its associated vector fields. It was generalized by Asimov [**3**], introducing round handles, that are not homotopically trivial. For 2-dimension singular sets must be generalized a bit more (see [**1**]). We will use the following definitions:

Definition 7. Let $i \geq 1$. An oriented i-th multi-round handle of dimension n and index k is a disk bundle over $\mathbb{T}^i$:

$$R_k^i = D^k \bigoplus_{\mathbb{T}^i} D^{n-k-i}.$$

If R_k^i is orientable it is a trivial disk bundle:

$$R_k^i = D^k \times D^{n-k-i} \times \mathbb{S}_1^1 \times \cdots \times \mathbb{S}_i^1.$$

The boundary of R_k^i contains two subsets

$$\partial_- R_k^i = \partial D^k \bigoplus_{\mathbb{T}^i} D^{n-k-1}, \quad \partial_+ R_k^i = D^k \bigoplus_{\mathbb{T}^i} \partial D^{n-k-1},$$

that in the orientable case can be reduced to:

$$\begin{aligned} \partial_- R_k^i &= \partial D^k \times D^{n-k-i} \times \mathbb{S}_1^1 \times \cdots \times \mathbb{S}_i^1, \\ \partial_+ R_k^i &= D^k \times \partial D^{n-k-i} \times \mathbb{S}_1^1 \times \cdots \times \mathbb{S}_i^1. \end{aligned} \tag{3.1}$$

Definition 8. Let $\mathbb{M}^n$ be a compact connected manifold. We say that $\mathbb{M}^n$ has a flow manifold structure if the boundary of $\mathbb{M}^n$ is a disjoint union of connected components such that $\partial\mathbb{M}^n = \partial_-\mathbb{M}^n \cup \partial_+\mathbb{M}^n$ and $\chi(\mathbb{M}^n) = \chi(\partial_-\mathbb{M}^n) = \chi(\partial_+\mathbb{M}^n)$.

Let s be a critical connected manifold of $S_b(F)$. We will denote by $H_b(F,s)$ a connected component of $I_{[b-\varepsilon,b+\varepsilon]}(F)$, or just $H(F,s)$ if it is not necessary to specify the singular values.

We want to study what kind of manifold is $H(F,s)$. We consider first that the singularity is a center.

Proposition 9. *$H_b(F,\varsigma^m)$ has a structure of flow round handle manifold, where ς^m is a center singularity of dimension $m = 1, 2$.*

Proof: Firstly we consider the case of the critical tori. In this case $H_b(F,s) = R_1^2 = D^1 \times D^0 \times \mathbb{S}^1 \times \mathbb{S}^1$ and $\partial_- R_1^2 = (p \times D^0 \times \mathbb{S}^1 \times \mathbb{S}^1) \cup (q \times D^0 \times \mathbb{S}^1 \times \mathbb{S}^1)$, where $D^1 = [p,q]$ and $\partial_+ R_1^2 = \emptyset$.

When we have a center periodic orbit, $H_b(F,s) = R_2^1 = D^2 \times D^0 \times \mathbb{S}^1$ and $\partial_- R_2^1 = \mathbb{S}^1 \times \{p\} \times \mathbb{S}^1$, where p is a point and $\partial_+ R_2^1 = \emptyset$.

As $\chi(R_k^i) = \chi(\partial_- R_k^i) = \chi(\partial_+ R_k^i) = 0$ since $\mathbb{S}^1$ is a factor of R_k^i for all i and k and $\partial_+ R_1^2 = \partial_+ R_2^1 = \emptyset$, it follows that $H_b(F,s) = R_2^1 = R_1^2$ are flow round handles. $\square$

We will denote $\Sigma(F)$ the manifold $\mathbb{M}$ minus the level sets of saddle singularities.

Definition 10. Consider a connected component $\Sigma(F)$ and ς a center singularity contained in $\Sigma(F)$. The basin of ς is in $\Sigma(F)$. It will be denoted by $B(\varsigma)$.

$B(\varsigma)$ is an open set since it is equal to $\mathbb{M}$ or the complementary set of the union of a finite number of closed sets.

Now we consider the case of a saddle singularity.

We will call $H_{b,l}(F,\sigma)$ the intersection of $R_1^1 = H_b(F,\sigma)$ with a closed regular neighborhood of σ in such a way that the border of $H_{b,l}(F,\sigma) = R_1^1$ is the union of two subsets. One of them, U, consists of the union of subsets of regular tori of the foliation. The second one, τ is transversal to the leaves. Let Π_p be a normal plane to s at p and $\Sigma_p = \Pi_p \cap H_{b,l}(F,\sigma)$. We will call B_i each connected component of U and $T(B_i)$ the torus that contains B_i, see Figure 3. Observe that the same torus can be attached to different B_i. By $B_i(c)$ we specify the level set $I_c(F)$ that contains B_i.

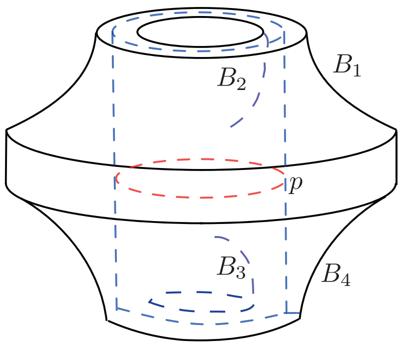

FIGURE 3. $H_{b,l}(F,\sigma)$, σ saddle.

For the local handle $H_{b,l}(F,\sigma)$ we can assume the normal form of the singularity. We characterize R_1^1 from the local handle. The construction of the global $H_b(F,\sigma)$ from $H_{b,l}(F,\sigma)$ will be done taking into account how many $T(B_i)$ exists and which B_i are contained in the same torus. In particular, as s is a Morse–Bott singularity there are at most three $T(B_i)$ different and if $T(B_i(b-\varepsilon)) = T(B_j(b-\varepsilon))$, $i<j$ then $j=i+2$.

- $H_b(F,\sigma)$ is orientable.

  To fix the notation we assume that $T(B_j)$ on $I_{b-\varepsilon}(F)$ are $T(B_1)$ and $T(B_3)$.

  Let p be a point of σ and consider on Σ_p, a local section π_p then, $\bigcup_{p\in\sigma} \pi_p \cap T(B_i)$, $i=1,3$ defines two closed circles parallel to s and located on $T(B_i)$. We will denote this two circles by $s_1(b-\varepsilon)$ and $s_3(b-\varepsilon)$. Since we are in the orientable case and π_p can be arbitrarily close to the separatrices, $s_1(b-\varepsilon) \neq s_3(b-\varepsilon)$. Moreover, $s_1(b-\varepsilon)$ and $s_3(b-\varepsilon)$ are essential (non homotopically trivial) in $T(B_1)$ and $T(B_3)$, respectively. Assume that $s_1(b-\varepsilon)$ is trivial.

Then s_1 bounds a disk $Ds_1(\varepsilon)$ on $T(B_1)$. As ε tends to zero $s_1(b-\varepsilon)$ tends to σ, the disks $Ds_1(\varepsilon)$ converge to a disk on $I_b(F)$ which is now invariant since it is bounded by σ. The interior of this disk must contain an equilibrium point in contradiction with the non singularity assumption on the vector field.

The set of the closed curves $s_i(b-\varepsilon)$ defines a cylinder C contractible to σ. On $T(B_1)$, $s_1(b-\varepsilon)$ and $s_2(b-\varepsilon)$ defines two complementary annuli A_i with respect $T(B_1)$. Each A_i jointly with C defines a solid torus. The union of the two tori for $i = 1, 3$ is $T(B_1)$ and they intersect only on C. The interior of these solid tori contains $T(B_2)$ and $T(B_4)$. See Figures 4, 5, 6 and 7.

By the last arguments $H_b(F,\sigma) \backsimeq N_2 \times \mathbb{S}^1$, where N_p a disk with $p \geq 1$ holes.

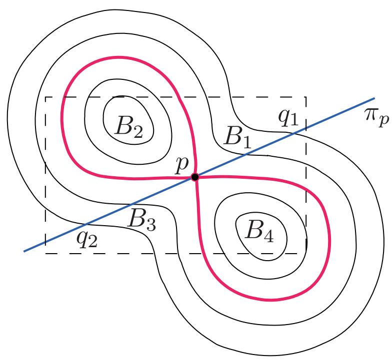

FIGURE 4. $H_{b,l}(F,\sigma) \cap \Pi_p$.

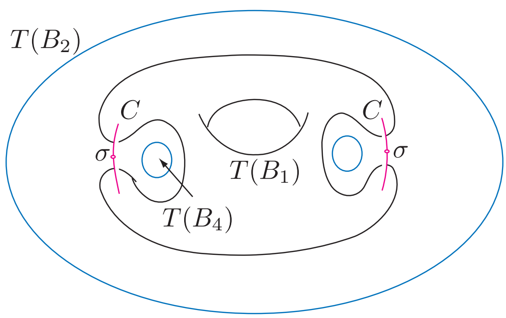

FIGURE 5. $T(B_i)$ and C.

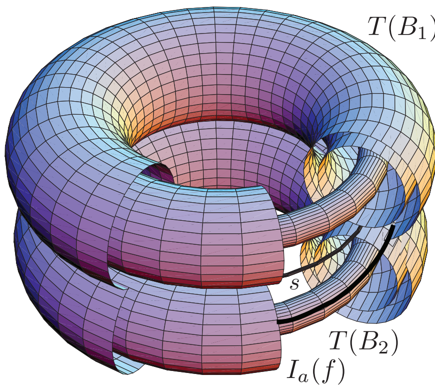

Figure 6. $H_b(F,s)$ orientable, $|T(B_j(b-\varepsilon))| = 1$, $s_1(b-\varepsilon)$ is not a parallel.

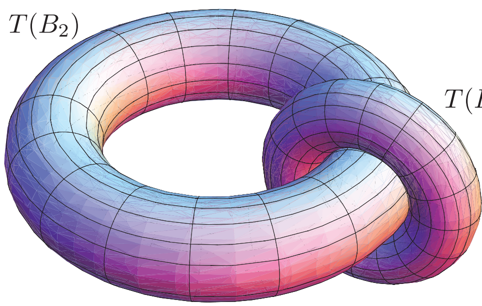

Figure 7. $H_b(F,\sigma)$ orientable, $|T(B_j(b-\varepsilon))| = 1$, $s_1(b-\varepsilon)$ is a parallel.

- $H_b(F,\sigma)$ is not orientable.

  In this case, $T(B_1) = T(B_3)$ and $T(B_2) = T(B_4)$. For each $p \in \sigma$, in the section π_p we have the same situation of the orientable case with a $(2,1)$ knot. See Figure 8. But now the fibration is not trivial and $H_b(F,\sigma)$ does not have an $\mathbb{S}^1$ factor. Then it must be characterized as $\left(N_2 \times \mathbb{S}^1\right) \setminus \Psi$, where Ψ is a regular neighborhood of a $(2,1)$-cable of $\{0\} \times \mathbb{S}^1$.

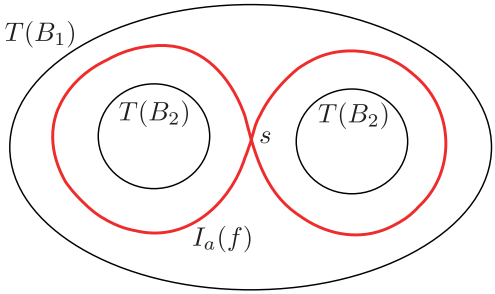

FIGURE 8. Section of $H_b(F, \sigma)$, case not orientable.

We resume our discussion in the next theorem:

Theorem 11. *Let σ be a saddle periodic orbit contained in $S_b(f)$ without connection by means of a separatrix with another saddle periodic orbit, then $H_b(F, \sigma)$ is homeomorphic to:*

a) *$N_2 \times \mathbb{S}^1$ if $H_b(F, \sigma)$ is orientable.*
b) *$\left(N_2 \times \mathbb{S}^1\right) \setminus \Psi$ where Ψ is a regular neighborhood of a $(2, 1)$-cable of $\{0\} \times \mathbb{S}^1$ if $H_b(F, \sigma)$ is not orientable.*

4. Link of singular periodic orbits

Recall (see [**15**]) that a subset K of a space $\mathbb{M}$ is a knot if it is homeomorphic to a sphere $\mathbb{S}^p$. More generally, K is a link if it is homeomorphic to a disjoint union $\mathbb{S}^{p_1} \cup \cdots \cup \mathbb{S}^{p_r}$ of one or more spheres. Two knots or links K and K' are equivalent if there is a homeomorphism $h \colon \mathbb{M} \to \mathbb{M}$ such that $h(K) = K'$.

The following union $\cup_{s \in \mathrm{Sing}(F)} H(F, s)$ defines a decomposition of $\mathbb{S}^3$ since F is a first integral. Moreover, as the sets $H(F, s)$ are attached in an essential way, as [**20**] or [**8**], one has the following result

Theorem 12. *The link of singular periodic orbits of a non singular Morse Bott integrable vector field on $\mathbb{S}^3$, without saddle connections, is made by applying operations IV, V and VI to a Hopf link.*

The Wada operations IV, V and VI to a Hopf link are the following:

Given indexed links l_1 and l_2, denote by $V(k, \mathbb{M})$ a regular neighborhood of k in $\mathbb{M}$ then

IV. To make $(l_1 \# l_2) \cup m$. The connected sum $l_1 \# l_2$ is obtained by composing a component k_1 of l_1 and a component k_2 of l_2 each of which has index 0 or 2. The index of the composed component

$k_1 \# k_2$ is equal to either $\mathrm{ind}(k_1)$ or $\mathrm{ind}(k_2)$. Finally, m is a meridian of $k_1 \# k_2$, and its index is 1.

V. Choose a component k_1 of l_1 of index 0 or 2, and replace $V(k_1, \mathbb{S}^3)$ by $D^2 \times \mathbb{S}^1$ with three indexed circles in it; $\{0\} \times \mathbb{S}^1$, k_2 and k_3. Here, k_2 and k_3 are parallel (p,q)-cables on $\partial V(\{0\} \times \mathbb{S}^1, D^2 \times \mathbb{S}^1)$. The indices of $\{0\} \times \mathbb{S}^1$ and k_2 are either 0 or 2, and one of them is equal to $\mathrm{ind}(k_1)$. The $\mathrm{ind}(k_3) = 1$.

VI. Choose a component k_1 of l_1 of index 0 or 2. Replace $V(k_1, \mathbb{S}^3)$ by $D^2 \times \mathbb{S}^1$ with two indexed circles in it; $\{0\} \times \mathbb{S}^1$, and the $(2,q)$-cable k_2 of $\{0\} \times \mathbb{S}^1$. So $\mathrm{ind}(\{0\} \times \mathbb{S}^1) = 1$ and $\mathrm{ind}(k_2) = \mathrm{ind}(k_1)$.

This result can also be obtained by applying the results of [**19**].

We will denote the link of periodic orbits defined by F by $\mathcal{L}(F)$. The associated knot of a singular periodic orbit is a generalized torus knot by [**8**] or [**9**]. Recall that a generalized torus knot is a knot obtained from iterated torus knots or connected sums of iterated torus knots. In particular, the eight knot is not an iterated torus knot.

If we consider also the periodic orbits of ν that are not singularities of the MB function f, the conclusion is the same since this periodic orbits lies on the tori of the system.

The next proposition and the realization properties can also be obtained with similar arguments to those in [**10**].

Proposition 13. *Let s be a periodic orbit of a vector field defined on $\mathbb{S}^3$ with a Morse–Bott first integral then, s is an generalized torus knot.*

Given a continuous function $F\colon \mathbb{M}^n \to \mathbb{R}$ the space obtained from $\mathbb{M}^n$ by contracting each connected component of the level sets to a point is called *Reeb graph of F* ([**14**] and [**18**]). When F is a MB function, each vertex corresponds to a critical value of F. Since the singular tori do not imply any topological change in the foliation defined by F we define the reduced Reeb graph $R_G(F)$ as the Reeb graph without the vertex associated to the singular tori.

Definition 14. The graph link of (ν, F) consists of the set of $R_G(F)$ and $\mathcal{L}(F)$ and a map from $R_G(F)$ to $\mathcal{L}(F)$ sending the vertex corresponding to the critical value b to the periodic orbits on $I_b(F)$.

5. Vector fields completely integrable

We will say that a vector field $v(\mathbb{M}^n)$ is completely integrable if it has $n-1$ first integrals $F_1, F_2, \ldots, F_{n-1}$ functionally independent. We will denote by $(\nu, F_1, \ldots, F_{n-1})$ the vector field and its integrals. From now on we will consider that $\mathbb{M}^n = \mathbb{S}^3$ and that F_i are MB-first integrals.

Now we focus on the singularities of $F = (F_1, F_2)$. A point p will be singular if the rank of the Jacobian of (F_1, F_2) is less than two. This happens if p is a singular point of F_1 or F_2 or if the gradients of both integrals are parallel.

Proposition 15. *The set T verifying that $I_{a_1}(F_1)$ intersects tangentially $I_{a_2}(F_2)$ is a closed set on each regular level set.*

Proof: In a point of tangential intersection the normal vectors of each level set are dependent. Consider an accumulation point q of T, since there exist a sequence of points of T tending to q where the normals are dependent, by continuity they must also be dependent on q. $\square$

Here to avoid degenerate cases we assume the following

I. F_i are MB functions.

II. The restriction of F_i to each regular level set $I_a(F_j)$ of F_j, where $j \neq i$ is a MB function.

Definition 16. We will say that (ν, F_1, F_2) is a MB completely integrable system if it is completely integrable and the assumptions I, II hold.

The regular level sets $I_a(F_1, F_2)$ are diffeomorphic to $\mathbb{S}^1$ and the set where $I_{a_1}(F_1)$ and $I_{a_2}(F_2)$ intersects tangentially, i.e. the singularities of F on regular torus, it is an invariant set. The proofs are similar to the case of one first integral.

On a regular torus of a MB completely integrable there is neither a Kneser ring nor a ring with asymptotic orbits since any continuous first integral must be constant on these rings.

Definition 17. We will say that ς^1 is a trivial periodic center if in a neighborhood of it all the orbits are periodic and parallel to ς^1.

Proposition 18. *Given a* MB *first integral F of a non singular vector field ν on $\mathbb{S}^3$ all saddle periodic orbits and non trivial periodic centers are singularities of F.*

Proof: Consider first a center of a system that has F_1 as MB first integral. If the proposition does not hold through ς^1 will pass a regular torus τ of the foliation defined by F_2. On it all orbits are equivalent knots. On the other hand, τ will cut all regular torus close to the center singularity. But this imply that also on this tori the orbits are equivalent to ς^1 in contradiction with the fact that ς^1 is not trivial center.

In the case of a saddle singularity, not only the periodic orbit σ^1 lies on the level set, also the invariant manifolds of σ^1 will lie on it. It must be a singular level set since the set of the invariant manifolds does not defines a manifold. □

Two first MB integrals of ν must share saddle and non trivial center one dimensional singularities, but they do not share necessarily the same trivial center singularities.

On the other hand, since the singular tori are not singularities of the foliation, if ς_1, ς_2 are trivial center periodic orbits in the same connected component of $\Sigma(f)$ (see Definition 10) then both center periodic orbits will define the same basins $B(\varsigma_1) = B(\varsigma_2)$. So we have:

Corollary 19. $R_G(F_1) = R_G(F_2)$.

Proposition 20. *All periodic orbits in the basin of a center* ς *are equivalent knots.*

Proof: If an orbit on a torus $\tau_1 \in B(\varsigma)(F_1)$ is dense on it, its rotation number is not rational and F_2 will be constant on τ_1 in contradiction with the assumption of (ν, F_1, F_2) be MB completely integrable. Assume that an orbit on τ_1 is a torus knot of type (p, q) with rotational number $\frac{p}{q}$. If the another torus τ_2 of $B(\varsigma)$ there is a periodic orbit of type $\frac{p_1}{q_1} \neq \frac{p}{q}$ as the rotation number varies continuously it will take a not rational number in some torus, between τ_1 and τ_2. So (ν, F_1, F_2) will be not MB completely integrable. □

As a consequence of this proposition we define:

Definition 21. $\mathcal{L}(F_1, F_2)$ will consists of $\mathcal{L}(F_1)$ and a periodic orbit of the basin of each ς^1.

Definition 22. The graph link of (ν, F_1, F_2) consists of the set of $R_G(F_1)$, $\mathcal{L}(F_1, F_2)$ and a map from $R_G(F_1)$ to $\mathcal{L}(F_1, F_2)$ sending each vertex to a periodic orbit of $\mathcal{L}(F_1)$ and an edge of $R_G(F_1)$ to an orbit of $\mathcal{L}(F_1, F_2) - \mathcal{L}(F_1)$.

As a consequence of this study we have that

Proposition 23. *The graph link is an invariant of orbital equivalent systems.*

In the other hand, the structure of the singular set determines the knotted type of the orbits. In the case of two foliations in a surface this is analyzed in [**12**]. The case of MB completely integrable system is similar. The intersection of the singular set with a regular torus consists of orbits of the system. As the type of knot will be the same on all the basin, it determines how the orbits are knotted.

6. Examples

In the last section we proved that any periodic orbit in a $\psi_{\mathrm{NMB}}(\mathbb{S}^3)$ is a generalized torus knot. Therefore a criterion of non-integrability of a system on $\mathbb{S}^3$ will be the presence of a knotted periodic orbit that was not of this type or if the link between them do not corresponds to the types described in Section 4. See in [**11**] an example of non integrable system in the context of Hamiltonian systems.

In this section we give several examples of vector field in $\psi_{\mathrm{NMB}}(\mathbb{S}^3)$ in particular an example of a non MB completely integrable system and another one MB completely integrable.

Example 1: Two center singularities and a singularity of the vector field. Consider the vector field defined in $\mathbb{R}^4$:

$$\begin{aligned}
\frac{dx_1}{dt} &= x_2\left(\left(x_1-\frac{1}{2}\right)^2+x_2^2\right),\\
\frac{dx_2}{dt} &= -x_1\left(\left(x_1-\frac{1}{2}\right)^2+x_2^2\right),\\
\frac{dx_3}{dt} &= x_4,\\
\frac{dx_4}{dt} &= -x_3.
\end{aligned} \tag{6.1}$$

Except for the factor on the first equations it is the harmonic oscillator. It has the following first integrals:

$$\begin{aligned}
h_1(x_1,x_2,x_3,x_4) &= x_1^2+x_2^2+x_3^2+x_4^2,\\
h_2(x_1,x_2,x_3,x_4) &= x_3^2+x_4^2,\\
h_3(x_1,x_2,x_3,x_4) &= x_1^2+x_2^2.
\end{aligned} \tag{6.2}$$

Fix $h_1(x_1,x_2,x_3,x_4)=1$ and the system is defined on the standard $\mathbb{S}^3$. Consider the MB function $F=h_2$ then the singularities on F are two center periodic orbits:

- $I_1(F)$ in the orbit $x_1^2+x_2^2=0$ where the family of tori $x_3^2+x_4^2=\epsilon$ concentrate;
- $I_0(F)$ in the orbit $x_1^2+x_2^2=1$ where the family of tori $x_1^2+x_2^2=\epsilon$ concentrate.

From this it follows that the reduced Reeb graph associated has two vertex and one edge. Moreover the link $\mathcal{L}(F)$ is the link of Hopf since each periodic orbit is sending to a trivial unknot.

This system also has the periodic orbit $x_1 = \frac{1}{2}$, $x_2 = 0$ on $I_{\frac{3}{4}}(F)$ as a singularity of the vector field. It is important to observe that this is not a MB completely integrable because any additional first integral is constant over the torus $I_{\frac{3}{4}}(F)$.

Example 2: The hat function. Consider the vector field defined in $\mathbb{R}^4$:

$$\begin{aligned}
\frac{dx_1}{dt} &= x_2 - \frac{23}{2}x_1^2 x_2 - 9x_2^3,\\
\frac{dx_2}{dt} &= -\frac{7}{2}x_1 + 14x_1^3 + \frac{23}{2}x_1 x_2^2,\\
\frac{dx_3}{dt} &= x_4 g(x_1, x_2, x_3, x_4),\\
\frac{dx_4}{dt} &= -x_3 g(x_1, x_2, x_3, x_4),
\end{aligned} \tag{6.3}$$

where $g = g(x_1, x_2, x_3, x_4)$ is a strictly positive function.

Then

$$h_1(x_1, x_2) = -21x_1^2(-1 + x_1^2 + x_2^2) - 27(x_1^2 + x_2^2)^2 + 6(x_1^4 + (1 + x_1^2)x_2^2),$$

$$h_2(x_3, x_4) = x_3^2 + x_4^2$$

are two first integrals for system (6.3).

Proposition 24. *The subset ζ^3 defined implicitly on $\mathbb{R}^4$ by*

$$h_1(x_1, x_2) + x_3^2 + x_4^2 = -21$$

is a manifold homeomorphic to $\mathbb{S}^3$ and invariant for the system defined by (6.1)*.*

Proof: The set ζ^3 projects on the (x_1, x_2)-plane on the domain D

$$h_1(x_1, x_2) + 21 \geq 0.$$

Since $h_1(x_1, x_2) + 21$ simplifies to $-3(-1 + x_1^2 + x_2^2)(7 + 14x_1^2 + 9x_2^2)$ and $7 + 14x_1^2 + 9x_2^2$ is always positive, the domain D is the unit disk.

Fix a point $p = (x_1, x_2)$ in the interior of D. ζ^3 on the fixed point is the $\mathbb{S}^1$ sphere $x_3^2 + x_4^2 = -21 - h_1(p)$. On ∂D the sphere collapses to a point. Then ζ^3 is a pinched fibration of $\mathbb{S}^1$ on the unit disk, i.e. a $\mathbb{S}^3$. The second assertion is a consequence of $h_1(x_1, x_2)$ and $x_3^2 + x_4^2$ be first integrals of the vector field. $\square$

We restrict the domain of definition of the system (6.1) to the manifold ζ^3 and consider the Morse–Bott first integral:

$$F(x_1, x_2, x_3, x_4) = (h_1(x_1, x_2) + 21)^2, \quad (x_1, x_2, x_3, xfa_4) \in \zeta^3.$$

In Figure 9 we have represented this function.

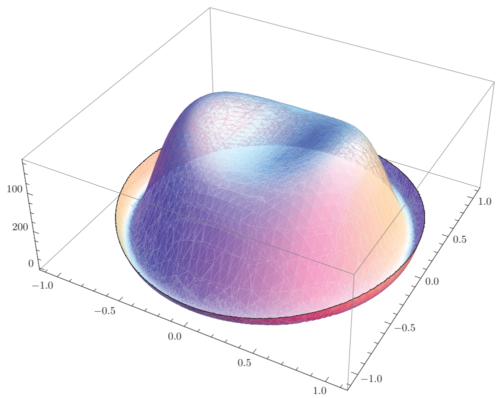

FIGURE 9. MB function for system (6.1).

On D, the subsystem:

$$(6.4) \qquad \begin{aligned} \frac{dx_1}{dt} &= x_2 - \frac{23}{2}x_1^2x_2 - 9x_2^3, \\ \frac{dx_2}{dt} &= -\frac{7}{2}x_1 + 14x_1^3 + \frac{23}{2}x_1x_2^2 \end{aligned}$$

has three center equilibrium points $(-\frac{1}{2}, 0)$, $(0,0)$ and $(\frac{1}{2}, 0)$ and two saddle equilibrium points $(0, \frac{1}{3})$, $(0, -\frac{1}{3})$. The flow is represented in Figure 10.

Moreover, in an increasing sense of F, the singularities that we meet are:

- ς_1 periodic orbit with $x_3 = x_4 = 0$, $F = 0$.
- ς_2 center periodic orbit at $x_1 = 0$, $x_2 = 0$, $F = 441$.
- σ_1, σ_2 saddle periodic orbits at $x_1 = 0$, $x_2 = \pm\frac{1}{3}$, $F = \frac{4096}{9} \cong 455.111$.
- ς_1, ς_2 center periodic orbits at $x_1 = \pm\frac{1}{2}$, $x_2 = 0$, $F = \frac{35721}{64} \cong 558.141$.

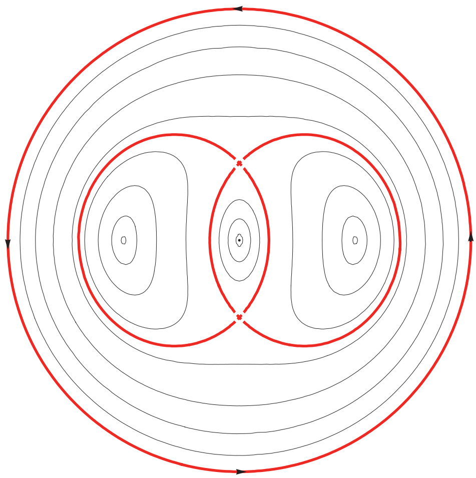

Figure 10. The flow of system (6.4).

We can re-write system (6.3) in coordinates (x_1, x_2, α) in the interior of D, where α is the angle of the circle defined by the coordinates (x_3, x_4) after fixing a point in the plane defined by (x_1, x_2).

Let $\tau(h_1)$ the period of the orbit given by $h_1 = C$, where C is a constant. The period of this orbit in the coordinates (x_3, x_4) depends on α and $\dot{\alpha} = -g$. So choosing $g = g(x_1, x_2, x_3, x_4)$ in such way that $g(x_1, x_2, x_3, x_4) = (p/q).\tau(h_1(x_1, x_2))$, p, q integers, the system (6.3) is a MB completely integrable system. All periodic orbits are (p, q) torus knots.

Acknowledgements

The first author is supported by the MINECO/FEDER grant number MTM2012-33073 and the grant number HPB-2009-0005. The second author is supported by FAPESP project 2008/54222-6, CAPES-DGU grant number 222/2010 and by FP7-PEOPLE-2012-IRSES-316338.

References

[1] J. Adachi, Surgeries of 4-dimensional manifolds and Engel structure, Kyoto University RIMSD **1664** (2009), 105–116.

[2] S. Kh. Aranson and V. Z. Grines, Topological classification of flows on closed two-dimensional manifolds, (Russian), *Uspekhi Mat. Nauk* **41** (1986), no. 1(247), 149–169, 240; translation in: *Russian Math. Surveys* **41(1)** (1986), 183–208.

[3] D. Asimov, Daniel Round handles and non-singular Morse–Smale flows, *Ann. of Math. (2)* **102(1)** (1975), 41–54.
[4] A. V. Bolsinov and A. T. Fomenko, *"Integrable Hamiltonian systems"*, Geometry, topology, classification, Translated from the 1999 Russian original, Chapman & Hall/CRC, Boca Raton, FL, 2004. DOI: 10.1201/9780203643426.
[5] A. V. Bolsinov, S. V. Matveev, and A. T. Fomenko, Topological classification of integrable Hamiltonian systems with two degrees of freedom. A list of systems of small complexity, (Russian), *Uspekhi Mat. Nauk* **45** (1990), no. 2(272), 49–77, 240; translation in: *Russian Math. Surveys* **45(2)** (1990), 59–94. DOI: 10.1070/RM1990v045n02ABEH002344.
[6] R. Bott, Nondegenerate critical manifolds, *Ann. of Math. (2)* **60** (1954), 248–261.
[7] G. E. Bredon and J. W. Wood, Non-orientable surfaces in orientable 3-manifolds, *Invent. Math.* **7** (1969), 83–110. DOI: 10.1007/BF01389793.
[8] J. Casasayas, J. Martínez Alfaro, and A. Nunes, Knots and links in integrable Hamiltonian systems, *J. Knot Theory Ramifications* **7(2)** (1998), 123–153. DOI: 10.1142/S0218216598000097.
[9] A. T. Fomenko, *"Integrability and nonintegrability in geometry and mechanics"*, Translated from the Russian by M. V. Tsaplina, Mathematics and its Applications (Soviet Series) **31**, Kluwer Academic Publishers Group, Dordrecht, 1988.
[10] A. T. Fomenko and T. Z. Nguyen, Topological classification of integrable nondegenerate Hamiltonians on the isoenergy three-dimensional sphere, in: *"Topological classification of integrable systems"*, Adv. Soviet Math. **6**, Amer. Math. Soc., Providence, RI, 1991, pp. 267–296.
[11] J. Müller, Knots and transition to chaos in modified Hénon–Heiles System, Report University of Uppsala, Departament of Theoretical Physics (2004).
[12] R. D. S. Oliveira and F. Tari, On pairs of differential 1-forms in the plane, *Discrete Contin. Dynam. Systems* **6(3)** (2000), 519–536. DOI: 10.3934/dcds.2000.6.519.
[13] Á. Pelayo and S. V. Ngọc, Symplectic theory of completely integrable Hamiltonian systems, *Bull. Amer. Math. Soc. (N.S.)* **48(3)** (2011), 409–455. DOI: 10.1090/S0273-0979-2011-01338-6.
[14] G. Reeb, Sur les points singuliers d'une forme de Pfaff complètement intégrable ou d'une fonction numérique, *C. R. Acad. Sci. Paris* **222** (1946), 847–849.

[15] D. Rolfsen, *"Knots and links"*, Mathematics Lecture Series **7**, Publish or Perish, Inc., Berkeley, Calif., 1976.

[16] B. Scárdua and J. Seade, Codimension one foliations with Bott–Morse singularities. I, *J. Differential Geom.* **83(1)** (2009), 189–212.

[17] B. Scárdua and J. Seade, Codimension 1 foliations with Bott–Morse singularities. II, *J. Topol.* **4(2)** (2011), 343–382. DOI: 10.1112/jtopol/jtr004.

[18] V. V. Sharko, About Kronrod–Reeb graph of a function on a manifold, *Methods Funct. Anal. Topology* **12(4)** (2006), 389–396.

[19] M. C. Sullivan, Transverse foliations to nonsingular Morse–Smale flows on the 3-sphere and Bott-integrable Hamiltonian systems, *Qual. Theory Dyn. Syst.* **7(2)** (2009), 411–415. DOI: 10.1007/s12346-008-0023-0.

[20] M. Wada, Closed orbits of nonsingular Morse–Smale flows on S^3, *J. Math. Soc. Japan* **41(3)** (1989), 405–413. DOI: 10.2969/jmsj/04130405.

José Martínez-Alfaro:
Departament de Matemàtica Aplicada
Universitat de València
Spain
E-mail address: martinja@uv.es

Regilene Oliveira:
Departamento de Matemática
ICMC - USP
C.P 668, 13.560-970, São Carlos, SP
Brazil
E-mail address: regilene@icmc.usp.br