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POLYNOMIAL AND RATIONAL FIRST INTEGRALS FOR PLANAR HOMOGENEOUS POLYNOMIAL DIFFERENTIAL SYSTEMS

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Abstract: In this paper we find necessary and sufficient conditions in order that a planar homogeneous polynomial differential system has a polynomial or rational first integral. We apply these conditions to linear and quadratic homogeneous polynomial differential systems.

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1. Introduction and statement of the main results

One of the main problems in the qualitative theory of differential systems in $\mathbb{C}^2$ is to determine whether they have a global first integral, i.e. first integrals defined in a dense and open subset of $\mathbb{C}^2$. This problem goes back to Poincaré. In fact, Poincaré in 1891 started a series of three papers [**26, 27, 28**] in which he tried to answer the following question: Is it possible to decide if an algebraic differential equation in two variables is algebraically integrable? (in the sense that it has a rational first integral).

For an arbitrary polynomial differential system in $\mathbb{C}^2$ the existence of a rational first integral does not imply the existence of an analytic equation on the coefficients, and neither the degree of the integral nor the genus of the phase curve is bounded by a function of the degree of the differential system, see [**18**].

The characterization of polynomial or rational integrability for different particular differential systems has attracted the attention of many authors, see for instance [**1, 16, 21, 22, 23**] and references therein. In the present paper we give the characterization of polynomial or rational integrability for homogeneous polynomial differential systems. Moreover, for such systems when we control the polynomial first integrals, in fact, we control all analytical first integrals, see [**17, 21**]. Indeed in [**17**] it is shown that if the eigenvalues of the linear part of the differential system do not satisfy some form of resonance condition, then no analytic first integral exists (in a close similarity to the conditions for the existence of polynomial and rational first integral).

Let $\mathbb{C}[x,y]$ be the ring of all polynomials in the variables x and y with coefficients in $\mathbb{C}$. And let $\mathbb{C}(x,y)$ be its quotient field, that is, the field of rational functions in the variables x and y with coefficients in $\mathbb{C}$. As we have said, here we are interested in computing polynomial and rational first integrals of homogeneous polynomial differential systems in $\mathbb{C}^2$ i.e. differential systems of the form

$$\dot{x} = P_n(x,y), \quad \dot{y} = Q_n(x,y), \tag{1}$$

where $(x,y) \in \mathbb{C}^2$, $P_n(x,y), Q_n(x,y) \in \mathbb{C}[x,y]$ are *coprime* and homogeneous of degree n and the dot denotes derivative with respect to an independent variable t real or complex. Our aim is to characterize the systems of the form (1) which have a polynomial or a rational first integral.

Given a planar polynomial differential system

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y), \tag{2}$$

where $P(x,y), Q(x,y) \in \mathbb{C}[x,y]$, we denote by n the degree of the system, that is, $n = \max\{\deg P, \deg Q\}$. We say that a function $H\colon \mathcal{U} \subseteq \mathbb{C}^2 \to \mathbb{C}$, with $\mathcal{U}$ an open set, is a *first integral* of system (2) if H is continuous, not locally constant and constant on each trajectory of the system contained in $\mathcal{U}$. We note that if H is of class at least $\mathcal{C}^1$ in $\mathcal{U}$, then H is a first integral if it is not locally constant and

$$P(x,y)\frac{\partial H}{\partial x} + Q(x,y)\frac{\partial H}{\partial y} \equiv 0$$

in $\mathcal{U}$. We call the *integrability problem* the problem of finding such a first integral and the functional class where it belongs. We say that the system has a *polynomial first integral* if there exists a first integral $H(x,y) \in \mathbb{C}[x,y]$. Analogously, we say that the system has a *rational first integral* if there exists a first integral $H(x,y) \in \mathbb{C}(x,y)$.

We say that a function $V\colon \mathcal{W} \subseteq \mathbb{C}^2 \to \mathbb{C}$, with $\mathcal{W}$ an open set, is an *inverse integrating factor* of system (2) if V is of class $\mathcal{C}^1$, not locally zero and satisfies the following linear partial differential equation

$$P(x,y)\frac{\partial V}{\partial x} + Q(x,y)\frac{\partial V}{\partial y} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) V(x,y)$$

in $\mathcal{W}$. The knowledge of an inverse integrating factor defined in $\mathcal{W}$ allows the computation of a first integral in $\mathcal{U} = \mathcal{W} \setminus \{V = 0\}$ by the following line integral

$$H(x,y) = \int_{(x_0,y_0)}^{(x,y)} \frac{P(x,y)\,dy - Q(x,y)\,dx}{V(x,y)},$$

where $(x_0, y_0) \in \mathcal{U}$ is any point. An easy computation shows that the homogeneous polynomial

$$V_{n+1}(x,y) = yP_n(x,y) - xQ_n(x,y)$$

is an inverse integrating factor for system (1). A proof of this well-known fact can be found in Lemma 3 of [**4**]. If $V_{n+1}(x,y) \equiv 0$ and $n > 1$ we have that the polynomials $P_n(x,y)$ and $Q_n(x,y)$ are not coprime in contradiction with our hypothesis. We detail the integrability for the case $n = 1$ of system (1) in Proposition 7. To see the relation between the functional classes of the inverse integrating factors and their associated first integrals see Theorem 3 of [**14**], see also [**7**].

Many works deal with the integrability problem of a system of the form (2), see for instance [**3, 5, 7, 20, 24**] and references therein. The main idea of these works is to consider the singular points of the system and to give necessary conditions on the eigenvalues associated to the linear part of each singular point in order that the system has a first integral of a particular functional class. The paper [**25**] deals with linear differential systems defined in k^N, where k is a field of characteristic zero and $N \geq 2$ is an integer, and the author gives a characterization of the linear differential systems which have a polynomial or a rational first integral in terms of the eigenvalues of the constant matrix which defines the linear differential system.

Our main result is a characterization of the homogeneous polynomial differential systems (1) which have a polynomial or a rational first integral in terms of the eigenvalues associated to the singular points of the system which lie on the "exceptional divisor", once the origin has been blow-up.

We consider the homogeneous polynomial $V_{n+1}(x,y) = yP_n(x,y) - xQ_n(x,y)$ and its factorization in linear factors

$$V_{n+1}(x,y) = \prod_{i=1}^{n+1} v_i(x,y), \tag{3}$$

where $v_i(x,y) \in \mathbb{C}[x,y]$ are homogeneous polynomials of degree 1, for $i = 1, 2, \ldots, n+1$. We say that two linear factors are *equal* if they are proportional by a scalar value $c \in \mathbb{C}$. We note that $V_{n+1}(x,y)$ is *square-free* if $v_i(x,y)$ is different from $v_j(x,y)$ for all $i \neq j$.

Our first result establishes the non-existence of a rational first integral, and in consequence the non existence of a polynomial first integral, for system (1) when $V_{n+1}(x, y)$ factorizes with repeated linear factors.

Proposition 1. *If the homogeneous polynomial* $V_{n+1}(x, y) = yP_n(x, y) - xQ_n(x, y)$ *is not square-free in* $\mathbb{C}[x, y]$*, then system* (1) *has no rational first integral.*

The proof of this result is given in Section 2.

We analyze system (1) with $n = 1$ in Proposition 7. We note that the origin for $n > 1$ of system (1) is a degenerate singular point. We make use of the blow-up technique for studying its local phase portrait and to characterize the integrability properties of the system. For a detailed explanation of the blow-up technique, see for instance Chapter 3 of [**11**]. The idea of a blow-up is to separate the directions at which the orbits of a system get to a singular point. In order to do that, we consider the change of coordinates $(x, y) \to (u, v)$ with $x = u$ and $y = u\,v$. We observe that in all the points of $\mathbb{C}^2$ except those with $x = 0$, this change is a one-to-one correspondence and that all the points (x, y) in the same straight line through the origin $y = mx$, with $m \in \mathbb{C}$, are transformed to the horizontal straight line $v = m$. In order to "see" also the points such that $x = 0$, we also consider the directional blow-up given by $(x, y) \to (u, v)$ with $x = u\,v$, $y = v$, which is in one-to-one correspondence for all the points of $\mathbb{C}^2$ except those with $y = 0$. This blow-up can be interpreted as a change which transforms the disk in a neighborhood of the origin into a Moebius band and the origin into a circle $\mathbb{S}^1$ in the middle of this Moebius band. This circle, which turns out to be the union of the orbits of the transformed system after dividing by u^{n-1} (or v^{n-1} in the case of the second directional blow-up), is called the *exceptional divisor*. The two considered directional blow-ups are two local charts which cover the Moebius band. In the first one $(x, y) \to (u, v)$ with $x = u$ and $y = u\,v$, the exceptional divisor is given by $u = 0$ and in the second one $(x, y) \to (u, v)$ with $x = u\,v$, $y = v$, the exceptional divisor is given by $v = 0$.

We deal with the singular points of system (1) which lie on the exceptional divisor $\mathbb{S}^1$. We will show that if $p \in \mathbb{S}^1$ is a singular point of system (1) once transformed, then the matrix corresponding to the linear part of the system in a neighborhood of p is diagonal. We denote its eigenvalues by λ_p and μ_p, where λ_p is the eigenvalue whose eigenvector is orthogonal to the exceptional divisor $\mathbb{S}^1$ and μ_p is the eigenvalue whose eigenvector is tangent to the exceptional divisor $\mathbb{S}^1$. We also define the quotient of these two eigenvalues by $\gamma_p = \lambda_p/\mu_p$.

Lemma 2. *Consider the factorization of the polynomial* $V_{n+1}(x,y) = yP_n(x,y) - xQ_n(x,y)$ *given in* (3)*. Each* $v_i(x,y)$ *defines a singular point on the exceptional divisor having eigenvalues* λ_i *and* μ_i*, where* λ_i *is the eigenvalue whose eigenvector is orthogonal to the exceptional divisor and* μ_i *is the eigenvalue whose eigenvector is tangent to the exceptional divisor. If* $V_{n+1}(x,y)$ *is square-free, then* $\lambda_i\,\mu_i \neq 0$*, for* $i = 1, 2, \ldots, n+1$*.*

The proof of Lemma 2 is given in Section 2.

The following result provides the explicit expression of a first integral of system (1) when $V_{n+1}(x,y)$ is square-free.

Theorem 3. *Using the same notation than in Lemma 2, if the homogeneous polynomial* $V_{n+1}(x,y)$ *is square-free, then*

$$H(x,y) = \prod_{i=1}^{n+1} v_i(x,y)^{\gamma_i},$$

with $\gamma_i = \frac{\lambda_i}{\mu_i}$*, is a first integral of system* (1)*. Furthermore,* $\sum_{i=1}^{n+1} \gamma_i = -1$*.*

The function of the first integral given in Theorem 3 is called of *Darboux* type, and the first integral is called *Darboux first integral.* To know more about the Darboux theory of integrability, see for instance [**6, 8, 9, 19**] and the references therein. The proof of Theorem 3 is given in Section 2. There are several papers which give results about general planar polynomial differential systems of the form (2). Using these results for the general case, one could give alternative proofs to Proposition 1 and Theorem 3 in the particular case of the homogeneous system (1). This could be done in the framework of remarkable values, see [**6, 12, 13**] and the references therein, and using the necessary conditions for the existence of invariant algebraic curves, see [**7**] and the references therein. In particular, in [**12**], there are several results about systems of the form (2) which have a Darboux first integral characterizing the existence of a polynomial inverse integrating factor. We note that the particular homogeneous systems (1) that we consider have the Darboux first integral provided in Theorem 3 and the polynomial inverse integrating factor $V_{n+1}(x,y)$ defined in Proposition 1.

The following theorem is the main result of this paper and characterizes when system (1) has a polynomial or a rational first integral. As usual $\mathbb{Q}$ denotes the set of rational numbers, and $\mathbb{Q}^+$ (resp. $\mathbb{Q}^-$) the set of positive (resp. negative) rational numbers.

Theorem 4. *Using the same notation than in Lemma 2, the following statements hold.*

(a) *System* (1) *has a rational first integral if and only if* $V_{n+1}(x, y)$ *is square-free and* $\gamma_i \in \mathbb{Q}$ *for* $i = 1, 2, \ldots, n + 1$.

(b) *System* (1) *has a polynomial first integral if and only if* $V_{n+1}(x, y)$ *is square-free and* $\gamma_i \in \mathbb{Q}^-$ *for* $i = 1, 2, \ldots, n + 1$.

In relation with statement (a) we remark that, as $\sum_{i=1}^{n+1} \gamma_i = -1$, we only need to have that n (out of $n+1$) of the values γ_i are rational and we deduce that all of them are rational. In relation with statement (b) we need that the $n + 1$ values γ_i are in $\mathbb{Q}^-$. Theorem 4 is proved in Section 2.

As an application of Theorem 4, we characterize all the linear and quadratic homogeneous polynomial differential systems of the form (1) which admit a rational or a polynomial first integral, see Section 3.

The following statement provides some necessary conditions for some planar polynomial differential systems of the form (2) to have a rational first integral.

Corollary 5. *Consider a system* (2) *of degree* n *and let* P_n *and* Q_n *be the terms of degree* n *of the polynomials* P *and* Q. *Assume that* P_n *and* Q_n *are coprime. We define the polynomial* $V_{n+1} = yP_n - xQ_n$ *and the values* γ_i *defined in Theorem 3. Then the following statements hold.*

(i) *If* V_{n+1} *is not square-free, then system* (2) *has no rational first integral.*

(ii) *If* V_{n+1} *is square-free and there is a value* $\gamma_i \notin \mathbb{Q}$, *for* $i=1, 2, \ldots, n+1$, *then system* (2) *has no rational first integral.*

(iii) *If* V_{n+1} *is square-free and there is a value* $\gamma_i \notin \mathbb{Q}^-$, *for* $i = 1, 2, \ldots, n + 1$, *then system* (2) *has no polynomial first integral.*

Corollary 5 is proved in Section 2.

2. Proofs of the main results

Proof of Proposition 1: If the homogeneous polynomial $V_{n+1}(x, y) = yP_n(x, y) - xQ_n(x, y)$ is not square-free in $\mathbb{C}[x, y]$, we can assume, without loss of generality, that it has the multiple factor y with multiplicity $m > 1$. That is, $V_{n+1}(x, y) = y^m R(x, y)$ where $R(x, y)$ is a homogeneous polynomial of degree $n+1-m$ and such that $R(x, 0) \not\equiv 0$. To have this assumption we consider, if necessary, a rotation of the variables so as to get y as the multiple factor. Indeed, in the new coordinates, we get that y divides $Q_n(x,y)$ and, since $P_n(x,y)$ and $Q_n(x,y)$ are coprime polynomials, we have that $P_n(x,0) \not\equiv 0$.

Assume that system (1) has a rational first integral $H(x,y) = \frac{A(x,y)}{B(x,y)}$, with $A, B \in \mathbb{C}[x,y]$. Hence,

$$P_n(x,y)\frac{\partial H}{\partial x} + Q_n(x,y)\frac{\partial H}{\partial y} \equiv 0.$$

By deriving the quotient $H(x,y) = A(x,y)/B(x,y)$ and multiplying by $B(x,y)^2$, we get

$$P_n(x,y)\left(\frac{\partial A}{\partial x}B(x,y) - A(x,y)\frac{\partial B}{\partial x}\right) \\ + Q_n(x,y)\left(\frac{\partial A}{\partial y}B(x,y) - A(x,y)\frac{\partial B}{\partial y}\right) \equiv 0,$$

or equivalently

$$\left(P_n(x,y)\frac{\partial A}{\partial x} + Q_n(x,y)\frac{\partial A}{\partial y}\right)B(x,y) \\ = A(x,y)\left(P_n(x,y)\frac{\partial B}{\partial x} + Q_n(x,y)\frac{\partial B}{\partial y}\right). \tag{4}$$

We denote by $A_a(x,y)$ the homogeneous terms of highest order a in the polynomial $A(x,y)$ and by $B_b(x,y)$ the homogeneous terms of highest order b in the polynomial $B(x,y)$. We denote by $\widetilde{A}(x,y)$ (resp. $\widetilde{B}(x,y)$) the sum of lower terms in $A(x,y)$ (resp. $B(x,y)$), that is, $A(x,y) = \widetilde{A}(x,y) + A_a(x,y)$ (resp. $B(x,y) = \widetilde{B}(x,y) + B_b(x,y)$). In the particular case that $a = b$ and there exists a constant $c \in \mathbb{C}$ such that $A_a(x,y) = c\,B_b(x,y)$, we consider the first integral $H(x,y) - c$ instead of $H(x,y)$. Then

$$\begin{aligned} H(x,y) - c &= \frac{\widetilde{A}(x,y) + A_a(x,y)}{\widetilde{B}(x,y) + B_b(x,y)} - c \\ &= \frac{\widetilde{A}(x,y) + A_a(x,y) - c\widetilde{B}(x,y) - cB_b(x,y)}{\widetilde{B}(x,y) + B_b(x,y)} \\ &= \frac{\widetilde{A}(x,y) - c\widetilde{B}(x,y)}{\widetilde{B}(x,y) + B_b(x,y)}. \end{aligned}$$

Thus, we can assume, without loss of generality, that the quotient $A_a(x,y)/B_b(x,y)$ is not a constant. The equation of the highest order terms in expression (4) gives

$$\left(P_n(x,y)\frac{\partial A_a}{\partial x}+Q_n(x,y)\frac{\partial A_a}{\partial y}\right)B_b(x,y)$$
$$=A_a(x,y)\left(P_n(x,y)\frac{\partial B_b}{\partial x}+Q_n(x,y)\frac{\partial B_b}{\partial y}\right),$$

which implies that the quotients $A_a(x,y)/B_b(x,y)$ and $B_b(x,y)/A_a(x,y)$ are also rational first integrals of system (1). Hence, if $H(x,y)$ is a rational first integral of system (1), we can assume without loss of generality that $H(x,y)$ is a homogeneous function of degree $d \geq 0$, where $d=|a-b|$.

We consider the blow-up $(x,y)\to(u,v)$ with $x=u$ and $y=uv$. The transformed system, after dividing by u^{n-1}, is

$$\dot{u}=u\,P_n(1,v),\quad \dot{v}=Q_n(1,v)-vP_n(1,v). \tag{5}$$

We remark that, since $V_{n+1}(x,y)=y^m\,R(x,y)$, we get $V_{n+1}(u,uv)=u^{n+1}\,v^m\,R(1,v)$. We define $r(v):=R(1,v)$ and since $V_{n+1}(x,y)=yP_n(x,y)-xQ_n(x,y)$, we get that the expression of $\dot{v}$ in the above system can we rewritten as $\dot{v}=-v^m\,r(v)$ with $r(0)\neq 0$. Indeed, as we have argued in the first paragraph of the proof of Proposition 1, $P_n(1,0)\neq 0$.

This change of variables gives that $H(u,u\,v)=u^dH(1,v)$ is a rational first integral of the system

$$\dot{u}=u\,P_n(1,v),\quad \dot{v}=-v^m\,r(v).$$

We define $h(v):=H(1,v)$ which is, by assumption, a rational function of the variable v. We have that

$$u\,P_n(1,v)\left(du^{d-1}h(v)\right)-v^m\,r(v)\left(u^d\,h'(v)\right)\equiv 0.$$

We note that if $d=0$ in the previous expression, we get that $h'(v)=0$, which implies that $h(v)$ is a constant. In this case, we can assume that $h(v)\equiv 1$. This fact implies that $H(u,u\,v)=u^0$ which means that $H(x,y)$ is a constant, in contradiction with the fact that it is a first integral. Therefore, $d>0$. The previous identity gives that

$$\frac{h'(v)}{h(v)}=\frac{d\,P_n(1,v)}{v^m\,r(v)}. \tag{6}$$

We develop the right-hand side of this identity in simple fractions of v, that is,

$$\frac{d\,P_n(1,v)}{v^m\,r(v)}=\frac{c_m}{v^m}+\frac{c_{m-1}}{v^{m-1}}+\cdots+\frac{c_1}{v}+\frac{\alpha_1(v)}{r(v)}+\alpha_0(v),$$

where $c_i\in\mathbb{C}$, for $i=1,2,\ldots,m$, and $\alpha_0(v)$, $\alpha_1(v)$ are polynomials with $\alpha_1(v)$ a polynomial of degree at most the degree of $r(v)$ minus 1.

Equating both expressions, we get that $c_m = d\, P_n(1,0)/r(0)$. Therefore, $c_m \in \mathbb{C}\backslash\{0\}$. We integrate identity (6) with respect to v, we exponentiate and we get that

$$h(v) = C \exp\left[\frac{c_m}{1-m}\frac{1}{v^{m-1}}\right] \times \exp\left[\int\left(\frac{c_{m-1}}{v^{m-1}} + \cdots + \frac{c_1}{v} + \frac{\alpha_1(v)}{r(v)} + \alpha_0(v)\right) dv\right],$$

where C is a constant of integration, which cannot be zero. We note that $\exp\left[\frac{c_m}{1-m}\frac{1}{v^{m-1}}\right]$ is not a rational function because c_m cannot be zero. This exponential function cannot be simplified by any part of the second factor. Thus, we get a contradiction with the fact that $h(v)$ is a rational function. We conclude that if $V_{n+1}(x,y)$ is not square-free, then there cannot exist a rational first integral for system (1). $\square$

Proof of Lemma 2: We can assume, without loss of generality, that x is not a divisor of the homogeneous polynomial $V_{n+1}(x,y) = yP_n(x,y) - xQ_n(x,y)$. If it was, we consider an affine change of variables (a rotation) to avoid it. Therefore, the homogeneous polynomial $V_{n+1}(x,y)$ factorizes in $\mathbb{C}[x,y]$ as

$$V_{n+1}(x,y) = c\,(y-\alpha_1 x)\,(y-\alpha_2 x)\cdots(y-\alpha_{n+1}x)\,,$$

where $\alpha_i \in \mathbb{C}$ for $i = 1,2,\ldots,n+1$ and $c \in \mathbb{C} - \{0\}$. In this way, we only need to consider the directional blow-up $(x,y) \to (u,v)$ with $x = u$, $y = u\,v$ in order to see all the singular points of the exceptional divisor. This directional blow-up transforms system (1) into (5). We observe that the singular points of system (5) on the exceptional divisor $u = 0$ are exactly those with $v = \alpha_i$, for $i = 1,2,\ldots,n+1$, because

$$\begin{aligned}\dot{v} &= Q_n(1,v) - vP_n(1,v) = -V_{n+1}(1,v)\\ &= -c\,(v-\alpha_1)\,(v-\alpha_2)\cdots(v-\alpha_{n+1})\,.\end{aligned}$$

Straightforward computations show that the linear matrix of system (5) in a neighborhood of the singular point $(u,v) = (0,\alpha_i)$ is

$$\left.\begin{pmatrix} P_n(1,v) & 0 \\ 0 & \dfrac{\partial}{\partial v}\left(Q_n(1,v) - vP_n(1,v)\right)\end{pmatrix}\right|_{v=\alpha_i}.$$

We observe that this matrix is diagonal and following the notation introduced in the statement of Lemma 2 we have $\lambda_i = P_n(1,\alpha_i)$ and

$$\mu_i = \left.\frac{\partial}{\partial v}\left(Q_n(1,v) - vP_n(1,v)\right)\right|_{v=\alpha_i}.$$

We remark that $\lambda_i \neq 0$ because if $P_n(1,\alpha_i)=0$ then, since $Q_n(1,\alpha_i)-\alpha_i P_n(1,\alpha_i)=0$, this would imply that $Q_n(1,\alpha_i)=0$ and, hence, the polynomials $P_n(x,y)$ and $Q_n(x,y)$ would not be coprime, in contradiction with the hypothesis. Therefore, $\lambda_i \neq 0$.

On the other hand, we see that

$$\mu_i = \frac{\partial}{\partial v}\left(-V_{n+1}(1,v)\right)\Big|_{v=\alpha_i} = -c \prod_{j=1,\, j\neq i}^{n+1} (\alpha_i-\alpha_j). \tag{7}$$

If $V_{n+1}(x,y)$ is square-free then $\alpha_i \neq \alpha_j$ when $i \neq j$ and, thus, $\mu_i \neq 0$, for $i=1,2,\ldots,n+1$. □

Proof of Theorem 3: We use the notation detailed in the statement and in the proof of Lemma 2 and we will first show that

$$\sum_{i=1}^{n+1} \gamma_i = \sum_{i=1}^{n+1} \frac{\lambda_i}{\mu_i} = -1. \tag{8}$$

This equality appears as a corollary of the results given in [**2, 24**] but we include here a proof for the sake of completeness. We can assume, without loss of generality, that x is not a divisor of the homogeneous polynomial $V_{n+1}(x,y)=yP_n(x,y)-xQ_n(x,y)$. We recall that

$$\begin{aligned} -V_{n+1}(1,v) &= Q_n(1,v) - vP_n(1,v) \\ &= -c\,(v-\alpha_1)\,(v-\alpha_2)\cdots(v-\alpha_{n+1})\,. \end{aligned} \tag{9}$$

Since the $V_{n+1}(x,y)$ is square-free, we have that $\alpha_i \neq \alpha_j$ when $i \neq j$. We use the decomposition in simple fractions of a rational function, then

$$\frac{P_n(1,v)}{Q_n(1,v)-vP_n(1,v)} = \sum_{i=1}^{n+1} \frac{b_i}{v-\alpha_i},$$

where $b_i \in \mathbb{C}$. We take the common denominator in the right-hand side, we multiply both members of the equality by it, i.e. by $\prod_{i=1}^{n+1}(v-\alpha_i)$, and we get that

$$\frac{P_n(1,v)}{-c} = \sum_{i=1}^{n+1} b_i \prod_{j=1,\, j\neq i}^{n+1} (v-\alpha_j).$$

We take the value $v=\alpha_i$ and we obtain that

$$\frac{P_n(1,\alpha_i)}{-c} = b_i \prod_{j=1,\, j\neq i}^{n+1} (\alpha_i-\alpha_j).$$

As we have seen in the proof of Lemma 2, we have that $\lambda_i = P_n(1, \alpha_i)$ and $\mu_i = -c \prod_{j=1,\, j\neq i}^{n+1} (\alpha_i - \alpha_j)$, see (7). Hence, $b_i = \lambda_i/\mu_i = \gamma_i$. Therefore, we have that

$$\frac{P_n(1, v)}{Q_n(1, v) - vP_n(1, v)} = \sum_{i=1}^{n+1} \frac{\gamma_i}{v - \alpha_i}. \tag{10}$$

We take again common denominator in the right-hand side of the previous identity and we multiply both members by $\prod_{i=1}^{n+1}(v - \alpha_i)$. We observe that the coefficient of v^n in the left-hand side is the coefficient of v^n of $P_n(1, v)$ divided by $-c$. By (9), we deduce that c is the coefficient of v^n in the polynomial $P_n(1, v)$. Thus, the coefficient of v^n in the left-hand side is -1 whereas the same coefficient in the right-hand side is $\sum_{i=1}^{n+1} \gamma_i$. Thus, we conclude (8).

Now we consider the expression $H(x, y) = \prod_{i=1}^{n+1} v_i(x, y)^{\gamma_i}$ and we have that it is a first integral of system (1) if and only if $H(u, u\,v)$ is a first integral of system (5), that is, if the following identity is satisfied

$$\frac{\partial H(u, u\,v)}{\partial u} uP_n(1, v) + \frac{\partial H(u, u\,v)}{\partial v} (Q_n(1, v) - vP_n(1, v)) \equiv 0. \tag{11}$$

By (9) and (8), we have that

$$H(u, u\,v) = u^{\sum_{i=1}^{n+1} \gamma_i} \prod_{i=1}^{n+1} (v - \alpha_i)^{\gamma_i} = \frac{1}{u} \prod_{i=1}^{n+1} (v - \alpha_i)^{\gamma_i}.$$

We can deduce, by simplifying $H(u, u\,v)$, from identity (11)

$$(-1)P_n(1, v) + \sum_{i=1}^{n+1} \gamma_i \frac{Q_n(1, v) - vP_n(1, v)}{v - \alpha_i} \equiv 0. \tag{12}$$

We see by (9) that

$$\frac{Q_n(1, v) - vP_n(1, v)}{v - \alpha_i} = -c \prod_{j=1,\, j\neq i}^{n+1} (v - \alpha_j).$$

We use this expression to rewrite the second term in the left-hand side of (12) and also using that $\gamma_i = \lambda_i/\mu_i$, $\lambda_i = P_n(1, \alpha_i)$ and $\mu_i = -c \prod_{j=1,\, j\neq i}^{n+1} (\alpha_i - \alpha_j)$ (see (7)) we have that

$$\sum_{i=1}^{n+1} \gamma_i \frac{Q_n(1, v) - vP_n(1, v)}{v - \alpha_i} = \sum_{i=1}^{n+1} P_n(1, \alpha_i) \prod_{j=1,\, j\neq i}^{n+1} \frac{v - \alpha_j}{\alpha_i - \alpha_j},$$

which is the expression of the Lagrange polynomial which interpolates the $n+1$ points $(\alpha_i, P_n(1,\alpha_i))$, $i = 1, 2, \ldots, n+1$, see for more details [**15**]. Therefore, this polynomial is $P_n(1,v)$ and we conclude that identity (12) is satisfied and, hence, identity (11) is also satisfied. $\square$

Proof of Theorem 4: As we have proved in Proposition 1, the fact that $V_{n+1}(x,y)$ is square-free is a necessary condition for system (1) to have a rational (or polynomial) first integral. We will assume that $V_{n+1}(x,y)$ is square-free for the rest of the proof and we will also assume that x is not a divisor of $V_{n+1}(x,y)$, by doing a rotation in the variables if necessary.

Assume that system (1) has a rational first integral $H(x,y)$. As we have shown in the proof of Proposition 1, we can assume that $H(x,y)$ is a homogeneous function of degree $d > 0$. Indeed, we have that $H(u, u\,v)$ is a first integral of system (5). We can write $H(u, u\,v) = u^d h(v)$ with $h(v) := H(1,v)$. Since it is a first integral of system (5), we have that

$$uP_n(1,v)\left(du^{d-1}h(v)\right) + \left(Q_n(1,v) - vP_n(1,v)\right)u^d h'(v) \equiv 0,$$

which implies that

$$\frac{h'(v)}{h(v)} = -d\frac{P_n(1,v)}{Q_n(1,v) - vP_n(1,v)}.$$

By (10) we have that

$$\frac{h'(v)}{h(v)} = -d\sum_{i=1}^{n+1}\frac{\gamma_i}{v-\alpha_i},$$

which implies that

$$h(v) = k\left(\prod_{i=1}^{n+1}(v-\alpha_i)^{\gamma_i}\right)^{-d},$$

where $k \in \mathbb{C}\setminus\{0\}$ is an integration constant. Therefore,

$$H(u, u\,v) = k\left(\frac{u}{\prod_{i=1}^{n+1}(v-\alpha_i)^{\gamma_i}}\right)^{d},$$

and thus

$$H(x,y) = k\left(\frac{x}{\prod_{i=1}^{n+1}(y/x-\alpha_i)^{\gamma_i}}\right)^{d} = k\left(\frac{x^{1+\sum_{i=1}^{n+1}\gamma_i}}{\prod_{i=1}^{n+1}(y-\alpha_i x)^{\gamma_i}}\right)^{d}.$$

Using that $\sum_{i=1}^{n+1} \gamma_i = -1$ as we have shown in Theorem 3, we get that

$$H(x,y) = k \left(\prod_{i=1}^{n+1} (y - \alpha_i x)^{\gamma_i} \right)^{-d}.$$

This fact implies that any first integral of system (1) needs to be the sum of powers of the expression given in Theorem 3.

In short, the only possibility for system (1) to have a rational first integral is that the expression given in Theorem 3 is a power of a rational function, which means that $\gamma_k/\gamma_1 \in \mathbb{Q}$ for $k = 2, 3, \ldots, n+1$. In this case, we denote by $q_k = \gamma_k/\gamma_1$ for $k = 2, 3, \ldots, n+1$. Since $\sum_{i=1}^{n+1} \gamma_i = -1$, we have that

$$\gamma_1 \left(1 + q_2 + q_3 + \cdots + q_{n+1}\right) = -1.$$

Therefore, the fact that $q_k \in \mathbb{Q}$ for $k = 2, 3, \ldots, n+1$ is equivalent to $\gamma_i \in \mathbb{Q}$ for $i = 1, 2, \ldots, n+1$.

Moreover, the only possibility for system (1) to have a polynomial first integral is that the expression given in Theorem 3 is the power of a polynomial, which means that all the γ_i belong to $\mathbb{Q}$ and are of the same sign. Again, since $\sum_{i=1}^{n+1} \gamma_i = -1$, we conclude that $\gamma_i \in \mathbb{Q}^-$ for $i = 1, 2, \ldots, n+1$. $\square$

Proof of Corollary 5: We first prove that if system (2) has a rational first integral, then system $\dot{x} = P_n(x,y)$, $\dot{y} = Q_n(x,y)$, where P_n and Q_n are the terms of degree n of the polynomials P and Q, has a rational first integral. We assume that system (2) has the rational first integral $H(x,y) = A(x,y)/B(x,y)$ with $A(x,y), B(x,y) \in \mathbb{C}[x,y]$ and we denote by $A_a(x,y)$, $B_b(x,y)$ the highest order terms of the polynomials $A(x,y)$, $B(x,y)$, respectively. We have that $P(\partial H/\partial x) + Q(\partial H/\partial y) \equiv 0$. We can reorder and simplify this expression to get

$$\left(P(x,y)\frac{\partial A}{\partial x} + Q(x,y)\frac{\partial A}{\partial y} \right) B(x,y) = A(x,y) \left(P(x,y)\frac{\partial B}{\partial x} + Q(x,y)\frac{\partial B}{\partial y} \right).$$

We equate the highest degree terms in the previous expression and we have

$$\left(P_n(x,y)\frac{\partial A_a}{\partial x} + Q_n(x,y)\frac{\partial A_a}{\partial y} \right) B_b(x,y) = A_a(x,y) \left(P_n(x,y)\frac{\partial B_b}{\partial x} + Q_n(x,y)\frac{\partial B_b}{\partial y} \right),$$

which implies that the quotients $A_a(x,y)/B_b(x,y)$ and $B_b(x,y)/A_a(x,y)$ are also rational first integrals of system $\dot{x} = P_n(x,y)$, $\dot{y} = Q_n(x,y)$. We can assume that these quotients are not constant using the same reasoning as in the proof of Proposition 1.

The statements of the corollary are direct consequences of Proposition 1 and Theorem 4. □

3. Linear and quadratic homogeneous polynomial differential systems

This section contains the characterization of the linear and quadratic homogeneous polynomial differential systems with a rational or a polynomial first integral. To do so, we shall use the canonical forms of linear and quadratic homogeneous polynomial differential systems. Lemma 6 describes the canonical forms of linear systems of the form (1) in $\mathbb{C}^2$. The canonical forms of quadratic homogeneous polynomial differential systems are given in [**10**] and are done for real quadratic systems. Thus, we give two statements: one in the real case and another one in the complex case.

The following lemma provides the canonical forms of linear homogeneous differential systems of the form (1).

Lemma 6. *By an affine change of variables and a rescaling of time, any linear homogeneous differential system of the form* (1) *is equivalent to one of the following linear systems:*

(a) $\dot{x} = x$, $\dot{y} = ax + y$;

(b) $\dot{x} = x$, $\dot{y} = ay$;

where $a \in \mathbb{C}$.

Proof: In order to avoid a confusion with the names of the variables, we start with a system with variables u and v and we apply an affine change of variables to get one of the systems (a) or (b) of the statement. That is, we consider constants $a_{10}, a_{01}, b_{10}, b_{01} \in \mathbb{C}$ which are the coefficients of a linear homogeneous differential system

$$\dot{u} = a_{10}u + a_{01}v, \quad \dot{v} = b_{10}u + b_{01}v. \tag{13}$$

We consider the homogeneous polynomial $\widetilde{V}(u,v) := v(a_{10}u + a_{01}v) - u(b_{10}u + b_{01}v)$ which is an inverse integrating factor of (13). The polynomial $\widetilde{V}(u,v)$ splits in two linear factors (equal or different) and we can assume, by a rotation of the variables if necessary, that u is one of these factors. Thus, we can assume that $a_{01} = 0$ without loss of generality and since the polynomials which define system (13) are assumed to be coprime, we have that $a_{10}\, b_{01} \neq 0$. Thus, we can take a time-rescaling and we get the system

$$\dot{u} = u, \quad \dot{v} = \frac{b_{10}}{a_{10}} u + \frac{b_{01}}{a_{10}} v.$$

We consider two cases, either $b_{01} = a_{10}$ or $b_{01} \neq a_{10}$.

Case (a): If $b_{01} = a_{10}$ we parameterize the coefficient b_{10} by $b_{10} = a\, a_{10}$ and renaming (u, v) with (x, y) we get system (a) of Lemma 6.

Case (b): If $b_{01} \neq a_{10}$, we parameterize the coefficient b_{01} by $b_{01} = a\, a_{10}$ and, hence, we are under the hypothesis that $a \neq 1$. We consider the affine change of variables $(u, v) \to (x, y)$ with $u = (1-a)a_{10}x$, $v = b_{10}x + y$ and we get system (b) of Lemma 6. □

The following proposition contains the characterization of linear homogeneous differential systems with a rational or a polynomial first integral.

Proposition 7. *By an affine change of variables and a rescaling of time, a linear homogeneous differential system of the form* (1) *has a rational (resp. polynomial) first integral if and only if it is equivalent to a linear system of the following form:*

$$\dot{x} = x, \quad \dot{y} = ay,$$

with $a \in \mathbb{Q}$ (resp. with $a \in \mathbb{Q}^-$).

Proof: We only need to consider the canonical forms provided in Lemma 6. For the system (a), we have that the polynomial $V_2(x, y) = ax^2$. We observe that if $a = 0$, we have a particular system of case (b). If $a \neq 0$ we get that the system has no rational first integral as a consequence of Theorem 4 because $V_2(x, y)$ is not square-free.

For system (b) in Lemma 6, we have that $V_2(x, y) = (a-1)\, xy$. If $a = 1$, we see that $H(x, y) = y/x$ is a rational first integral and there is no polynomial first integral. If $a \neq 1$, as an application of Theorem 3, we get the following first integral $H(x, y) = x^{\gamma} y^{-1-\gamma}$ with

$$\gamma = \frac{a}{1-a}.$$

Thus, the system has a rational first integral if and only if $a \in \mathbb{Q}$ by Theorem 4. Indeed, we see that $\gamma < 0$ and $-1-\gamma < 0$ if and only if $a < 0$. Therefore, also by Theorem 4, the system has a polynomial first integral only if $a < 0$. □

The following results contains the canonical forms of real quadratic homogeneous polynomial differential systems.

Lemma 8 ([**10**]). *Any real quadratic homogeneous polynomial differential system of the form* (1) *is affine-equivalent to one and only one of the following real quadratic systems:*

(i) $\dot{x} = ax^2 + xy$, $\dot{y} = (a+3)xy + y^2$;

(ii) $\dot{x} = xy$, $\dot{y} = x^2 + y^2$;

(iii) $\dot{x} = -xy$, $\dot{y} = x^2 - y^2$;

(iv) $\dot{x} = -2xy + \frac{2}{3}x(ax+by)$, $\dot{y} = x^2 + y^2 + \frac{2}{3}y(ax+by)$;

(v) $\dot{x} = -2xy + \frac{2}{3}x(ax+by)$, $\dot{y} = -x^2 + y^2 + \frac{2}{3}y(ax+by)$;

where $a, b \in \mathbb{R}$.

As a direct consequence of this lemma, we have the following result in the complex case.

Lemma 9. *Any quadratic homogeneous polynomial differential system of the form* (1) *is affine-equivalent to one and only one of the following quadratic systems:*

(a) $\dot{x} = ax^2 + xy$, $\dot{y} = (a+3)xy + y^2$;

(b) $\dot{x} = xy$, $\dot{y} = x^2 + y^2$;

(c) $\dot{x} = -2xy + \frac{2}{3}x(ax+by)$, $\dot{y} = x^2 + y^2 + \frac{2}{3}y(ax+by)$;

where $a, b \in \mathbb{C}$.

Proof: In order to avoid a confusion with the names of the variables, we start with a system with variables u and v and we apply an affine change of variables to get one of the systems (a), (b) or (c) of the statement. That is, we consider constants $a_{20}, a_{11}, a_{02}, b_{20}, b_{11}, b_{02} \in \mathbb{C}$ which are the coefficients of a quadratic homogeneous polynomial differential system

$$\dot{u} = a_{20}u^2 + a_{11}u\,v + a_{02}v^2, \quad \dot{v} = b_{20}u^2 + b_{11}u\,v + b_{02}v^2. \tag{14}$$

We consider the homogeneous polynomial $\widetilde{V}(u,v) := v(a_{20}u^2 + a_{11}u\,v + a_{02}v^2) - u(b_{20}u^2 + b_{11}u\,v + b_{02}v^2)$ which is an inverse integrating factor of (14). The polynomial $\widetilde{V}(u,v)$ splits in three linear factors (equal or different) and we can assume, by a rotation of the variables if necessary, that u is one of these factors. Thus, we can assume that $a_{02} = 0$ without loss of generality and since the polynomials which define system (14) are assumed to be coprime, we have that $b_{02} \neq 0$. We have three possible ways of splitting the polynomial $\widetilde{V}(u,v)$ taking into account the multiplicity of the factors. It may have three different linear factors, or a double one (which we will assume that it is u) with a simple one, or a triple one (which we will assume that it is u).

Case (a): We assume that $\widetilde{V}(u,v)$ has u as a double factor, that is, we assume that $a_{11} = b_{02}$ and $a_{20} - b_{11} \neq 0$. In this case, we parameterize the coefficient b_{20} by a value a such that

$$b_{20} = -\frac{(a_{20} - b_{11})^2}{3b_{02}} a + \frac{a_{20}(a_{20} - b_{11})}{b_{02}}.$$

We consider the affine change of variables $(u,v) \to (x,y)$ with

$$u = \frac{-3}{a_{20} - b_{11}} x, \quad v = \frac{-3b_{20}}{(a_{20} - b_{11})^2} x + \frac{1}{b_{02}} y,$$

and we get system (a) of Lemma 9.

Case (b): We assume that $\widetilde{V}(u,v)$ has u as a triple factor, that is, we assume that $a_{11} = b_{02}$ and $b_{11} = a_{20}$. In this case, since the polynomials which define system (14) are assumed to be coprime, we have that $b_{20} \neq 0$. We consider the affine change of variables $(u,v) \to (x,y)$ with

$$u = \frac{1}{\sqrt{b_{02}b_{20}}} x, \quad v = \frac{-a_{20}}{b_{02}\sqrt{b_{02}b_{20}}} x + \frac{1}{b_{02}} y,$$

and we get system (b) of Lemma 9.

Case (c): We assume that $\widetilde{V}(u,v)$ has three different linear factors, that is, we assume that $a_{11} \neq b_{02}$ and $\Delta := (a_{20} - b_{11})^2 + 4(a_{11} - b_{02})b_{20} \neq 0$. We consider the affine change of variables $(u,v) \to (x,y)$ with

$$u = \frac{2\sqrt{-3}}{\sqrt{\Delta}} x, \quad v = \frac{\sqrt{-3}(b_{11} - a_{20})}{\sqrt{\Delta}} x - \frac{3}{a_{11} - b_{02}} y.$$

Indeed, we parameterize the coefficients a_{11} and b_{11} by values a and b such that $b \neq -3/2$ and $(a,b) \neq (0,3)$ with

$$a_{11} = 2b_{02}\frac{b-3}{3+2b}, \quad b_{11} = \frac{a_{20}(b+6) + a\sqrt{-3\Delta}}{b-3},$$

and we get system (c) of Lemma 9. We note that if $b = -3/2$ or $(a,b) = (0,3)$ we have that the polynomials which define system (c) of Lemma 9 are not coprime, in contradiction with our hypothesis. □

The following results establish all the real quadratic homogeneous polynomial differential systems with a rational or a polynomial first integral.

Proposition 10. *A real quadratic homogeneous polynomial differential system of the form* (1) *has a rational first integral if and only if it is affine-equivalent to one and only one of the following forms:*

(a) $\dot{x} = -2xy + \frac{2}{3}x(ax+by)$, $\dot{y} = x^2 + y^2 + \frac{2}{3}y(ax+by)$, *with* $a = 0$ *and* $b \in \mathbb{Q}$.

(b) $\dot{x} = -2xy + \frac{2}{3}x(ax+by)$, $\dot{y} = -x^2 + y^2 + \frac{2}{3}y(ax+by)$, *with* $a = \sqrt{3}\,c$ *and* $b, c \in \mathbb{Q}$.

In what follows, as usual, we denote by $i = \sqrt{-1}$ the imaginary unit.

Proof of Propostion 10: By an affine change of variables, we only need to consider the five families of systems detailed in Lemma 8. In the family (i) we have that the homogeneous polynomial $V_3(x,y) = -3x^2y$ which is not square-free and, thus, the system has no rational first integral. In the families (ii) and (iii), the inverse integrating factor is $V_3(x,y) = -x^3$ which is neither square-free and the same conclusion follows.

In the family (iv):

$$\dot{x} = -2xy + \frac{2}{3}x(ax+by), \quad \dot{y} = x^2 + y^2 + \frac{2}{3}y(ax+by), \tag{15}$$

with $a, b \in \mathbb{R}$, we have that the inverse integrating factor is $V_3(x,y) = -x(x^2+3y^2)$. In order to compute the values of the γ_i for the three points on the exceptional divisor provided by the factors of $V_3(x,y)$ we consider the change of variables $(x,y) \to (u,v)$ with $x = uv$ and $y = v$. We take this blow-up because x is a divisor of $V_3(x,y)$ and y is not a divisor of this homogeneous polynomial. This blow-up leads to the system

$$\dot{u} = -u(3+u^2), \quad \dot{v} = \frac{1}{3}\left(3 + 2b + 2au + 3u^2\right)v.$$

This system has the singular points $p_1 = (0,0)$, $p_2 = (i\sqrt{3}, 0)$ and $p_3 = (-i\sqrt{3}, 0)$ on the exceptional divisor $v = 0$. The matrices associated to the linear part of the system on each of these points are:

$$A_{p_1} = \begin{pmatrix} -3 & 0 \\ 0 & \frac{1}{3}(3+2b) \end{pmatrix},$$

$$A_{p_2} = \begin{pmatrix} 6 & 0 \\ 0 & \frac{2}{3}(-3+i\sqrt{3}a+b) \end{pmatrix},$$

$$A_{p_3} = \begin{pmatrix} 6 & 0 \\ 0 & \frac{2}{3}(-3-i\sqrt{3}a+b) \end{pmatrix}.$$

For each singular point, we define its eigenvalues λ_i and μ_i, where λ_i is the eigenvalue whose eigenvector is orthogonal to the exceptional divisor and μ_i is the eigenvalue whose eigenvector is tangent to the exceptional divisor. Then we compute $\gamma_i = \lambda_i/\mu_i$ and we have that

$$\gamma_1 = -\frac{3+2b}{9}, \quad \gamma_2 = \frac{1}{9}\left(-3 + i\sqrt{3}a + b\right), \quad \gamma_3 = \frac{1}{9}\left(-3 - i\sqrt{3}a + b\right).$$

By Theorem 4, the necessary and sufficient condition for system (15) to have a rational first integral is that $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{Q}$ which implies that $b \in \mathbb{Q}$ and $a = 0$. In this case, a rational first integral is

$$H(x,y) = \left(x^{3+2b}\left(x^2+3y^2\right)^{3-b}\right)^m,$$

where m is the denominator of the rational number b.

In the family (v):

$$\dot{x} = -2xy + \frac{2}{3}x(ax+by), \quad \dot{y} = -x^2 + y^2 + \frac{2}{3}y(ax+by), \tag{16}$$

with $a, b \in \mathbb{R}$, we have that the homogeneous polynomial $V_3(x,y) = x(x^2-3y^2)$ is square-free. As in the previous case, we consider the change of variables $(x,y) \to (u,v)$ with $x = uv$ and $y = v$ which leads to the system

$$\dot{u} = u(u^2-3), \quad \dot{v} = \frac{1}{3}\left(3+2b+2au-3u^2\right)v.$$

This system has three singular points on $v = 0$ which are $p_1 = (0,0)$, $p_2 = (\sqrt{3}, 0)$ and $p_3 = (-\sqrt{3}, 0)$. By analogous computations as the ones performed in the previous case, we get that the quotients of eigenvalues for each singular point are

$$\gamma_1 = -\frac{1}{9}(3+2b), \quad \gamma_2 = \frac{1}{9}\left(\sqrt{3}a + b - 3\right), \quad \gamma_3 = \frac{1}{9}\left(-\sqrt{3}a + b - 3\right).$$

By Theorem 4, system (16) has a rational first integral if and only if $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{Q}$. We note that $\gamma_1 \in \mathbb{Q}$ iff $b \in \mathbb{Q}$. If we substract $\gamma_2 - \gamma_3$ we get $2\sqrt{3}a/9$ which also needs to be a rational number. Therefore we conclude that $a = \sqrt{3}\,c$ with $c \in \mathbb{Q}$. A rational first integral in this case is:

$$H(x,y) = \left(x^{3+2b}\left(x+\sqrt{3}y\right)^{3-b+3c}\left(x-\sqrt{3}y\right)^{3-b-3c}\right)^m,$$

where m is the least common multiple of the denominators of $\{3+2b, 3-b+3c, 3-b-3c\}$ with b, $c \in \mathbb{Q}$. □

Proposition 11. *A real quadratic homogeneous polynomial differential system of the form* (1) *has a polynomial first integral if and only if it is affine-equivalent to one and only one of the following forms:*

(a) $\dot{x} = -2xy + \frac{2}{3}x(ax+by)$, $\dot{y} = x^2 + y^2 + \frac{2}{3}y(ax+by)$, *with* $a = 0$, $b \in \mathbb{Q}$ *and* $-3/2 < b < 3$.

(b) $\dot{x} = -2xy + \frac{2}{3}x(ax+by)$, $\dot{y} = -x^2 + y^2 + \frac{2}{3}y(ax+by)$, *with* $a = \sqrt{3}\,c$, $b, c \in \mathbb{Q}$ *and* (b,c) *belong to the triangle* $b > -3/2$, $b < 3 - 3c$, $b < 3 + 3c$.

Proof: If a system of the form (1) has a polynomial first integral, in particular it has a rational first integral. Thus we are under the hypothesis of Proposition 10.

We recall that system (15) has the associated quotients of eigenvalues of singular points in the exceptional divisor

$$\gamma_1 = -\frac{3+2b}{9}, \quad \gamma_2 = \frac{1}{9}\left(-3 + i\sqrt{3}a + b\right), \quad \gamma_3 = \frac{1}{9}\left(-3 - i\sqrt{3}a + b\right)$$

as we have seen in the proof of Proposition 10. We have that $a = 0$ and $b \in \mathbb{Q}$ by Proposition 10. As a consequence of Theorem 4 we have that $\gamma_i \in \mathbb{Q}^-$ for $i = 1,2,3$ is the necessary and sufficient condition to have a polynomial first integral. This fact implies that $-3/2 < b < 3$.

For system (16) we have the associated values

$$\gamma_1 = -\frac{1}{9}(3+2b), \quad \gamma_2 = \frac{1}{9}\left(\sqrt{3}a + b - 3\right), \quad \gamma_3 = \frac{1}{9}\left(-\sqrt{3}a + b - 3\right),$$

as we have seen in the proof of Proposition 10. We have that $a = \sqrt{3}\,c$ and $b, c \in \mathbb{Q}$ again by Proposition 10. To have a polynomial first integral, we need that $\gamma_i \in \mathbb{Q}^-$ for $i = 1,2,3$, which implies that (b,c) belong to the triangle $b > -3/2$, $b < 3 - 3c$, $b < 3 + 3c$. □

The following results establish all the complex quadratic homogeneous polynomial differential systems with a rational or a polynomial first integral.

Proposition 12. *A quadratic homogeneous polynomial differential system of the form* (1) *has a rational first integral if and only if it is affine-equivalent to a quadratic system of the following form*

$$\dot{x} = -2xy + \frac{2}{3}x(ax+by), \quad \dot{y} = x^2 + y^2 + \frac{2}{3}y(ax+by),$$

with $a = i\sqrt{3}\,c$ *and* $b, c \in \mathbb{Q}$.

Proof: We only need to consider the systems appearing in the statement of Lemma 9. In the case (a) the polynomial $V_3(x,y)$ is $-3x^2y$ and in the case (b) it is $-x^3$. Since these polynomials are not square-free, we discard these two systems by Theorem 4. For system (c):

$$\dot{x} = -2xy + \frac{2}{3}x(ax+by), \quad \dot{y} = x^2 + y^2 + \frac{2}{3}y(ax+by), \tag{17}$$

with $a, b \in \mathbb{C}$, we have that $V_3(x,y) = -x(x^2+3y^2)$ which is square-free. Indeed, as we have seen in the proof of Proposition 10, we have that the quotients of eigenvalues of the singular points in the exceptional divisor are

$$\gamma_1 = -\frac{3+2b}{9}, \quad \gamma_2 = \frac{1}{9}\left(-3 + i\sqrt{3}a + b\right), \quad \gamma_3 = \frac{1}{9}\left(-3 - i\sqrt{3}a + b\right).$$

By Theorem 4, system (17) has a rational first integral if and only if $\gamma_i \in \mathbb{Q}$ for $i = 1, 2, 3$. We observe that $\gamma_1 \in \mathbb{Q}$ if and only if $b \in \mathbb{Q}$. If we consider $\gamma_2 - \gamma_3 = 2i\sqrt{3}a/9$, we see that a needs to be of the form $a = i\sqrt{3}\,c$ with $c \in \mathbb{Q}$. We denote by m the least common multiple of the denominators of $\{3+2b, 3-b+3c, 3-b-3c\}$. A rational first integral in this case is:

$$H(x,y) = \left(x^{3+2b}\left(x - i\sqrt{3}y\right)^{3-b+3c}\left(x + i\sqrt{3}y\right)^{3-b-3c}\right)^m. \qquad \square$$

Proposition 13. *A quadratic homogeneous polynomial differential system of the form* (1) *has a polynomial first integral if and only if it is affine-equivalent to a quadratic system of the following form*

$$\dot{x} = -2xy + \frac{2}{3}x(ax+by), \quad \dot{y} = x^2 + y^2 + \frac{2}{3}y(ax+by),$$

with $a = i\sqrt{3}\,c$, $b, c \in \mathbb{Q}$ and (b,c) belong to the triangle $b > -3/2$, $b < 3 - 3c$, $b < 3 + 3c$.

Proof: If a complex quadratic homogeneous polynomial differential system has a polynomial first integral, then in particular it has a rational first integral. Thus, we are under the hypothesis of Proposition 12. We consider system (17) with $a = i\sqrt{3}\,c$ and $b, c \in \mathbb{Q}$. Thus, the values of the quotients of eigenvalues of the singular points on the exceptional divisor are:

$$\gamma_1 = -\frac{3+2b}{9}, \quad \gamma_2 = \frac{1}{9}\left(-3 - 3c + b\right), \quad \gamma_3 = \frac{1}{9}\left(-3 + 3c + b\right),$$

as we have seen in the proof of Proposition 10 and where we have substituted $a = i\sqrt{3}\,c$. By Theorem 4, system (17) has a polynomial first integral if and only if $\gamma_i \in \mathbb{Q}^-$ for $i = 1, 2, 3$, which implies that (b,c) belong to the triangle $b > -3/2$, $b < 3 - 3c$, $b < 3 + 3c$. $\square$

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References

[1] J. C. ARTÉS, J. LLIBRE, AND N. VULPE, Quadratic systems with a polynomial first integral: a complete classification in the coefficient space $\mathbb{R}^{12}$, *J. Differential Equations* **246(9)** (2009), 3535–3558. DOI: 10.1016/j.jde.2008.12.010.

[2] C. CAMACHO AND P. SAD, Invariant varieties through singularities of holomorphic vector fields, *Ann. of Math. (2)* **115(3)** (1982), 579–595. DOI: 10.2307/2007013.

[3] D. CERVEAU AND A. LINS NETO, Holomorphic foliations in $\mathbb{CP}(2)$ having an invariant algebraic curve, *Ann. Inst. Fourier (Grenoble)* **41(4)** (1991), 883–903. DOI: 10.5802/aif.1278.

[4] J. CHAVARRIGA, I. A. GARCÍA, AND J. GINÉ, On integrability of differential equations defined by the sum of homogeneous vector fields with degenerate infinity, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **11(3)** (2001), 711–722. DOI: 10.1142/S0218127401002390.

[5] J. CHAVARRIGA, H. GIACOMINI, J. GINÉ, AND J. LLIBRE, On the integrability of two-dimensional flows, *J. Differential Equations* **157(1)** (1999), 163–182. DOI: 10.1006/jdeq.1998.3621.

[6] J. CHAVARRIGA, H. GIACOMINI, J. GINÉ, AND J. LLIBRE, Darboux integrability and the inverse integrating factor, *J. Differential Equations* **194(1)** (2003), 116–139. DOI: 10.1016/S0022-0396(03)00190-6.

[7] J. CHAVARRIGA, H. GIACOMINI, AND M. GRAU, Necessary conditions for the existence of invariant algebraic curves for planar polynomial systems, *Bull. Sci. Math.* **129(2)** (2005), 99–126. DOI: 10.1016/j.bulsci.2004.09.002.

[8] C. CHRISTOPHER, J. LLIBRE, C. PANTAZI, AND X. ZHANG, Darboux integrability and invariant algebraic curves for planar polynomial systems, *J. Phys. A* **35(10)** (2002), 2457–2476. DOI: 10.1088/0305-4470/35/10/310.

[9] G. Darboux, Mémoire sur les équations différentielles algébriques du premier ordre et du premier degré, *Bulletin des sciences mathématiques et astronomiques, Sér. 2* **2(1)** (1878), 60–96, 123–144, 151–200.

[10] T. Date and M. Iri, Canonical forms of real homogeneous quadratic transformations, *J. Math. Anal. Appl.* **56(3)** (1976), 650–682. DOI: 10.1016/0022-247X(76)90031-7.

[11] F. Dumortier, J. Llibre, and J. C. Artés, *"Qualitative theory of planar differential systems"*, Universitext, Springer-Verlag, Berlin, 2006.

[12] A. Ferragut, Some new results on Darboux integrable differential systems, *J. Math. Anal. Appl.* **394(1)** (2012), 416–424. DOI: 10.1016/j.jmaa.2012.04.075.

[13] A. Ferragut, J. Llibre, and A. Mahdi, Polynomial inverse integrating factors for polynomial vector fields, *Discrete Contin. Dyn. Syst.* **17(2)** (2007), 387–395. DOI: 10.3934/dcds.2007.17.387.

[14] J. Giné and J. Llibre, On the planar integrable differential systems, *Z. Angew. Math. Phys.* **62(4)** (2011), 567–574. DOI: 10.1007/s00033-011-0116-5.

[15] E. Isaacson and H. B. Keller, *"Analysis of numerical methods"*, Corrected reprint of the 1966 original, Dover Publications, Inc., New York, 1994.

[16] W. Li, J. Llibre, and X. Zhang, Planar analytic vector fields with generalized rational first integrals, *Bull. Sci. Math.* **125(5)** (2001), 341–361. DOI: 10.1016/S0007-4497(01)01083-1.

[17] W. Li, J. Llibre, and X. Zhang, Local first integrals of differential systems and diffeomorphisms, *Z. Angew. Math. Phys.* **54(2)** (2003), 235–255. DOI: 10.1007/s000330300003.

[18] A. Lins Neto, Some examples for the Poincaré and Painlevé problems, *Ann. Sci. École Norm. Sup. (4)* **35(2)** (2002), 231–266. DOI: 10.1016/S0012-9593(02)01089-3.

[19] J. Llibre and C. Pantazi, Polynomial differential systems having a given Darbouxian first integral, *Bull. Sci. Math.* **128(9)** (2004), 775–788. DOI: 10.1016/j.bulsci.2004.04.001.

[20] J. Llibre, C. Pantazi, and S. Walcher, First integrals of local analytic differential systems, *Bull. Sci. Math.* **136(3)** (2012), 342–359. DOI: 10.1016/j.bulsci.2011.10.003.

[21] J. Llibre and X. Zhang, Polynomial first integrals for quasi-homogeneous polynomial differential systems, *Nonlinearity* **15(4)** (2002), 1269–1280. DOI: 10.1088/0951-7715/15/4/313.

[22] J. MOULIN-OLLAGNIER, Polynomial first integrals of the Lotka–Volterra system, *Bull. Sci. Math.* **121(6)** (1997), 463–476.
[23] J. MOULIN-OLLAGNIER, Rational integration of the Lotka–Volterra system, *Bull. Sci. Math.* **123(6)** (1999), 437–466. DOI: 10.1016/S0007-4497(99)00111-6.
[24] J. MOULIN-OLLAGNIER, Simple Darboux points of polynomial planar vector fields, *J. Pure Appl. Algebra* **189(1–3)** (2004), 247–262. DOI: 10.1016/j.jpaa.2003.10.027.
[25] A. NOWICKI, On the nonexistence of rational first integrals for systems of linear differential equations, *Linear Algebra Appl.* **235** (1996), 107–120. DOI: 10.1016/0024-3795(94)00122-7.
[26] H. POINCARÉ, Sur l'intégration algébrique des équations différentielles du premier ordre et du premier degré, *C. R. Acad. Sci., Paris* **112** (1891), 761–764.
[27] H. POINCARÉ, Sur l'intégration algébrique des équations différentielles du premier ordre et du premier degré, *Palermo Rend.* **5** (1891), 161–191.
[28] H. POINCARÉ, Sur l'intégration algébrique des équations différentielles du premier ordre et du premier degré, *Palermo Rend.* **11** (1897), 193–239.

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