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HALF-REEB COMPONENTS, PALAIS–SMALE CONDITION AND GLOBAL INJECTIVITY OF LOCAL DIFFEOMORPHISMS IN $\mathbb{R}^3$

Francisco Braun and Jean Venato-Santos

Abstract: Let $F = (F_1, F_2, F_3)\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a C^∞ local diffeomorphism. We prove that each of the following conditions are sufficient to the global injectivity of F:

A) The foliations $\mathcal{F}_{F_i}$ made up by the connected components of the level surfaces $F_i =$ constant, consist of leaves without half-Reeb components induced by F_j, $j \in \{1,2,3\} \setminus \{i\}$, for $i \in \{1,2,3\}$.

B) For each $i \neq j \in \{1,2,3\}$, $F_i|_L\colon L \to \mathbb{R}$ satisfy the Palais–Smale condition, for all $L \in \mathcal{F}_{F_j}$.

We also prove that B) implies A) and give examples to show that the converse is not true. Further, we give examples showing that none of these conditions is necessary to the global injectivity of F.

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1. Introduction

The problem of establishing conditions to ensure that a local diffeomorphism $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism remounts to the beginning of the twentieth century with the work of Hadamard ([**13**], see also [**21**]) and its classical hypothesis $\int_0^\infty \inf_{|x|\leq s} \left\|DF(x)^{-1}\right\|^{-1} ds = \infty$, or even the result of Banach and Mazur saying that F is a diffeomorphism if and only if F is proper (a good reference is again [**21**]). Since then, many areas of Mathematics have been asking for different conditions (maybe more simple). In 1939, for example, Keller conjectured that if F is polynomial, then no additional hypothesis needed to be made. This problem is known as Real Jacobian Conjecture and was proved false in 1994 by Pinchuk (see [**20**]). In $\mathbb{R}^2$, if the degree of F is low, then F is a diffeomorphism (see [**3**]). This polynomial case of the global invertibility problem is closely related to the famous Jacobian Conjecture: very polynomial local diffeomorphism $F\colon \mathbb{C}^n \to \mathbb{C}^n$ is an automorphism (invertible with polynomial inverse). In this case, it is enough to prove the injectivity of F to establish the conjecture (see [**2**] and [**5**]). Anyway Jacobian Conjecture remains open up to now. References can be found in [**1**] and in [**7**].

Now returning to the general case (F does not to be polynomial), we can ask just for the *injectivity* of F. In $\mathbb{R}^2$, an interesting result is the following, due to Fernandes, Gutierrez, and Rabanal [**8**]:

Theorem 1.1. *Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable map. If there exists $\varepsilon > 0$ such that*

$$\operatorname{Spec}(F) \cap [0, \varepsilon) = \varnothing,$$

then F is globally injective.

Here $\operatorname{Spec}(F)$ stands for the set of all the (complex) eigenvalues of $DF(x)$, when x varies in $\mathbb{R}^2$. A first (C^1) version of this theorem appeared in [**9**], where Gutierrez solved the bi-dimensional case of the Markus–Yamabe Problem (see [**16**]). Indeed, in [**18**], Olech had already proved the equivalence between Markus–Yamabe Conjecture and the injectivity of F in $\mathbb{R}^2$. Recently, in [**12**] and [**22**], analogous results were obtained when the equilibrium point 0 is not an attractor, but a hyperbolic saddle and a center, respectively.

The essential tool to prove Theorem 1.1 is the concept of half-Reeb component (hRc for short) of planar foliations that we recall in Definition 3.1. The connection between hRc and injectivity is given by Proposition 1.2 right below. But before seeing this, let us introduce a notation: given $f\colon M \to \mathbb{R}$ a C^k-submersion, $k \geq 1$, where M is a differentiable manifold, then the connected components of the level sets of f give rise to a C^k-foliation of codimension 1 of M. *We will denote this foliation by $\mathcal{F}_f$.*

Proposition 1.2. *Let $F = (F_1, F_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable map such that $DF(x)$ is non singular for all $x \in \mathbb{R}^2$. If F is not injective, then $\mathcal{F}_{F_i}$ has a* hRc*, for both $i = 1, 2$.*

The proof of this proposition can be found in [**8**]. With this tool, the proof of Theorem 1.1 depicted in [**8**] uses Proposition 1.2 after filling in the following two items:

(i) It is shown that it is enough to prove the result under the stronger condition $\operatorname{Spec}(F) \cap (-\varepsilon, \varepsilon) = \varnothing$.
(ii) Under the spectral condition of (i), it is shown the non-existence of hRc for $\mathcal{F}_{F_i}$, $i = 1, 2$.

Gutierrez and Maquera introduced in [**11**] the concept of half-Reeb component for a foliation $\mathcal{F}_f$ when $f\colon \mathbb{R}^3 \to \mathbb{R}$ is a differentiable submersion. We recall this in Definition 3.4 and call it *half-Reeb component of type* 2 (hRc_2 for short). With this concept, they proved

Theorem 1.3. *Let $F = (F_1, F_2, F_3)\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a C^2 local diffeomorphism. If $\mathrm{Spec}(F) \cap (-\varepsilon, \varepsilon) = \varnothing$, for some $\varepsilon > 0$, then $\mathcal{F}_i$ does not have any hRc_2, for $i = 1, 2, 3$. In particular, all the leaves of $\mathcal{F}_{F_i}$ are diffeomorphic to $\mathbb{R}^2$.*

As we said above this is one of the steps to obtain the global injectivity of Theorem 1.1 in the bidimensional case (by Proposition 1.2). However, as Examples 3.5 and 3.8 below show, Proposition 1.2 is not true to maps $F\colon \mathbb{R}^3 \to \mathbb{R}^3$, when we change hRc by hRc_2. So the injectivity of F can not be obtained exactly as in Theorem 1.1.

Despite this obstruction, Gutierrez and Maquera [**11**] used the non-existence of hRc_2 in $\mathcal{F}_{F_i}$, for $i = 1, 2, 3$, to obtain a global injectivity result when F is polynomial and has the set of not proper points with codimension greater than or equal to 2. This is a weak version of the Real Jacobian Conjecture of Jelonek (see [**14**]) in dimension 3 and was recently generalized by Maquera and Venato-Santos in [**15**] to the n-dimensional case.

These facts motivated us to study the consequences of non-injectivity of $F = (F_1, F_2, F_3)\colon \mathbb{R}^3 \to \mathbb{R}^3$ on the associated foliations $\mathcal{F}_{F_i}$ of $\mathbb{R}^3$, for $i = 1, 2, 3$. Our investigation lead us to the following: if F is a not injective C^∞ local diffeomorphism then there exist $i \neq j \in \{1, 2, 3\}$ and a leaf $L \in \mathcal{F}_{F_i}$ such that $\mathcal{F}_{F_j|_L}$ has a hRc (observe $\mathcal{F}_{F_j|_L}$ is a foliation of dimension 1 of L, since $F_j|_L\colon L \to \mathbb{R}$ is a submersion). Putting this consequence more precise, let us define that $\mathcal{F}_{F_j}$ *has a half-Reeb component of type* 0 (hRc_0 for short) if there exists a leaf L of one of the foliations $\mathcal{F}_{F_i}$, $i \in \{1, 2, 3\} \setminus \{j\}$ such that $\mathcal{F}_{F_j|_L}$ has a hRc. With this definition, our conclusion is the following:

Theorem A. *Let $F = (F_1, F_2, F_3)\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a C^∞ local diffeomorphism. If there is no hRc_0 in the foliations $\mathcal{F}_{F_j}$, for all $j \in \{1, 2, 3\}$, then F is globally injective.*

Now as a consequence of Theorem A, we obtain a sufficient condition for global injectivity based on the concept of Palais–Smale (PS) condition: let $f\colon M \to \mathbb{R}$ be a C^1 map, where M is a C^1 manifold. We say that f satisfies the *Palais–Smale condition* (or simply *PS condition*) if any sequence $\{x_n\}$ in M, such that $\{f(x_n)\}$ is bounded and $Df(x_n) \to 0$, has a convergent subsequence. In the special case $Df(x) \neq 0$, for all $x \in M$, we can state f satisfies PS condition when for any sequence $\{x_n\}$, such that $x_n \to \infty$ and $f(x_n) \to c$,[1] there exists $\varepsilon > 0$ with $||Df(x_n)|| \geq \varepsilon$, for all $n \in \mathbb{N}$, where $||\cdot||$ is a Riemannian Metric of M. The use of PS type condition as a mechanism to globalize the injectivity of local diffeomorphisms has been exploited in many recent works, see for instance [**4, 10, 17, 23, 24, 27**] and the references therein. Theorem B bellow generalizes to $\mathbb{R}^3$ the bidimensional result of [**10**]: "Given $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ a C^∞ local diffeomorphism, if $F_i\colon \mathbb{R}^2 \to \mathbb{R}$ satisfies the PS condition, then $\mathcal{F}_{F_i}$ has no hRc". Then Proposition 1.2 gives the global injectivity of F.

Theorem B. *Let $F = (F_1, F_2, F_3)\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a C^∞ local diffeomorphism. If for each $i \neq j \in \{1, 2, 3\}$, $F_i|_L\colon L \to \mathbb{R}$ satisfy PS condition, for all $L \in \mathcal{F}_{F_j}$, then F is globally injective.*

The paper is organized as follows. In Section 2, we recall the known concept of global solvability of vector fields and see a known result of global injectivity using this concept, which is a different generalization of Proposition 1.2 in all dimensions.

In the third section, we recall the definitions of hRc in a 2 manifold and hRc_2 in $\mathbb{R}^3$. We also give examples that motivate the definitions of hRc_1 and hRc_0, and Theorem A.

Finally, Section 4 is devoted to prove Theorem A after proving the more general Propositions 4.1 and 4.2. In fact the last one guarantees that hRc_2 implies the existence of hRc_0. We then finish with the proof of Theorem B, which is a direct consequence of Theorem A and Proposition 4.4, where we prove (adapting ideas of [**10**]) that the existence of hRc in $\mathcal{F}_{F_i|_L}$, $L \in \mathcal{F}_{F_j}$, guarantees that $F_i|_L\colon L \to \mathbb{R}$ does not satisfy PS condition.

The very simple Example 3.11 shows that the reciprocal of Proposition 4.4 is not true and that the condition in Theorem B is not necessary for injectivity.

2. Preliminaries

We now recall the definition of global solvability of vector fields.

Definition 2.1. Let M be a C^∞ manifold and $X\colon C^\infty(M) \to C^\infty(M)$ be a vector field. We say that X is globally solvable when X is surjective, i.e. given $f\colon M \to \mathbb{R}$, there is $u\colon M \to \mathbb{R}$ such that $Xu = f$ (f and u are C^∞).

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[1] $x_n \to \infty$ means that $\{x_n\}$ does not have any convergent subsequence.

The next result gives a geometric characterization of global solvability and is part of Theorem 6.4.2 of [**6**], due to Duistermaat and Hörmander.

Lemma 2.2. *Let M be a C^∞ manifold and $X\colon C^\infty(M) \to C^\infty(M)$ be a vector field. The items bellow are equivalent:*

(1) *X is globally solvable.*

(2) (a) *No integral curve of X is contained in a compact subset of M.*
   (b) *For all compact $K \subset M$, there exists a compact $K' \subset M$ such that every compact interval on an integral curve of X with end points in K is contained in K'.*

Now given a C^∞ map $F = (F_1, \dots, F_n)\colon \mathbb{R}^n \to \mathbb{R}^n$, we define n vector fields ν_i, $i = 1, \dots, n$, as follows:

$$\nu_i(\phi) = \det D(F_1, \dots, F_{i-1}, \phi, F_{i+1}, \dots, F_n),$$

and recall the following result of [**26**] (see also [**25**]):

Theorem 2.3. *Let $F\colon \mathbb{R}^n \to \mathbb{R}^n$ be a C^∞ local diffeomorphism. If ν_i is globally solvable for $n-1$ different indices $i \in \{1, \dots, n\}$, then F is injective.*

In the next result, we will see some useful properties of the vector fields ν_i.

Lemma 2.4. *Let $F\colon \mathbb{R}^n \to \mathbb{R}^n$ be a C^∞ map such that $\det DF(x) \neq 0$, $\forall\, x \in \mathbb{R}^n$. For each $i \in \{1, \dots, n\}$, the following holds:*

(1) *The integral curves of ν_i are the non-empty connected components of the intersections*

$$\cap_{\substack{j=1\\ j\neq i}}^{n} F_j^{-1}(c_j), \quad c_j \in \mathbb{R}.$$

(2) *F_i is strictly monotone along the integral curves of ν_i.*

(3) *The alpha and omega limit sets of each integral curve of ν_i are empty.*

Proof: Given $\gamma(t)$ an integral curve of ν_i, we have that for each $j \in \{1, \dots, n\}$, $(F_j(\gamma))'(t) = \delta_{ij} \det DF(\gamma(t))$, where δ_{ij} is the Kronecker delta. This shows that F_i is strictly monotone along γ (proving item (2)) and that γ is contained in a connected component of one of the intersections of item (1). Since these connected components are C^∞ curves (by the Implicit Function Theorem), we have by maximality of $\gamma(t)$ that it must coincide with this curve, proving item (1). To prove item (3), observe that item (1) guarantees that each integral curve of ν_i is a closed set, so it contains its alpha and omega limit sets. If the alpha (or the omega) limit set of one integral curve of ν_i is non-empty, we will have a periodic integral curve, what is impossible by item (2). $\square$

3. Examples and half-Reeb components of types 0, 1 and 2

We start this section recalling the concepts of hRc in foliations of two-dimensional manifolds and hRc_2 in foliations of $\mathbb{R}^3$.

Definition 3.1. Let M be a C^∞ manifold of dimension 2 and $f\colon M \to \mathbb{R}$ be a C^∞ submersion. We say that $\mathcal{A} \subset M$ is a *half-Reeb component* (or simply a hRc) of $\mathcal{F}_f$ if there is a homeomorphism $G\colon B \to \mathcal{A}$ which is a topological equivalence between $\mathcal{F}_f|_\mathcal{A}$ and $\mathcal{F}_{f_0}|_B$, where $B = \{(x,y) \in [0,2] \times [0,2];\ 0 < x+y \leq 2\}$ and $f_0\colon \mathbb{R}^2 \to \mathbb{R}$ is defined by $f_0(x,y) = xy$. Moreover, G satisfies

(1) The segment $\{(x,y) \in B \mid x+y = 2\}$ is sent by G onto a curve which gives two transversal sections for the foliation $\mathcal{F}_f$ in the complement of $G(1,1)$; this curve is called the *compact edge* of $\mathcal{A}$.
(2) Both segments $\{(x,y) \in B \mid x = 0\}$ and $\{(x,y) \in B \mid y = 0\}$ are sent by G onto full half-trajectories of $\mathcal{F}_f$. These two half-trajectories of $\mathcal{F}_f$ are called the *non-compact edges* of $\mathcal{A}$.

Remark 3.2. Note that, given a C^∞ map $F = (F_1, F_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$, such that $\det DF(x) \neq 0$, $\forall\, x \in \mathbb{R}^2$, each connected component of the intersections given by Lemma 2.4, for $i = 1$ or 2, are exactly the leaves of the foliations $\mathcal{F}_{F_2}$ or $\mathcal{F}_{F_1}$, respectively. Then Lemma 2.2 shows that $\mathcal{F}_{F_i}$ does not have any hRc if and only if ν_j is globally solvable, for $i \neq j \in \{1,2\}$. So in $\mathbb{R}^2$, Theorem 2.3 and Proposition 1.2 are equivalent.[2]

Now the hRc_2 concept (according to [**11**]):

Definition 3.3. Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be a C^2 submersion. We say that a C^2-embedding $H_0\colon S^1 \to \mathbb{R}^3$ is a *vanishing cycle* for $\mathcal{F}_f$ if it satisfies:

(1) $H_0(S^1)$ is contained in a leaf L_0 but is not homotopic to a point in L_0.
(2) H_0 can be extended to a C^2-embedding $H\colon [-1,2] \times S^1 \to \mathbb{R}^3$ such that for all $t \in (0,1]$, there is a 2-disc D_t contained in a leaf $L_t \in \mathcal{F}_f$ with $\partial D_t = H(\{t\} \times S^1)$.
(3) For all $x \in S^1$, the curve $t \in [-1,2] \mapsto H(t,x)$ is transverse to the foliation $\mathcal{F}_f$ and, for all $t \in (0,1]$, the disc D_t depends continuously on t.

We say that the leaf L_0 *supports* the vanishing cycle H_0 and that the map H is associated to H_0.

[2]Observe that when $n = 2$, $\nu_i = (-1)^i H_{F_j}$, $i \neq j \in \{1,2\}$, where H_{F_j} stands for the Hamiltonian vector field associated to F_j, $H_{F_j} = -\partial_2 F_j \partial_1 + \partial_1 F_j \partial_2$.

Definition 3.4. A *half-Reeb component of type* 2 (that we denote by hRc_2) of $\mathcal{F}_f$ associated to the vanishing cycle H_0 is the region

$$\mathcal{A} = \left(\bigcup_{t\in(0,1]} D_t \right) \cup L \cup H_0(S^1),$$

where L is the connected component of $L_0 \setminus H_0(S^1)$ contained in the closure of $\cup_{t\in(0,1]} D_t$. We say that the transversal $H([0,1]\times S^1)$ (to the foliation $\mathcal{F}_f$) is the *compact face* of $\mathcal{A}$ and the half-leaf $L \cup H_0(S^1)$ is the *non-compact face* of $\mathcal{A}$.

As we said in the introduction, just the non-existence of hRc_2 in the foliations $\mathcal{F}_{F_i}$, $i = 1, 2, 3$ does not imply the injectivity of F in dimension 3. Here we give some examples:

Example 3.5. Consider the map $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$F(x_1, x_2, x_3) = \big(F_1(x_1, x_2), F_2(x_1, x_2), x_3\big),$$

where $(F_1, F_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a differentiable non-injective local diffeomorphism (for example, take $F_1 = e^{x_1}\cos(x_2)$ and $F_2 = e^{x_1}\sin(x_2)$). The leaves of the foliations $\mathcal{F}_{F_i}$, for $i = 1, 2, 3$, are all diffeomorphic to the plane, so it is clear that each of these foliations has a hRc_2.

More precisely, the leaves of the foliation $\mathcal{F}_{F_3}$ are the planes $x_3 = c$, $c \in \mathbb{R}$. Now the leaves of $\mathcal{F}_{F_i}$, for $i = 1$ or 2 are of the form $l \times \mathbb{R}$, where l are the leaves of the foliations of $\mathbb{R}^2$, $\mathcal{F}_{F_1}$ and $\mathcal{F}_{F_2}$, respectively.

It is simple to modify the map of the example above in a way that one of the foliations $\mathcal{F}_{F_i}$ presents hRc_2:

Example 3.6. Consider the map $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$F(x_1, x_2, x_3) = \big(F_1(x_1, x_2), F_2(x_1, x_2), x_1^2 + x_2^2 - e^{x_3}\big),$$

where $(F_1, F_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a differentiable non-injective local diffeomorphism. The foliations $\mathcal{F}_{F_i}$, for $i = 1$ and 2 have the same property than the ones of Example 3.5. But the foliation $\mathcal{F}_{F_3}$ has a hRc_2.

In these two examples, as we said, the foliations $\mathcal{F}_{F_i}$, for $i = 1, 2$, have its leaves of the form $l \times \mathbb{R}$, where l are the leaves of the foliation $\mathcal{F}_{F_i}$, respectively. Since the map $(x_1, x_2) \mapsto (F_1(x_1, x_2), F_2(x_1, x_2))$ is not injective, we get by Proposition 1.2 that $\mathcal{F}_{F_i}$, for $i = 1$ and 2, exhibit hRc.

Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be a C^∞ submersion such that all the leaves of $\mathcal{F}_f$ are diffeomorphic to $\mathbb{R}^2$ (so $\mathcal{F}_f$ does not have hRc_2). By Corollary 1 of [**19**], there is a diffeomorphism $H\colon \mathbb{R}^3 \to \mathbb{R}^3$ that take leaves of $\mathcal{F}_f$

onto leaves of a foliation of type $\mathcal{F}_0 \times \mathbb{R}$, where $\mathcal{F}_0$ is a foliation of $\mathbb{R}^2$. We say that $\mathcal{F}_f$ has a *half-Reeb component of type* 1, (or simply hRc_1) if $\mathcal{F}_0$ has a hRc (it is possible to prove that $\mathcal{F}_0$ is defined by the connected components of a submersion – see the second part of Remark 3.7). See Figure 1(b).

Remark 3.7. Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be a C^∞ submersion such that all the leaves of $\mathcal{F}_f$ are diffeomorphic to $\mathbb{R}^2$. Then $\mathcal{F}_f$ has a hRc_1 if, and only if, f has a disconnected level set. Indeed, suppose first that $\mathcal{F}_f$ has a hRc_1 and consider $H\colon \mathbb{R}^3 \to \mathbb{R}^3$ and $\mathcal{F}_0$ as in the definition above. Since $\mathcal{F}_0$ has a hRc, there are points $a, b \in \mathbb{R}^2$ in different leaves of $\mathcal{F}_0$ and a sequence l_n of leaves of $\mathcal{F}_0$ each of them containing a_n, b_n points of $\mathbb{R}^2$ such that $a_n \to a$ and $b_n \to b$. Since $f\left(H^{-1}(a_n, 0)\right) = f\left(H^{-1}(b_n, 0)\right)$, we get by continuity that $H^{-1}(a, 0)$ and $H^{-1}(b, 0)$ are in the same level set of f. This level set must be disconnected (since its image by H is disconnected).

On the other hand, since the leaves of $\mathcal{F}_f$ are all diffeomorphic to $\mathbb{R}^2$, we have by Corollary 1 of [**19**] that there is $H\colon \mathbb{R}^3 \to \mathbb{R}^2 \times \mathbb{R}$ a diffeomorphism such that $\mathcal{F}_f$ is taken by H in a foliation of type $\mathcal{F}_0 \times \mathbb{R}$, where $\mathcal{F}_0$ is a foliation of $\mathbb{R}^2$. Then it is not difficult to see that $\mathcal{F}_0$ is the foliation $\mathcal{F}_g$ where $g\colon \mathbb{R}^2 \to \mathbb{R}$ is the *submersion* defined by $g(x) = f\left(H^{-1}(x, 0)\right)$. If f has a disconnected level set then so does g. Then (by Theorem 2.4 of [**3**], for example, or, for a different argument, see the proof of Proposition 1.4 of [**8**]), $\mathcal{F}_0 = \mathcal{F}_g$ has a hRc and so $\mathcal{F}_f$ has a hRc_1 by definition.

With this concept, we ask if the non injectivity of $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ implies that one of the foliations $\mathcal{F}_{F_i}$ has hRc_1 or hRc_2. Next example shows it is not the case:

Example 3.8. Let $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$F(x_1, x_2, x_3) = (x_3 - e^{x_1} \cos x_2, x_3 - e^{x_1} \sin x_2, x_3)\,.$$

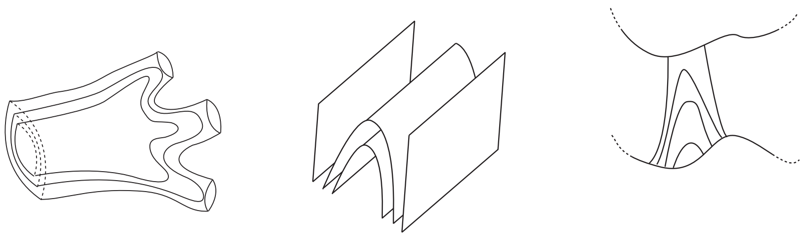
F is clearly not injective and $\det DF(x) = e^{2x_1}$.

Moreover, all the level sets of F_i are clearly connected and diffeomorphic to $\mathbb{R}^2$, for $i = 1, 2, 3$ (all of them are graphics of C^∞ maps carrying $\mathbb{R}^2$ to $\mathbb{R}$). Hence, the foliations $\mathcal{F}_{F_i}$, $i = 1, 2, 3$, do not have hRc_2 neither hRc_1 (by Remark 3.7).

Remark 3.9. Example 3.8 above also shows that $F_i\colon \mathbb{R}^3 \to \mathbb{R}$ satisfying PS condition, for $i = 1, 2, 3$, is not sufficient for global injectivity. Indeed, we have that $||\nabla F_i(x)|| \geq 1$, for all $i = 1, 2, 3$.

So, what is the influence of the non-injectivity of F on the foliations $\mathcal{F}_{F_i}$? Observe that in the three examples above we have the (opposite) property of our main Theorem A: in Example 3.5, any leaf L of $\mathcal{F}_{F_3}$ satisfies that $\mathcal{F}_{F_i|_L}$ have hRc for $i = 1, 2$. In Example 3.6, take the leaves $L = F_3^{-1}(c)$, with $c < 0$, of $\mathcal{F}_{F_3}$. It is also clear that $\mathcal{F}_{F_i|_L}$ have hRc for $i = 1, 2$. Now in Example 3.8, take $L = F_3^{-1}(0)$ the leaf of $\mathcal{F}_{F_3}$ to be considered. It is also simple to see that $\mathcal{F}_{F_i|_L}$ have hRc, for $i = 1, 2$. So the common property is that at least one of the foliations $\mathcal{F}_{F_i}$ has a hRc_0 as defined in the introduction.

See Figure 1 for the hRc_0, hRc_1 and hRc_2 in $\mathbb{R}^3$.



(a) hRc_2 (b) hRc_1 (c) hRc_0

FIGURE 1.

Remark 3.10. In the next section we will prove, in Proposition 4.2, that the existence of hRc_2 implies the occurrence of hRc_0. Hence, modifying Example 3.6 such that $F_1(x_1, x_2) = x_1$ and $F_2(x_1, x_2) = x_2$, for example, we have that $\mathcal{F}_{F_3}$ has a hRc_0 despite of the injectivity of F. Proving that the non existence of hRc_0 is not necessary to the injectivity in $n = 3$.

Example 3.11. Let $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$F(x_1, x_2, x_3) = (\arctan x_1, x_2, x_3),$$

where we choose the values of $\arctan x_1$ in the interval $(-\pi/2, \pi/2)$. It is clear that $\det DF(x) = \frac{1}{1+x_1^2} > 0$ and F is globally injective. Moreover, there is no hRc_0 in the foliations $\mathcal{F}_{F_j}$, for all $j \in \{1, 2, 3\}$.

Now, consider the leaf $L = \mathbb{R} \times \{0\} \times \mathbb{R}$ of $\mathcal{F}_{F_2}$ and the sequence $p_n = (n, 0, 0) \to \infty$ in L. We have that $F_1(p_n) \to \pi/2$ and $D(F_1|_L)(p_n) = \left(\frac{1}{1+n^2}, 0, 0\right) \to 0$, when $n \to \infty$. That is, $F_1|_L$ does not satisfies PS condition. Indeed, it is simple to observe that $F_1|_M$ does not satisfy PS condition for any $M \in \mathcal{F}_{F_j}$, $j = 2$ and 3. This shows that condition A) is weaker than B).

4. Proof of main results

We will prove the following stronger proposition, which by Theorem 2.3 and Proposition 4.2 will result in Theorem A.

Proposition 4.1. *Let $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a C^∞ local diffeomorphism. If there is $i \in \{1,2,3\}$ such that ν_i is not globally solvable, then there is $j \in \{1,2,3\} \setminus \{i\}$ such that $\mathcal{F}_{F_j}$ has a* hRc$_0$. *More precisely, there is $L \in \mathcal{F}_{F_k}$, with $k \in \{1,2,3\} \setminus \{i,j\}$, such that $\mathcal{F}_{F_j|L}$ has a* hRc*.*

Proof: Let us suppose without loss of generality that ν_3 is not globally solvable.

We will first make the proof supposing that $\mathcal{F}_{F_1}$ and $\mathcal{F}_{F_2}$ are foliations whose leaves are diffeomorphic to $\mathbb{R}^2$.

By Lemma 2.2 there exists a compact $K \subset \mathbb{R}^3$ such that for all $n \in \mathbb{N}$, there exist γ_n, an integral curve of ν_3, and $0 < s_n < t_n \in \mathbb{R}$ such that $\gamma_n(0), \gamma_n(t_n) \in K$ but $|\gamma_n(s_n)| > n$. Since K is compact, passing to subsequences, if necessary, we may assume that $\gamma_n(0) \to a \in K$ and $\gamma_n(t_n) \to b \in K$. Furthermore, since $\gamma_n(0), \gamma_n(t_n) \in F_1^{-1}\{c_{1n}\} \cap F_2^{-1}\{c_{2n}\}$, for some $c_{1n}, c_{2n} \in \mathbb{R}$ (by Lemma 2.4), we have that a and b are in the same level set of both F_1 and F_2. Let us denote L_a^i and L_b^i the leaves of $\mathcal{F}_{F_i}$ which contain a and b, respectively, for $i = 1, 2$. We have two possibilities:

(1) $L_a^1 = L_b^1$ or $L_a^2 = L_b^2$;

(2) $L_a^1 \neq L_b^1$ and $L_a^2 \neq L_b^2$.

In case (1), suppose that $L_a^2 = L_b^2$. We *assert a and b can not be in the same integral curve of* ν_3, since if this is so, by the Flow Box Theorem, we can construct a tubular neighborhood Γ along the arc of integral curve γ from a to b. Since each leaf of $\mathcal{F}_{F_3}$ is a local transversal section of ν_3, this tubular neighborhood can be built such that the leaf of $\mathcal{F}_{F_3}$ passing through a is a global transversal section of $\nu_3|_\Gamma$. Then by item (2) of Lemma 2.4, we have that each integral curve of ν_3 enters Γ just once. So for n big enough, the arcs of trajectories γ_n from $\gamma_n(0)$ to $\gamma_n(t_n)$ are entirely contained in Γ. But this is a clear contradiction with the fact stated above that there exists s_n in $(0, t_n)$ such that $|\gamma(s_n)| \to \infty$.

This assertion then issues that $F_1|_{L_a^2}\colon L_a^2 \to \mathbb{R}$ is a submersion with a disconnected level set (again by Lemma 2.4). We then take $h\colon L_a^2 \to \mathbb{R}^2$ a diffeomorphism and consider the submersion $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f = F_1 \circ h^{-1}$. It is clear f has a disconnected level set and so $\mathcal{F}_f$ has a hRc (again by Theorem 2.4 of [**3**] or by the proof of Proposition 1.4 of [**8**]). So by definition, $\mathcal{F}_{F_1|_{L_a^2}}$ has a hRc. Consequently, $\mathcal{F}_{F_1}$ has a hRc$_0$.

Now for case (2), call L_n^1 and L_n^2 the leaves of $\mathcal{F}_{F_1}$ and $\mathcal{F}_{F_2}$, which contains γ_n, respectively. We *assert there is* n_0 *such that for all* $n \geq n_0$, $L_n^2 \cap L_a^1 \neq \varnothing$ *and* $L_n^2 \cap L_b^1 \neq \varnothing$.

Before proceeding to the proof of the assertion, we observe it already implies the existence of hRc of $\mathcal{F}_{F_1|_{L_n^2}}$ on each of the leaves L_n^2 (i.e. $\mathcal{F}_{F_1}$ has a hRc$_0$), for all $n \geq n_0$, by the same reason above, since $F_1|_{L_n^2} : L_n^2 \to \mathbb{R}$ has a disconnected level set.

So let us prove the assertion. In a neighborhood of a, we can trivialize the foliation $\mathcal{F}_{F_2}$. So since L_a^1 intersects transversally L_a^2, this transversal intersection also occurs with leaves of $\mathcal{F}_{F_2}$ which contain points near enough of a. Since $\gamma_n(0) \to a$ as $n \to \infty$, it is clear L_n^2 will intersect transversally L_a^1 for n big enough. The same can be done to show $L_n^2 \cap L_b^1 \neq \emptyset$, as we wanted.

Now in the case there is a leaf of one of the foliations $\mathcal{F}_{F_1}$ or $\mathcal{F}_{F_2}$ which is not diffeomorphic to $\mathbb{R}^2$, Proposition 2.2 of [**11**] asserts the existence of a hRc$_2$ of the foliation $\mathcal{F}_{F_1}$ or $\mathcal{F}_{F_2}$, respectively. The proof then follows by next result. □

Proposition 4.2. *Let* $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ *be a* C^1 *local diffeomorphism such that* $\mathcal{F}_{F_i}$ *has a* hRc$_2$, *for some* $i \in \{1,2,3\}$. *Then* $\mathcal{F}_{F_i}$ *has a* hRc$_0$. *More precisely, for each* $k \in \{1,2,3\} \setminus \{i\}$, *there exists a leaf* $L \in \mathcal{F}_{F_k}$ *such that* $\mathcal{F}_{F_i|_L}$ *has a* hRc.

Proof: Let us suppose without loss of generality that $i = 1$ and that $k = 2$. Consider so $\mathcal{A}$ a hRc$_2$ of $\mathcal{F}_{F_1}$. In our arguments we will use the notations of Definitions 3.3 and 3.4.

For each n, choose $x_n \in D_{1/n}$ in such a way that $x_n \to \infty$ when $n \to \infty$. Consider γ_n the integral curve of the vector field ν_3 through x_n. This curve will cut the compact edge of $\mathcal{A}$ in two points a_n, b_n, for each $n \in \mathbb{N}$. It is clear that $a_n \to a$ and $b_n \to b$, where a and b are points in the evanescent cycle of $\mathcal{A}$. Take now L_a^2 the leaf of $\mathcal{F}_{F_2}$ passing trough a. *We assert that the intersection of* L_a^2 *with* L_0 *will give at least two connected components, i.e. at least two distinct integral curves of* ν_3. *Moreover, each of these integral curves has one end entirely contained in* L. Indeed, let us first prove that one end of the integral curve of ν_3 passing through a, γ_a, is entirely contained in L. This is so because if there is a compact arc of trajectory of γ_a in L with ends in the evanescent cycle of $\mathcal{A}$, then we use the Flow Box Theorem, as in the proof of last proposition, to construct a tubular neighborhood along this arc and get a contradiction with the fact that $x_n \to \infty$ (recall that the integral curves of ν_3 are connected components of the intersections of the level sets of F_1 and F_2 by Lemma 2.4). Now to finish the proof of the assertion, observe that L_a^2 has to intersect the evanescent cycle of $\mathcal{A}$ in at least another point different of a. Then repeat the argument for this point.

Let us suppose, without loss of generality, that there is an arc β in the evanescent cycle connecting a and b such that $\beta \cap L_a^2 = \{a, b\}$. So the intersection of L_a^2 with the compact face of $\mathcal{A}$ contains two curves $\alpha_a(t)$ and $\alpha_b(t)$ in such a way that $\alpha_a(0) = a$ and $\alpha_b(0) = b$. Moreover, $\alpha_a(t)$ and $\alpha_b(t)$ are points in ∂D_t with the property that the connected component of the intersection of L_a^2 with D_t containing the point $\alpha_a(t)$ is a curve with ends $\alpha_a(t)$ and $\alpha_b(t)$, for $t \in (0, c]$, for some $c \in (0, 1)$. Let us call this curve γ_t. We will suppose, without loss of generality that $c = 1/2$.

By construction, α_a and α_b are transversal sections of the foliation given by the connected components of $D_t \cap L_a^2$, $t \in (0, 1/2)$.

Consider now $B = \{(x, y) \in [0, 2] \times [0, 2] \mid 0 < x + y \leq 2\}$ and $f_0 \colon B \to \mathbb{R}$ given by $f_0(x, y) = xy$, as in Definition 3.1. It is clear we can define a C^1 map $H \colon B \setminus \{(x, y) \mid f_0(x, y) = s,\ s \in (1/2, 1]\} \to L_a^2$ satisfying the following:

(1) $H\left(f_0^{-1}(t(2 - t))\right) = \gamma_t$, for each $t \in (0, 1/2]$,

(2) $H(\{0\} \times (0, 2)) = \gamma_a$ and $H((0, 2) \times \{0\}) = \gamma_b$, where γ_a and γ_b are the half-trajectories of ν_3 passing through a and b, respectively, and contained in L_0.

Denote $\mathcal{C} = H\big(B \setminus \{(x, y) \mid f_0(x, y) = s,\ s \in (1/2, 1]\}\big)$. Now by a composition, we clearly obtain $G \colon B \to \mathcal{C}$ satisfying Definition 3.1. So $\mathcal{C}$ is a hRc of $\mathcal{F}_{F_1|_{L_a^2}}$, i.e. we have that $\mathcal{F}_{F_1}$ has a hRc$_0$. □

Before proving Theorem B, let us make some remarks.

Given a C^1 map $f \colon \mathbb{R}^3 \to \mathbb{R}$, let $\nabla f \colon \mathbb{R}^3 \to \mathbb{R}^3$ denote the usual gradient vector of f, that is $\nabla f = (\partial_1 f, \partial_2 f, \partial_3 f)$. Let L be a 2-dimensional C^1-submanifold of $\mathbb{R}^3$. It is clear that $D(f|_L)(p)$ is the restriction of $\nabla f(p)$ to the subspace T_pL, for each $p \in L$. In other words, it is the projection of $\nabla f(p)$ at the tangent space T_pL, for each $p \in L$.

Remark 4.3. If $F \colon \mathbb{R}^3 \to \mathbb{R}^3$ is a C^1 local diffeomorphism and L is a leaf of $\mathcal{F}_{F_j}$, for some $j = 1, 2$ or 3, then $D(F_i|_L)(p) \neq 0$, for all $p \in L$ and $i \neq j$. In fact, since F is a local diffeomorphism, $\{\nabla F_i(p) \mid i = 1, 2, 3\}$ is a basis for $\mathbb{R}^3$, for each $p \in \mathbb{R}^3$. Then since $\nabla F_j(p)$ is normal to T_pL, we conclude that $D(F_i|_L)(p) \neq 0$, for all $p \in L$.

It is simple to see that in Example 3.8 above we have F_i satisfying PS condition for $i = 1, 2$ and 3. So this is not sufficient for global injectivity. On the other hand, the next result with Theorem A show that the new PS condition introduced above is sufficient (and so prove Theorem B):

Proposition 4.4. *Let $F = (F_1, F_2, F_3)\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a C^1 local diffeomorphism. If $F_i|_L\colon L \to \mathbb{R}$ satisfy PS condition, where L is a leaf of $\mathcal{F}_{F_j}$, for $j \neq i \in \{1,2,3\}$, then $\mathcal{F}_{F_i}$ has no* hRc$_0$. *More precisely, $\mathcal{F}_{F_i|_L}$ has no* hRc.

Proof: Suppose, by contradiction, that there is a hRc $\mathcal{A}$ of $\mathcal{F}_{F_i|_L}$. We will adapt the argument of Theorem 4(i) of [**10**] to construct a sequence $p_n \to \infty$ in L such that $F_i(p_n) \to c \in \mathbb{R}$ but $\|D(F_i|_L)(p_n)\| \to 0$, when $n \to \infty$. This contradiction with PS condition of $F_i|_L$ will prove the proposition.

Let L_p and L_q the leaves of $\mathcal{F}_{F_i|_L}$ containing the non compact edges of $\mathcal{A}$ and let $c \in \mathbb{R}$ such that $F_i(L_p) = F_i(L_q) = c$. Let Σ_p and Σ_q be one-sided compact transversal sections of L_p and L_q passing through p and q, respectively, and contained at the canonical region Ω_0 of $\mathcal{F}_{F_i}|_L$ between the leaves L_p and L_q. Furthermore, we can choose Σ_p and Σ_q in such a way that the Poincaré map $\pi\colon \Sigma \setminus \{p\} \to \Sigma \setminus \{q\}$ is defined and satisfies $\lim_{x\to p} \pi(x) = q$. Taking $p_0 \in \Sigma_p$ and $q_0 \in \Sigma_q$ such that $\pi(p_0) = q_0$, denote by Ω the open subset of Ω_0 determined by the segment of leaf joining p_0 and q_0, the segments of the two one-sided transversal sections $[p, p_0] \subset \Sigma_p$ and $[q_0, q] \subset \Sigma_q$, and the two leaves L_p and L_q (see Figure 2).

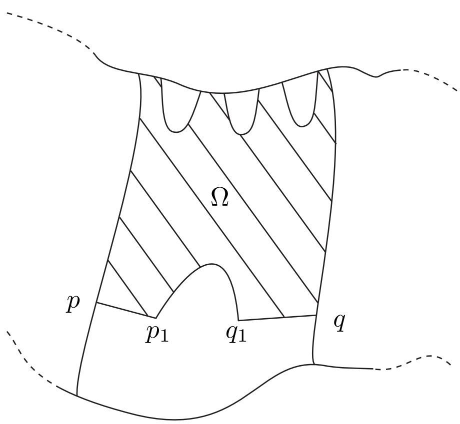


Figure 2. The subset $\Omega \subset \Omega_0$.

Observe that Ω is an unbounded set and that F_i assume values different than c in Ω, so we can assume that F_i takes values strictly less than c

on it. In fact, concerning the canonical region Ω_0, the function F_i assume the value c only in the leaves L_p, L_q and perhaps in other separatrices of Ω_0.

Consider an increasing sequence (δ_n) in $\mathbb{R}$ such that $\delta_n \to \infty$ when $n \to \infty$ and the corresponding sequence of 3-dimensional balls $B(\delta_n)$ centered at the origin of $\mathbb{R}^3$ with radius δ_n. Note that this sequence can be chosen in a such way that for each n we can choose a point r_n lying in $\Omega \cap (B(\delta_n) \setminus B(\delta_{n-1}))$ and with $F_i(r_n)$ being a strictly increasing sequence. Since the leaves containing the sequence r_n accumulate at L_p and L_q, we have $F_i(r_n) \to c$ when $n \to \infty$. For each point n, consider the solution $\phi_t(r_n)$ of the vector field $D(F_i|_L)$ passing through r_n in $t = 0$ and defined for $t \in (-1/4, 1/4)$. If such a solution is not entirely contained at Ω, change r_n by the point $\phi_{-1/4}(r_n)$ which is in Ω and by construction the solution of the vector field $D(F_i|_L)$ passing through it belongs now to Ω for all $t \in (-1/4, 1/4)$. For simplicity, let us rename all these points again by r_n.

Let $\gamma\colon \mathbb{R} \to \Omega$ be a C^1 parametrized curve that coincides with $\phi_t(r_n)$ around each point r_n. We can assume that $\gamma(n) = r_n$ for all $n \geq 0$ and that the parametrization is by arc length, i.e. $\|\gamma'(t)\| = 1$.

Applying the Mean Value Theorem to $g(t) = F_i(\gamma(t))$ around each n in the intervals $I_n = (n - 1/4, n + 1/4)$ we may conclude that there is a c_n such that:

$$g(n + 1/6) - g(n) = \frac{1}{6} g'(c_n).$$

Note that

$$\begin{aligned} g'(c_n) &= DF_i(\gamma(c_n))(\gamma'(c_n)) \\ &= \langle \nabla F_i(\gamma(c_n)), \gamma'(c_n) \rangle = \|\nabla F_i(\gamma(c_n))\| \|\gamma'(c_n)\| \cos\theta_n, \end{aligned}$$

where θ_n is the angle between the vectors $\nabla F_i(\gamma(c_n))$ and $\gamma'(c_n)$.

Since $g(n + 1/6) - g(n) \to 0$, we have that $g'(c_n) \to 0$ when $n \to \infty$. As $\|\gamma'(c_n)\| = 1$, we have two possibilities: $\|\nabla F_i(\gamma(c_n))\| \to 0$ or $\theta_n \to \pi/2$. The second case means that $\nabla F_i(\gamma(c_n))$ is becoming orthogonal to $T_{\gamma(c_n)}L$ (since $\gamma'(c_n)$ is the solution of the orthogonal projection of $\nabla F_i(p)$ at T_pL). In any case we will have $D(F_i|_L)(\gamma(c_n)) \to 0$, since this vector is the orthogonal projection of $\nabla F_i(p)$ at T_pL. Hence, the sequence $p_n = \gamma(c_n)$ satisfies $p_n \to \infty$, $F_i(p_n) \to c$ but $\|D(F_i|_L)(p_n)\| \to 0$, as desired. □

Another consequence of Proposition 4.2 and the proof of Proposition 4.4 is: if $F_i|_L\colon L \to \mathbb{R}$ satisfy the PS condition, for all $L \in \mathcal{F}_{F_j}$, $j \neq i$, then $\mathcal{F}_{F_i}$ has no hRc$_2$.

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Francisco Braun:
Departamento de Matemática
Universidade Federal de São Carlos
Rod. Washington Luis, Km 235 - C.P. 676
13565-905 São Carlos, SP
Brasil
E-mail address: franciscobraun@dm.ufscar.br

Jean Venato-Santos:
Faculdade de Matemática
Universidade Federal de Uberlândia
Av. João Naves de Ávila 2121
1F120 - CEP: 38408-100 Uberlândia
Brazil
E-mail address: jvenatos@famat.ufu.br